

ARITHMETIC FUNCTIONS EVALUATED AT POLYNOMIAL VALUES

Jean-Marie De Koninck (Québec, Canada)

Nicolas Doyon (Québec, Canada)

I. Kátai (Budapest, Hungary)

*This paper is dedicated to Professor János Galambos
on the occasion of his seventieth anniversary*

1. Introduction

Let $f : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ be a multiplicative function such that $f(p^a)$ depends only on a for all prime powers p^a , and let F_1, F_2, \dots, F_t be polynomials with integer coefficients. We establish minimal conditions on the polynomials F_i 's which guaranty that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : f(F_j(n)) \mid f(F_{j+1}(n)) \text{ for } i = 1, 2, \dots, t-1\} \text{ exists.}$$

Given $g \in \mathbb{Z}[x]$, we let $\rho_g(m) = \#\{u \bmod m : g(u) \equiv 0 \pmod{m}\}$ and we write $\text{Discr}(g)$ to denote the discriminant of g . Given $Q_1, Q_2 \in \mathbb{Z}[x]$, we let $\text{Res}(Q_1, Q_2)$ stand for their resultant.

Given a positive integer n , we let $\tau(n)$ stand for the number of divisors of n and, for any fixed integer $k \geq 1$, we let $\tau_k(n)$ stand for the number of ways one can write n as the product of k positive integers taking into account the order in which they are written. For each $n \geq 2$, let $\beta(n)$ stand for the product of the exponents in the prime factorization of n , with $\beta(1) = 1$. Let $\omega(n)$ stand for the number of distinct prime factors of $n \geq 2$, with $\omega(1) = 0$.

Let $\pi(x; k, \ell)$ stand for the number of primes $p \leq x$ such that $p \equiv \ell \pmod{k}$.

We denote by $\text{LCM}(a_1, \dots, a_k)$ the least common multiple of the positive integers a_1, \dots, a_k . In what follows, c_1, c_2, \dots stand for absolute positive constants, while c and C also denote constants, but not necessarily the same at

each occurrence. Finally, p and q , with or without subscripts, always stand for prime numbers.

At times, we shall also write x_1 for $\log x$, x_2 for $\log \log x$, and so on.

2. Main results

Theorem 1. *Let $f : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ be a multiplicative function such that $f(p^a)$ depends only on a for all prime powers p^a . Let Q_1, Q_2, \dots, Q_h be distinct irreducible primitive monic polynomials each of degree no larger than 3. For each $\nu = 1, 2, \dots, t$, let $c_1^{(\nu)}, c_2^{(\nu)}, \dots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^h Q_j(x + c_j^{(\nu)})$ ($\nu = 1, 2, \dots, t$). Let us assume that $(F_\nu(x), F_\mu(x)) = 1$ if $\nu \neq \mu$. Then, there exists a non negative constant d_0 such that*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : f(F_\ell(n)) \text{ divides } f(F_{\ell+1}(n)) \text{ for } \ell = 1, 2, \dots, t-1\} = d_0.$$

Remark 1. The condition $(F_\nu(x), F_\mu(x)) = 1$ for $\nu \neq \mu$ holds if the numbers $c_1^{(\nu)}, \dots, c_h^{(\nu)}$ ($\nu = 1, \dots, t$) are chosen in such a manner that $Q_j(x + c_j^{(\nu)}) \neq Q_i(x + c_i^{(\mu)})$ holds whenever $i \neq j$ for arbitrary values of ν and μ .

Remark 2. Interesting arithmetic functions to which one can apply Theorem 1 are $\tau(n)$, $\tau_k(n)$, $\beta(n)$ and also $a(n)$, the number of finite non isomorphic abelian groups with n elements (studied in particular by Ivić [4]).

Remark 3. From the proof of Theorem 1, the following assertion follows:

If there exists at least one positive integer n_0 such that

$$f(F_\ell(n_0)) \text{ divides } f(F_{\ell+1}(n_0)) \quad (\ell = 1, 2, \dots, t-1),$$

then $d_0 > 0$.

Theorem 2. *Let f be as in Theorem 1 and let Q_1, Q_2, \dots, Q_h be distinct irreducible primitive monic polynomials of degree no larger than 2. Then, define $F_\nu(x)$ ($\nu = 1, 2, \dots, t$) as in Theorem 1. Then, there exists a non negative constant e_0 such that*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \#\{p \leq x : f(F_\ell(p)) \text{ divides } f(F_{\ell+1}(p)) \text{ for } \ell = 1, 2, \dots, t-1\} = e_0.$$

3. Preliminary lemmas

Lemma 1. *Given $F_1, F_2 \in \mathbb{Z}[x]$, which are relatively prime, then the congruences*

$$F_1(m) \equiv 0 \pmod{a} \quad \text{and} \quad F_2(m) \equiv 0 \pmod{a}$$

have common roots for at most finitely many a 's.

Proof. A proof of this result was established by Tanaka [8]. ■

Lemma 2. *Let $F(m)$ be an arbitrary primitive polynomial with integer coefficients and of degree ν . Let D be the discriminant of F and assume that $D \neq 0$. Let $\rho(m)$ be the number of solutions n of $F(n) \equiv 0 \pmod{m}$. Then ρ is a multiplicative function whose values on the prime powers satisfy*

$$\rho(p^\alpha) \quad \begin{cases} = \rho(p) & \text{if } p \nmid D, \\ \leq 2D^2 & \text{if } p \mid D. \end{cases}$$

Moreover, there exists a positive constant $c = c(f)$ such that $\rho(p^\alpha) \leq c$ for all prime powers p^α .

