

ON CONJUGATE MEANS OF n VARIABLES

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*Dedicated to Professor János Galambos
on the occasion of his 70th birthday*

Abstract. Let $I \subset \mathbb{R}$ be a nonvoid open interval and let $n \geq 3$ be a fixed natural number. The question is which conjugate means of n variables generated by the n -variable arithmetic mean are weighted quasi-arithmetic means of n variables at the same time? This question is a functional equation problem. We characterize the real valued continuous and strictly monotone functions φ, ψ defined on I and the parameters $p_1, \dots, p_n, q_1, \dots, q_n$ for which the equation

$$\begin{aligned} \varphi^{-1} \left(\sum_{i=1}^n p_i \varphi(x_i) + \left(1 - \sum_{i=1}^n p_i \right) \varphi \left(\frac{x_1 + \dots + x_n}{n} \right) \right) = \\ = \psi^{-1} \left(\sum_{i=1}^n q_i \psi(x_i) \right) \end{aligned}$$

holds for all $x_1, \dots, x_n \in I$, where

$$\begin{aligned} q_i > 0 \quad (i = 1, \dots, n), \quad \sum_{i=1}^n q_i = 1, \\ p_j > 0 \quad \text{and} \quad \sum_{i=1}^n p_i - p_j \leq 1 \quad (j = 1, \dots, n). \end{aligned}$$

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1. Introduction

Let $n \geq 2$ be a natural number. Denote by K_n the n -tuples (p_1, \dots, p_n) with the property below:

$$\text{If } \min\{x_i\} \leq M \leq \max\{x_i\} \quad (i = 1, \dots, n)$$

holds for the real numbers x_1, \dots, x_n, M then

$$\min\{x_i\} \leq \sum_{i=1}^n p_i x_i + \left(1 - \sum_{i=1}^n p_i\right) M \leq \max\{x_i\}$$

holds, too.

Daróczy–Páles ([2]) gave the necessary and sufficient condition for the n -tuple (p_1, \dots, p_n) to be an element of K_n :

Theorem 1. $(p_1, p_2, \dots, p_n) \in K_n$ ($n \geq 2$) if and only if

$$p_j \geq 0 \quad \text{and} \quad \sum_{i=1}^n p_i - p_j \leq 1 \quad (j = 1, 2, \dots, n).$$

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Denote by $\mathcal{CM}(I)$ the class of continuous and strictly monotone real valued functions defined on the interval I .

Furthermore, let M be a mean of n variables on I . Then for any $\varphi \in \mathcal{CM}(I)$

$$\min\{\varphi(x_i)\} \leq \varphi(M(x_1, x_2, \dots, x_n)) \leq \max\{\varphi(x_i)\}$$

holds for all $x_1, \dots, x_n \in I$, hence by Theorem 1 we have that

$$\min\{\varphi(x_i)\} \leq \sum_{i=1}^n p_i \varphi(x_i) + \left(1 - \sum_{i=1}^n p_i\right) \varphi(M(x_1, x_2, \dots, x_n)) \leq \max\{\varphi(x_i)\}$$

holds for all $(p_1, p_2, \dots, p_n) \in K_n$ and $x_1, \dots, x_n \in I$.

We get that

$$\begin{aligned} \varphi^{-1} \left(\sum_{i=1}^n p_i \varphi(x_i) + \left(1 - \sum_{i=1}^n p_i\right) \varphi(M(x_1, \dots, x_n)) \right) =: \\ =: M_\varphi^{p_1, p_2, \dots, p_n}(x_1, \dots, x_n) \end{aligned}$$

is always between $\min\{x_i\}$ and $\max\{x_i\}$, that is $M_\varphi^{p_1, p_2, \dots, p_n} : I^n \rightarrow I$ is a mean. We call this mean *the conjugate mean generated by the mean M with weights p_1, p_2, \dots, p_n* . This class of means includes the weighted quasi-arithmetic means and the quasi-arithmetic means. If for instance we consider the case $n = 2$ we get the following mean:

$$(1) \quad M_\varphi^{(p_1, p_2)}(x, y) := \varphi^{-1}(p_1\varphi(x) + p_2\varphi(y) + (1 - p_1 - p_2)\varphi(M(x, y))) \\ (x, y \in I),$$

where $(p_1, p_2) \in [0, 1]^2$ (see Theorem 1). If we put $p_1 + p_2 = 1$ in (1) we get the weighted quasi-arithmetic mean. If $p_1 = p_2 = \frac{1}{2}$ we get the quasi-arithmetic mean.

Let now M be the n -variable arithmetic mean:

$$M(x_1, \dots, x_n) := \frac{(x_1 + \dots + x_n)}{n}.$$

The question is which conjugate means generated by the arithmetic mean M are weighted quasi-arithmetic means at the same time?

In this paper we characterize the functions $\varphi, \psi \in \mathcal{CM}(I)$ and the parameters $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n$ for which the equation

$$\varphi^{-1} \left(\sum_{i=1}^n p_i \varphi(x_i) + \left(1 - \sum_{i=1}^n p_i \right) \varphi(M(x_1, \dots, x_n)) \right) = \psi^{-1} \left(\sum_{i=1}^n q_i \psi(x_i) \right)$$

holds for all $x_1, \dots, x_n \in I$, where $(p_1, p_2, \dots, p_n) \in K_n$, with $p_i > 0$ and $\sum_{i=1}^n q_i = 1, \quad q_i > 0 \quad (i = 1, \dots, n)$.

This problem has been solved in the special case $n = 2$ by Daróczy–Dascăl [4]. It is surprising that this