Proof. This assertion is well known. ■

Lemma 3. *If $g \in \mathbb{Q}[x]$ is an irreducible polynomial and $\rho(m)$ stands for the number of residue classes mod m for which $g(n) \equiv 0 \pmod{m}$, then*

$$(i) \quad \sum_{p \leq x} \rho(p) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right);$$

$$(ii) \quad \sum_{p \leq x} \frac{\rho(p)}{p} = \log \log x + C + O\left(\frac{1}{\log x}\right).$$

Proof. This result is due to Landau [6]. ■

Lemma 4. (Brun–Titchmarsh Inequality.) *There exists a positive constant c_3 such that*

$$\pi(x; k, \ell) < c_3 \frac{x}{\varphi(k) \log(x/k)} \quad \text{for all } k < x.$$

Proof. For a proof, see the book of Halberstam and Richert [2]. ■

Lemma 5. (Siegel–Walfisz Theorem.) *There exists a constant $c > 0$ such that for every fixed number $A > 0$, the estimate*

$$\pi(x; k, \ell) - \frac{li(x)}{\varphi(k)} = O\left(xe^{-c\sqrt{\log x}}\right)$$

holds uniformly, as $(\ell, k) = 1$, for $k \leq \log^A x$.

Proof. For a proof, see the book of Prachar [7]. ■

Lemma 6. (Bombieri–Vinogradov Theorem.) *Given any fixed number $A > 0$, there exists a number $B = B(A) > 0$ such that*

$$\sum_{k \leq \sqrt{x}/(\log^B x)} \max_{(k, \ell)=1} \max_{y \leq x} \left| \pi(x; k, \ell) - \frac{li(x)}{\varphi(k)} \right| = O\left(\frac{x}{\log^A x}\right).$$

Moreover, an appropriate choice for $B(A)$ is $2A + 6$.

Proof. For a proof, see the book of Iwaniec and Kowalski [5]. ■

Lemma 7. *Let F be a square-free polynomial with integer coefficients and of positive degree such that the degree of each of its irreducible factors is of degree no larger than 3. Let $Y(x)$ be a function which tends to $+\infty$ as $x \rightarrow +\infty$. Then*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \{n \leq x : p^2 | F(n) \text{ for some } p > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [3] (pp. 62–69). ■

Lemma 8. *Let F and Y be as in Lemma 7. Assume that each of the irreducible factors of F is of degree no larger than 2 and that $F(0) \neq 0$. Then*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \{p \leq x : q^2 | F(p) \text{ for some } q > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [3] (pp. 69–72). ■

Lemma 9. *Let $f(n)$ be a real valued non negative arithmetic function. Let a_n , $n = 1, \dots, N$, be a sequence of integers. Let r be a positive real number, and let $p_1 < p_2 < \dots < p_s \leq r$ be prime numbers. Set $Q = p_1 \cdots p_s$. If $d|Q$, then let*

$$(3.1) \quad \sum_{\substack{n=1 \\ a_n \equiv 0 \pmod{d}}}^N f(n) = \kappa(d)X + R(N, d),$$

where X and R are real numbers, $X \geq 0$, and $\kappa(d_1 d_2) = \kappa(d_1)\kappa(d_2)$ whenever d_1 and d_2 are co-prime divisors of Q .

Assume that for each prime p , $0 \leq \kappa(p) < 1$. Setting

$$I(N, Q) := \sum_{\substack{n=1 \\ (a_n, Q)=1}}^N f(n),$$

then the estimate

$$I(N, Q) = \{1 + 2\theta_1 H\} X \prod_{p|Q} (1 - \kappa(p)) + 2\theta_2 \sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} |R(N, d)|$$

holds uniformly for $r \geq 2$, $\max(\log r, S) \leq \frac{1}{8} \log z$, where $|\theta_1| \leq 1$, $|\theta_2| \leq 1$, and

$$H = \exp\left(-\frac{\log z}{\log r} \left\{ \log\left(\frac{\log z}{S}\right) - \log \log\left(\frac{\log z}{S}\right) - \frac{2S}{\log z} \right\}\right)$$

and

$$S = \sum_{p|Q} \frac{\kappa(p)}{1 - \kappa(p)} \log p.$$

When these conditions are satisfied, there exists an absolute positive constant c such that $2H \leq c < 1$.

Proof. This result is Lemma 2.1 in the book of Elliott [1]. ■

4. The first part of the proof of Theorem 1

Since $\rho_{Q_j(x)}(m) = \rho_{Q_j(x+c)}(m)$ for any constant c , it follows that $\rho_{F_\nu}(m) = \rho_{F_\mu}(m)$. Observe also that $\text{Res}(Q_i, Q_j) \neq 0$ if $i \neq j$. We shall now define four sets of primes, namely $\wp_1, \wp_2, \wp_3, \wp_4$, as follows.

First, as elements of \wp_1 , we include

1. the prime divisors of $\prod_{1 \leq i < j \leq h} \text{Res}(Q_i, Q_j)$,
2. the prime divisors of $\prod_{1 \leq i \leq h} \text{Discr}(Q_i)$,
3. those primes p for which $\rho_F(p) \geq p$,
4. and no other primes.

Then, let $\mathcal{N}(\wp_1)$ be the semigroup generated by the set of primes \wp_1 .

Observe that:

- (a) If $(m, \mathcal{N}(\wp_1)) = 1$, then $\rho_F(m) = \rho_{Q_1}(m) + \cdots + \rho_{Q_h}(m)$.
- (b) If $(m_1 m_2, \mathcal{N}(\wp_1)) = 1$ with $(m_1, m_2) = 1$, then $\rho_F(m_1 m_2) = \rho_F(m_1) + \rho_F(m_2)$.
- (c) If $p \notin \wp_1$, then $\rho(p^a) = \rho(p)$ for each $a \in \mathbb{N}$.

Let $Y = Y(x)$ be a large number. Moreover, let A_x and $\varepsilon(x)$ be such that $\varepsilon(x)A_x \rightarrow 0$ as $x \rightarrow \infty$, and define $r = r_x = x^{\varepsilon(x)}$.

We now define the other sets of primes \wp_2 , \wp_3 and \wp_4 (which depend on x) as follows:

$$\begin{aligned}\wp_2 &= \{p : p \leq Y, p \notin \wp_1\}, \\ \wp_3 &= \{p : Y < p \leq r\}, \\ \wp_4 &= \{p : p > r\}.\end{aligned}$$

Now consider the sets $\mathcal{N}(\wp_2)$, $\mathcal{N}(\wp_3)$, $\mathcal{N}(\wp_4)$, that is the semigroups generated respectively by the sets of primes \wp_2 , \wp_3 , \wp_4 .

For each positive integer ν , we now define $A(\nu)$, $B(\nu)$, $C(\nu)$ and $D(\nu)$ by

$$\nu = A(\nu)B(\nu)C(\nu)D(\nu),$$

where

$$A(\nu) \in \mathcal{N}(\wp_1), \quad B(\nu) \in \mathcal{N}(\wp_2), \quad C(\nu) \in \mathcal{N}(\wp_3), \quad D(\nu) \in \mathcal{N}(\wp_4).$$

Finally, let $T(u) := \prod_{p \leq u} p$.

We now choose $\xi_1, \xi_2, \dots, \xi_t \in \mathcal{N}(\wp_1)$ in such a way that there exists at least one solution $n = n_0$ of

$$(4.1) \quad F_\nu(n) \equiv 0 \pmod{\xi_\nu}, \quad \left(\frac{F_\nu(n)}{\xi_\nu}, \mathcal{N}(\wp_1) \right) = 1 \quad (\nu = 1, \dots, t).$$

Further define

$$\begin{aligned}\xi^* &= \text{LCM}(\xi_1, \dots, \xi_t), \\ \xi^{**} &= \xi^* \prod_{p \in \wp_1} p.\end{aligned}$$

Clearly, (4.1) holds for all those positive integers n for which $n \equiv n_0 \pmod{\xi^{**}}$.

Now let $\kappa = \kappa(\xi_1, \dots, \xi_t)$ be the number of those residue classes $r \pmod{\xi^{**}}$ for which

$$(4.2) \quad F_\nu(r) \equiv 0 \pmod{\xi_\nu}, \quad \left(\frac{F_\nu(r)}{\xi_\nu}, \mathcal{N}(\wp_1) \right) = 1 \quad (\nu = 1, \dots, t)$$

holds. Note that, in the case where (4.1) has no solutions, we simply set $\kappa(\xi_1, \dots, \xi_t) = 0$.

We now choose

$$\begin{aligned} m_1, \dots, m_t &\in \mathcal{N}(\wp_2), & (m_i, m_j) &= 1 \text{ if } i \neq j, \\ d_1, \dots, d_t &\in \mathcal{N}(\wp_3), & (d_i, d_j) &= 1 \text{ if } i \neq j. \end{aligned}$$

With these notations in mind, we introduce the set

$$(4.3) \quad \begin{aligned} \mathcal{M}_x &= \mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) = \\ &= \{n \leq x : A(F_\ell(n)) = \xi_\ell, B(F_\ell(n)) = m_\ell, C(F_\ell(n)) = d_\ell \text{ for } \ell = 1, \dots, t\}. \end{aligned}$$

Observe that if $(m_i, m_j) > 1$ or $(d_i, d_j) > 1$ for some $i \neq j$, then

$$\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) = \emptyset.$$

One can easily see that \mathcal{M}_x is the set of those integers $n \leq x$ for which

$$(4.4) \quad \xi_j m_j d_j | F_j(n), \quad \left(\frac{F_j(n)}{\xi_j m_j d_j}, T(r) \right) = 1 \quad \text{for } j = 1, \dots, t.$$

We now let $\mathcal{E}(\xi_1, \dots, \xi_t)$ be the set of those integers n for which (4.4) holds for $j = 1, \dots, t$ for an appropriate choice of $m_1, \dots, m_t, d_1, \dots, d_t$.

Let ν_1, \dots, ν_t be those residues mod ξ^{**} for which $\mathcal{E}(\xi_1, \dots, \xi_t)$ is covered exactly by

$$\bigcup_{u=1}^{\kappa} \{n \leq x : n \equiv \nu_u \pmod{\xi^{**}}\},$$

that is, if $n \in \mathcal{E}(\xi_1, \dots, \xi_t)$, then $n \equiv \nu_u \pmod{\xi^{**}}$ for some $u \in \{1, \dots, \kappa\}$, and

$$\{n \leq x : n \equiv \nu_u \pmod{\xi^{**}}\} \cap \mathcal{E}(\xi_1, \dots, \xi_t) \neq \emptyset.$$

Now let $\xi_1, \dots, \xi_t, \nu \in \{\nu_1, \dots, \nu_\kappa\}$, where $\kappa = \kappa(\xi_1, \dots, \xi_t)$ is fixed. Then the fact that $n \equiv \nu \pmod{\xi^{**}}$ guarantees that

$$\left(\frac{F_\ell(n)}{\xi_\ell}, \xi^{**} \right) = 1 \quad (\ell = 1, \dots, t)$$

holds.

We further define

$$\begin{aligned}\underline{m} &= (m_1, \dots, m_t), \\ \underline{d} &= (d_1, \dots, d_t),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{M}}_x(\nu \pmod{\xi^{**}}; \underline{m}, \underline{d}) &= \\ = \{n \leq x : n \equiv \nu \pmod{\xi^{**}}, B(F_j(n)) = m_j, C(F_j(n)) = d_j \text{ for } j = 1, \dots, t\}\end{aligned}$$

Observe that, for each $j = 1, \dots, t$, the number of solutions of $F_j(\nu + s\xi^{**}) \equiv 0 \pmod{m_j d_j} \pmod{m_1 \dots m_t d_1 \dots d_t}$ is equal to $\rho(m_1 \dots m_t) \rho(d_1 \dots d_t)$.

Let μ_0 be one of these solutions, that is let $0 \leq \mu_0 < m_1 \dots m_t d_1 \dots d_t$, $F_j(\nu + \mu_0 \xi^{**}) \equiv 0 \pmod{m_j d_j}$ ($j = 1, \dots, t$), and set

$$(4.5) \quad R = \xi^{**} m_1 \dots m_t d_1 \dots d_t.$$

We would like to estimate the size of the number of those integers $k \leq x/R$ for which

$$\varphi_j(k) := \frac{F_j(\nu + \mu_0 \xi^{**} + kR)}{\xi_j m_j d_j}$$

is coprime to $T(r)$ for every $j = 1, \dots, t$.

We shall only consider those $k \leq x/R$ for which m_j, d_j are both not very large, that is when $m_j \leq Y^{A_x}$ and $d_j \leq r^{A_x}$. Indeed, one can easily prove, in light of Lemma 7, that

$$(4.6) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : \max m_j > Y^{A_x} \text{ or } \max d_j > r^{A_x}\} = 0,$$

and we will therefore skip the proof. Now, define

$$(4.7) \quad \Phi(k) := \varphi_1(k) \cdots \varphi_t(k).$$

Since, if $p \in \wp_1$, then $\rho_{\varphi_j}(p^a) = \rho_{\varphi_j}(p) = 0$, it follows that $\rho_{\Phi}(p^a) = \rho_{\Phi}(p) = 0$.

Furthermore, if $p \in \wp_2 \cup \wp_3$, then $\rho_{\varphi_j}(p^a) = \rho_{\varphi_j}(p)$ and we shall prove that

(P1) if $p | d_j m_j$, then $\rho_{\varphi_j}(p) = 1$ and $\rho_{\varphi_\ell}(p) = 0$ for all $\ell \neq j$;

(P2) if $(p, d_1 m_1 \cdots d_t m_t) = 1$, then $\rho_{\varphi_j}(p) = \rho(p)$ for $j = 1, \dots, t$.

Consequently, assuming that (P1) and (P2) are true, and letting $\eta(M)$ stand for the number of those $k \pmod{M}$ for which $\Phi(k) \equiv 0 \pmod{M}$, we then have

$$(4.8) \quad \eta(p^a) = \begin{cases} 0 & \text{if } p \in \wp_1, \\ \rho_{\varphi_j}(p) = 1 & \text{if } p \in \wp_2 \cup \wp_3 \text{ and } p | d_j m_j, \\ t\rho(p) & \text{if } p \in \wp_2 \cup \wp_3 \text{ and } (p, d_1 m_1 \cdots d_t m_t) = 1. \end{cases}$$

We now prove (P1). We prove it only in the case $j = 1$, the general case being similar. Assume that $p|d_1$ (the same reasoning will work if one assumes that $p|m_1$). Let a be the positive integer defined by $p^a||d_1$. Then,

$$\varphi_1(k) \equiv 0 \pmod{p} \iff F_1(\nu + \mu_0\xi^{**} + kR) \equiv 0 \pmod{p^{a+1}},$$

meaning that, since $\rho_{\varphi_1}(p)$ stands for the number of solutions k of $\varphi_1(k) \equiv 0 \pmod{p}$, while $\rho_{F_1}(p^{a+1})$ stands for the number of solutions of $F_1(\nu + \mu_0\xi^{**} + kR) \equiv 0 \pmod{p^{a+1}}$, it follows that $\rho_{\varphi_1}(p) = \rho_{F_1}(p) = 1$. It remains to prove that $\rho_{\varphi_\ell}(p) = 0$ if $\ell \neq 1$. To do so, we assume that $\rho_{\varphi_\ell}(p) \neq 0$. In this case, we have that $p|\varphi_1(k_1)$ and $p|\varphi_\ell(k_2)$, in which case

$$\begin{aligned} F_1(\nu + \mu_0\xi^{**} + k_1R) &\equiv 0 \pmod{p}, \\ F_\ell(\nu + \mu_0\xi^{**} + k_2R) &\equiv 0 \pmod{p}. \end{aligned}$$

Now, in light of (4.5), we have that $p|R$, implying that

$$\begin{aligned} F_1(\nu + \mu_0\xi^{**}) &\equiv 0 \pmod{p}, \\ F_\ell(\nu + \mu_0\xi^{**}) &\equiv 0 \pmod{p}, \end{aligned}$$

which is an impossible situation in light of Lemma 1, because $F_1(a) = 0$ and $F_2(a)$ cannot occur simultaneously due to the fact that $p \notin \wp_1$. This completes the proof of (P1).

The proof of (P2) is almost obvious. Indeed,

$$\varphi_1(k) \equiv 0 \pmod{p} \iff F_1(\nu + \mu_0\xi^{**} + kR) \equiv 0 \pmod{p}.$$

Thus, $F_1(u) \equiv 0 \pmod{p}$ holds for $u = u_1, \dots, u_{\rho(p)} \pmod{p}$, and therefore, $\nu + \mu_0\xi^{**} + kR \equiv u_j \pmod{p}$ ($j = 1, \dots, \rho(p)$) can be solved in k .

We now move on to estimate $\#\mathcal{M}$ and $\#\widetilde{\mathcal{M}}$ using Lemma 9. We choose $Q = T(r)$, $f(k) = 1$, $a_k = \Phi(k)$ (which was defined in (4.7)), $X = x/R$ (where R is as in (4.5)) and $\kappa(d) = \eta(d)/d$ with η as defined in (4.8). We thus obtain

$$\sum_{\substack{k \leq X \\ a_k \equiv 0 \pmod{d}}} 1 = \frac{\eta(d)}{d} X + R(X, d)$$

with

$$(4.9) \quad |R(X, d)| \leq t\rho(d).$$

With $I(X, Q) := \#\{k \leq X : (a_k, Q) = 1\}$, we obtain from Lemma 9 that

$$(4.10) \quad I(X, Q) = (1 + O(H)) \frac{x}{R} \prod_{p|Q} \left(1 - \frac{\eta(p)}{p}\right) + O\left(\sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} |R(X, d)|\right).$$

In light of (4.9), we have that

$$(4.11) \quad \sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} |R(X, d)| \leq t \sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} \eta(d).$$

We shall prove that for $z \geq 2$,

$$(4.12) \quad \sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} \eta(d) \leq cz^3 (\log z)^K,$$

for a suitable large constant K .

$$(4.13) \quad \begin{aligned} \sum_{d \leq Y} 3^{\omega(d)} \eta(d) |\mu(d)| \log d &\leq \sum_{pu \leq Y} 3^{\omega(pu)} (\log p) \eta(p) \eta(u) |\mu(u)| \leq \\ &\leq 3 \sum_{u \leq Y} 3^{\omega(u)} \eta(u) |\mu(u)| \sum_{p \leq Y/u} \eta(p) \log p. \end{aligned}$$

Since $\sum_{p \leq Y/u} \eta(p) \log p \leq c \frac{Y}{u}$, (4.13) becomes

$$(4.14) \quad \begin{aligned} \sum_{d \leq Y} 3^{\omega(d)} \eta(d) |\mu(d)| \log d &\leq cY \sum_{u \leq Y} \frac{3^{\omega(u)} \eta(u)}{u} |\mu(u)| \leq \\ &\leq cY \prod_{p \leq Y} \left(1 + \frac{3\eta(p)}{p} \right) \leq cY \exp \left\{ 3 \sum_{p \leq Y} \frac{\eta(p)}{p} \right\} \leq \\ &\leq cY \exp(3th \log \log Y) = cY (\log Y)^{3th}. \end{aligned}$$

Let us write

$$(4.15) \quad \sum_{d \leq Y} 3^{\omega(d)} \eta(d) |\mu(d)| = \sum_{d \leq \sqrt{Y}} + \sum_{\sqrt{Y} < d \leq Y} = S_1 + S_2,$$

say. Clearly we have

$$(4.16) \quad S_1 \ll \sqrt{Y} \cdot Y^\varepsilon,$$

where $\varepsilon > 0$ can be taken arbitrarily small. On the other hand, in light of (4.14), we have

$$(4.17) \quad S_2 \leq \frac{2}{\log Y} \cdot cY (\log Y)^{3th} \ll Y (\log Y)^{3th-1}.$$

Setting $Y = z^3$ and using (4.16) and (4.17) in (4.14) proves (4.12).

We now move to obtain the size of S and to find an upper bound for H . First of all, since

$$0 < \frac{c_1}{p} < \frac{\eta(p)}{p - \eta(p)} < \frac{c_2}{p} \quad \text{if } p \in \wp_2 \cup \wp_2,$$

$$\text{while } \eta(p) = 0 \quad \text{if } p \in \wp_1,$$

it follows that

$$(4.18) \quad S \asymp \log r \asymp \varepsilon(x) \log x.$$

Let $B(x)$ be a real valued function satisfying $B(x) \rightarrow 0$ and $B(x)/\varepsilon(x) \rightarrow +\infty$ as $x \rightarrow \infty$, and set $z = x^{B(x)}$. Note that, in our context, the condition $\max(\log r, S) \geq \frac{1}{8} \log z$ (of Lemma 9) clearly holds for every large x .

Then, we have

$$H = \exp \left\{ -\frac{\log z}{\varepsilon(x) \log x} \left[\log \left(\frac{\log z}{\varepsilon(x) \log x} \right) - \log \log \left(\frac{\log z}{\varepsilon(x) \log x} \right) - \frac{2\varepsilon(x) \log x}{\log z} + O(1) \right] \right\}.$$

From this representation, it follows that

$$0 \leq H \leq C \exp \left\{ -\frac{1}{2} \frac{B(x)}{\varepsilon(x)} \log \left(\frac{B(x)}{\varepsilon(x)} \right) \right\} =: H_1,$$

for an appropriate constant C .

Hence, applying Lemma 9, we obtain

$$(4.19) \quad I(X, Q) = \{1 + O(H_1)\} \frac{x}{R} \prod_{p|Q} \left(1 - \frac{\eta(p)}{p} \right) + O \left(x^{3B(x)} [(\log x)B(x)]^K \right).$$

Now observe that

$$(4.20) \quad \prod_{p|Q} \left(1 - \frac{\eta(p)}{p} \right) = \prod_{\substack{p \in \wp_2 \\ (p, m_1 \cdots m_t) = 1}} \left(1 - \frac{t\rho_F(p)}{p} \right) \cdot \prod_{p|m_1 \cdots m_t} \left(1 - \frac{1}{p} \right) \cdot \prod_{\substack{p \in \wp_3 \\ (p, d_1 \cdots d_t) = 1}} \left(1 - \frac{t\rho_F(p)}{p} \right) \cdot \prod_{p|d_1 \cdots d_t} \left(1 - \frac{1}{p} \right).$$

Summing up over all the $\kappa = \kappa(\xi_1, \dots, \xi_t)$ residue classes mod ξ^{**} , we obtain that

$$\begin{aligned}
 & \# \mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) = \\
 & = \frac{\kappa(\xi_1, \dots, \xi_t) x \rho_F(m_1 \cdots m_t d_1 \cdots d_t)}{\xi^{**} m_1 \cdots m_t d_1 \cdots d_t} \cdot \frac{\varphi(m_1 \cdots m_t)}{m_1 \cdots m_t} \cdot \frac{\varphi(d_1 \cdots d_t)}{d_1 \cdots d_t} \\
 (4.21) \quad & \cdot \prod_{\substack{p \in \wp_2 \cup \wp_3 \\ (p, m_1 \cdots m_t d_1 \cdots d_t) = 1}} \left(1 - \frac{t \rho_F(p)}{p}\right) \cdot (1 + \theta_1 \kappa(\xi_1, \dots, \xi_t) H_1) + \\
 & + O\left(x^{3B(x)} [(\log x) B(x)]^K\right).
 \end{aligned}$$

Using the fact that

$$\sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_F(p)}{p} = \sum_{j=1}^h \sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_{Q_j}(p)}{p} = h \log \log x^{\varepsilon(x)} + c_2 + O\left(\frac{1}{\varepsilon(x) \log x}\right),$$

it follows that

$$\begin{aligned}
 (4.22) \quad & \prod_{p \in \wp_2 \cup \wp_3} \left(1 - \frac{t \rho_F(p)}{p}\right) = \exp\left\{-t \sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_F(p)}{p}\right\} + c_1 + O\left(\frac{1}{\varepsilon(x) \log x}\right) = \\
 & = \exp\{-th \log \log x^{\varepsilon(x)} + c_1 - t c_2\} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right) = \\
 & = \left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} e^{c_3} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right).
 \end{aligned}$$

Thus, by defining the strongly multiplicative function λ on $\mathcal{N}(\wp_1 \cup \wp_2 \cup \wp_3)$ by

$$\lambda(p) = \begin{cases} 1, & \text{if } p \in \wp_1 \\ \left(1 - \frac{t \rho_F(p)}{p}\right)^{-1} \cdot \left(1 - \frac{1}{p}\right), & \text{if } p \mid m_1 \cdots m_t d_1 \cdots d_t \\ 1, & \text{if } p \in \wp_2 \cup \wp_3 \text{ and } (p, m_1 \cdots m_t d_1 \cdots d_t) = 1 \end{cases}$$

then, in light in the above, (4.21) becomes

$$\begin{aligned}
 & \# \mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) = \\
 & = \frac{x}{R} \frac{\kappa(\xi_1, \dots, \xi_t)}{\xi^{**}} \cdot \frac{\varphi(m_1 \cdots m_t)}{m_1 \cdots m_t} \cdot \frac{\varphi(d_1 \cdots d_t)}{d_1 \cdots d_t} \cdot \\
 (4.23) \quad & \cdot \lambda(m_1 \cdots m_t) \lambda(d_1 \cdots d_t) \rho_F(m_1 \cdots m_t d_1 \cdots d_t) \cdot \\
 & \cdot \left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} e^{c_3} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right) \cdot \\
 & \cdot (1 + \theta_1 \kappa(\xi_1, \dots, \xi_t) H_1) + O\left(x^{3B(x)} [(\log x) B(x)]^K\right).
 \end{aligned}$$

Now, from the estimates (4.21) and (4.23), we can formulate the following straightforward and important assertions:

Proposition 1. *Let m'_1, \dots, m'_t and d'_1, \dots, d'_t be arbitrary permutations of m_1, \dots, m_t and d_1, \dots, d_t respectively. Then, for some constant $c > 0$,*

$$\begin{aligned} & |\#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) - \\ & - \#\mathcal{M}_x(\xi_1, \dots, \xi_t; m'_1, \dots, m'_t; d'_1, \dots, d'_t)| \leq \\ & \leq \frac{cx\kappa(\xi_1, \dots, \xi_t)}{\xi^{**}} \rho_F(m_1 \cdots m_t d_1 \cdots d_t) \cdot \\ & \cdot \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th} \left(\frac{1}{\varepsilon(x) \log x} + \kappa(\xi_1, \dots, \xi_t) H_1 \right) + \\ & + O\left(x^{3B(x)} [(\log x)B(x)]^K\right). \end{aligned}$$

Proposition 2. *Let $M \in \mathcal{N}(\wp_1)$ and $D \in \mathcal{N}(\wp_2)$ be two square-free integers satisfying $M \leq Y^{A_x}$ and $D \leq r^{A_x}$. Assume that $M = m_1 \cdots m_t$ and $D = d_1 \cdots d_t$. Let m'_1, \dots, m'_t and d'_1, \dots, d'_t be permutations of m_1, \dots, m_t and d_1, \dots, d_t respectively. Then, for some constant $c > 0$,*

$$\begin{aligned} & |\#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) - \\ & - \#\mathcal{M}_x(\xi_1, \dots, \xi_t; m'_1, \dots, m'_t; d'_1, \dots, d'_t)| \leq \\ & \leq \frac{cx\kappa(\xi_1, \dots, \xi_t)}{\xi^{**}} \rho_F(MD) \cdot \\ & \cdot \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th} \left(\frac{1}{\varepsilon(x) \log x} + \kappa(\xi_1, \dots, \xi_t) H_1 \right) + \\ & + O\left(x^{3B(x)} [(\log x)B(x)]^K\right). \end{aligned}$$

5. The second part of the proof of Theorem 1

Given a positive integer n , we write it as $n = M(n)S(n)$, where $M(n)$ is the squarefree part of n and $S(n)$ the squarefull part of n . Then we clearly have $f(n) = U^{\omega(S(n))} f(M(n))$, where $U = q_1^{\beta_1} \cdots q_v^{\beta_v}$, say. With this set up, we may write $f(m) = f_1(m)f_2(m)$, where $f_2(m) \in \mathcal{N}(\{q_1, \dots, q_v\})$ and $f_1(m) =$

$= f(m)/f_2(m)$ satisfies $(f_1(m), U) = 1$. Of course, f_1 and f_2 are easily seen to be multiplicative functions.

Writing $m = M(m) \cdot S(m)$, we have that

$$f(n)|f(m) \iff \begin{cases} (1) & f_1(S(n))|f_1(S(m)) \\ \text{and} \\ (2) & f_1(S(n)) \cdot U^{\omega(M(n))} | f_1(S(m)) \cdot U^{\omega(M(m))} \end{cases}$$

Define $L(S(n))$ as the smallest (nonnegative) integer for which $f_2(S(n))$ divides $U^{L(S(n))}$. Then, in order for the condition $f(n)|f(m)$ to be satisfied, it is sufficient that the conditions (1) and

$$(2)' \quad L(S(n)) + \omega(M(n)) \leq \omega(M(m))$$

be satisfied, while it is necessary that conditions (1) and

$$(2)'' \quad \omega(M(n)) \leq \omega(M(m)) + L(S(m))$$

hold.

From this, it follows that in order to have

$$f(F_\ell(n))|f(F_{\ell+1}(n)) \quad (\ell = 1, \dots, t-1),$$

the conditions

$$(5.1) \quad f_1(S(F_\ell(n)))|f_1(S(F_{\ell+1}(n))) \quad (\ell = 1, \dots, t-1)$$

and

$$(5.2) \quad L(S(f_\ell(n))) + \omega(M(F_\ell(n))) \leq \omega(M(F_{\ell+1}(n)))$$

are sufficient, while the conditions (5.1) and

$$(5.3) \quad \omega(M(F_\ell(n))) \leq \omega(M(F_{\ell+1}(n))) + L(S(f_{\ell+1})) \quad (\ell = 1, \dots, t-1)$$

are necessary.

Now, let S_1, \dots, S_t be squarefull numbers. By using a method developed by Hooley (see [3], Chapter 4) and using also the Eratosthenian sieve, one can prove that

$$(5.4) \quad \frac{1}{x} \#\{n \leq x : S(F_\ell(n)) = S_\ell, \ell = 1, \dots, t\} = d(S_1, \dots, S_t) + O\left(\frac{x}{\log \log x}\right),$$

for some nonnegative constant $d(S_1, \dots, S_t)$ which satisfy

$$\sum_{S_1, \dots, S_t} d(S_1, \dots, S_t) = 1,$$

and where the constant implied in the error term is absolute.

Let \mathcal{B} be the set of all those vectors (S_1, \dots, S_t) for which S_1, \dots, S_t are squarefull numbers and

$$(5.5) \quad f_1(S_\ell(n)) | f_1(S_{\ell+1}(n)) \quad (\ell = 1, \dots, t-1).$$

We will prove that

$$(5.6) \quad d_0 = \frac{1}{t!} \sum_{(S_1, \dots, S_t) \in \mathcal{B}} d(S_1, \dots, S_t).$$

Since

$$\sum_{\max(S_1, \dots, S_t) \geq Y} d(S_1, \dots, S_t) \rightarrow 0 \text{ as } Y \rightarrow \infty,$$

it is sufficient to prove that, for each fixed $(S_1, \dots, S_t) \in \mathcal{B}$,

$$(5.7) \quad \begin{aligned} & \frac{1}{x} \#\{n \leq x : S(F_\ell(n)) = S_\ell, f(F_\ell(n)) | f(F_{\ell+1}(n)) \text{ for } \ell = 1, \dots, t\} = \\ & = \frac{1}{t!} d(S_1, \dots, S_t) + o(1) \quad (x \rightarrow \infty). \end{aligned}$$

Let Y be large enough so that $\max(S_1, \dots, S_t) \leq Y$. We now move on to count the number of those integers $n \leq x$ for which both

$$(5.8) \quad S(F_\ell(n)) = S_\ell \quad (\ell = 1, \dots, t)$$

and

$$(5.9) \quad f(F_\ell(n)) | f(F_{\ell+1}(n)) \quad (\ell = 1, \dots, t)$$

hold. We must compute the number of those integers $n \leq x$ appearing in the set displayed in equation (4.3), with the additional condition $S(\xi_\ell m_\ell) = S_\ell$ and with also d_ℓ and $D(F_\ell(n))$ both squarefree for $\ell = 1, \dots, t$. But it is clear that, choosing $\varepsilon(x) = 1/\log \log \log x$, we have

$$\omega(A(F_\ell(n))B(F_\ell(n))D(F_\ell(n))) \leq \frac{c_4}{\varepsilon(x)} + Y \leq c_5 \log \log \log x.$$

In light of (4.6), we only need to consider those $n \leq x$ for which

$$(5.10) \quad \max_{i=1, \dots, t} m_j \leq Y^{A_x} \text{ and } \max_{i=1, \dots, t} d_j \leq r^{A_x}.$$

For now, fix $\xi_1, \dots, \xi_t, m_1, \dots, m_t, d_1, \dots, d_t$ and assume that $\omega(d_i) \neq \omega(d_j)$ when $i \neq j$. Under these conditions, there exists one and only one permutation d_1^*, \dots, d_t^* of d_1, \dots, d_t for which $\omega(d_1^*) < \dots < \omega(d_t^*)$.

In the event that $|\omega(d_i) - \omega(d_j)| \geq 2c_5 \log \log \log \log x$ whenever $i \neq j$, then (5.9) will hold for the corresponding number n .

Hence, it remains to prove that the sum of

$$\#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t)$$

running over all those $\xi_1, \dots, \xi_t, m_1, \dots, m_t, d_1, \dots, d_t$ for which $|\omega(d_i) - \omega(d_j)| < 2c_5 \log \log \log \log x$ holds for some $i \neq j$ is $o(x)$.

In order to prove this, observe that, in light of Proposition 1,

$$\begin{aligned} \#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) &\leq \\ &\leq \frac{cx\kappa(\xi_1, \dots, \xi_t)}{\xi^{**}} \rho_F(m_1 \cdots m_t d_1 \cdots d_t) \cdot \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th}. \end{aligned}$$

For short, let us write $d_1 \Delta d_2$ to express the condition $\omega(d_1) \leq \omega(d_2) \leq \omega(d_1) + c_5 \log \log \log \log x$.

We then have

$$\begin{aligned} &\sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \sum_{\substack{\xi_1, \dots, \xi_t \\ m_1, \dots, m_t \\ d_1, \dots, d_t}} \#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) \leq \\ &\leq c(Y) \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th} x \sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \frac{\rho_F(d_1 d_2)}{d_1 d_2} \sum_{d_3, \dots, d_t} \frac{\rho_f(d_3 \cdots d_t)}{d_3 \cdots d_t} \leq \\ (5.11) \quad &\leq c(Y) \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th} x \left(\sum_{d \in \mathcal{N}(\wp_3)} \frac{\rho_F(d)}{d} \right)^{t-2} \sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \frac{\rho_F(d_1 d_2)}{d_1 d_2} \leq \\ &\leq c(Y) \left(\frac{1}{\log x^{\varepsilon(x)}} \right)^{th} x \left(\log x^{\varepsilon(x)} \right)^{(t-2)h} \sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \frac{\rho_F(d_1 d_2)}{d_1 d_2}. \end{aligned}$$

Now, because

$$\sum_{\omega(d)=r} \frac{\rho_F(d)}{d} \leq \frac{1}{r!} \left(\sum_{p \in \wp_3} \frac{\rho_F(p)}{p} \right)^r \leq \frac{(h \log \log x + O(1))^r}{r!},$$

it follows, in light of Lemma 3, that, as $x \rightarrow \infty$,

$$(5.12) \quad \sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \frac{\rho_F(d_1 d_2)}{d_1 d_2} \leq \sum_{r=1}^{\infty} \frac{(h \log \log x + O(1))^r}{r!} \sum_{t=0}^{\lfloor c_5 x_4 \rfloor} \frac{(hx_2 + O(1))^{t+r}}{(t+r)!} = o(\log^{2h} x).$$

Using (5.12) in (5.11), it follows that

$$\sum_{\substack{d_1, d_2 \\ d_1 \Delta d_2}} \sum_{\substack{\xi_1, \dots, \xi_t \\ m_1, \dots, m_t \\ d_1, \dots, d_t}} \#\mathcal{M}_x(\xi_1, \dots, \xi_t; m_1, \dots, m_t; d_1, \dots, d_t) = o(x),$$

thus proving our claim and thereby completing the proof of Theorem 1.

6. The proof of Theorem 2

The proof of Theorem 2 can be obtained along the same lines as that of Theorem 1. We only provide here the main ideas. Indeed, Hooley [3] proved that

$$\limsup_{x \rightarrow \infty} \frac{1}{\pi(x)} \#\{p \leq x : \text{for some } \ell, C(F_\ell(p))D(F_\ell(p)) \neq \text{squarefree}\} = 0.$$

Using this result, the Siegel-Walfisz form of the Prime Number Theorem (Lemma 5) as well as the Bombieri-Vinogradov Inequality (Lemma 6), one can proceed as earlier and prove the analogues of Propositions 1 and 2, thereby easily completing the proof of Theorem 2.

7. Further applications

It is interesting to observe that the following two results are consequences of Proposition 1. Here, $\phi(y)$ is the standard cumulative distribution function defined by $\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$.

Theorem 3. *Let F_1, \dots, F_t be as in Theorem 1. Then,*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : \frac{\omega(F_\ell(n)) - h \log \log x}{\sqrt{h \log \log x}} < y_\ell, \ell = 1, \dots, t\} = \phi(y_1) \dots \phi(y_t).$$

Theorem 4. *Let F_1, \dots, F_t be as in Theorem 2. Then,*

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \#\{p \leq x : \frac{\omega(F_\ell(p)) - h \log \log x}{\sqrt{h \log \log x}} < y_\ell, \ell = 1, \dots, t\} = \phi(y_1) \dots \phi(y_t).$$

8. Final remarks

We now state a few remarks shedding some light on the value of d_0 and whether it is strictly positive.

The main idea is that the value of d_0 as well as the fact that it is positive or zero depends on the values taken by f on squarefull numbers.

Remark 4. Let $u(n)$ stand for the squarefull part of n , and $v(n)$ for the part of n which is coprime to $f(p)$, that is

$$v(n) = \prod_{q^a \parallel n, (q, f(p))=1} q^a.$$

Under the additional assumption that $f(p) > 1$, then

$$d_0 = \frac{1}{t!} \lim_{x \rightarrow \infty} \frac{1}{x} \# \{n \leq x : v(f(u(F_j(n)))) | v(f(u(F_{j+1}(n))))\}, j = 1, \dots, t-1\}.$$

Indeed, let $\omega_k(n)$ be defined as

$$\omega_k(n) := \sum_{p^k \parallel n} 1.$$

Then, except for a set of density zero,

$$\min_{1 \leq j < r \leq t} |\omega_1(F_j(n)) - \omega_1(F_r(n))| > \left(\log \log \left(\max_{1 \leq j \leq t} F_j(n) \right) \right)^{1/3}.$$

Furthermore for any function $g(n)$ tending to infinity with n , if $u(n)$ stands for the squarefull part of n ,

$$\lim_{x \rightarrow \infty} \frac{1}{x} \# \left\{ n \leq x : \max_{1 \leq j \leq t} f(u(F_j(n))) > g(n) \right\} = 0.$$

From the above equations, it follows that except on a set of density zero,

$$f(F_j(n)) | f(F_{j+1}(n)) \iff v(f(u(F_j(n)))) | v(f(u(F_{j+1}(n))))),$$

thus completing the proof of our claim.

Remark 5. The constant d_0 will be positive regardless of the polynomials F if and only if for any values of A and n , there exists a number m such that $p|m \Rightarrow p > A$ and $v(f(u(n))) | v(f(u(m)))$.

In order to prove this, first assume that the assumption does not hold. Let n_0 and A_0 be such that if $m > n_0$ and $v(f(u(n_0)))|v(f(u(m)))$, then there is prime $p < A_0$ such that $p|m$. Set $t = (u(n_0) \prod_{p < A_0} p)^2$ and $F_j(n) = n + j$. Then for at least one j , $f(u(n_0))|f(n + j)$ while $(n + j + 1, \prod_{p < A} p) = 1$. It follows that $f(n + j)$ does not divide $f(n + j + 1)$. Assume now that the assumption holds. Let $F_j(n)$ be a suitable family of polynomials. Choose Y large enough so that for any t -uple m_1, \dots, m_t of integers, and for any prime $p > Y$, there exists n such that $F_j(n) \equiv m_j \pmod{p}$ for $j = 1, \dots, t$. It follows that n can be chosen so that

$$v(f(u(F_j(n)))) \text{ divides } v(f(u(F_{j+1}(n)))) \quad (j = 1, \dots, t),$$

thus completing the proof.

Remark 6. Assume that f is such that on prime powers p^a , we have $f(p^a) = g(a)$ for a certain function g . Then, for any value of t and any family of polynomials F_1, \dots, F_t , we have that d_0 is strictly positive.

Indeed, this is an easy corollary of Remark 5.

The following remark provides perhaps the simplest instance for which $d_0 = 0$.

Remark 7. Let f be a multiplicative function such that $f(p) = 1$ and $f(p^a) = p^a$ if $a \geq 2$. Then there exists no integer n such that

$$f(n)|f(n+1)|f(n+2)|f(n+3)|f(n+4).$$

Indeed, for exactly one value of $j = 0, 1, 2, 3$, we have that $n + j$ is divisible by 4. It follows that $f(n + j)$ is even while $f(n + j + 1)$ is odd, a non sense.

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J.-M. De Koninck and N. Doyon

Dép. de mathématiques
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca
nicolas.doyon@mat.ulaval.ca

I. Kátai

Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
katali@compalg.inf.elte.hu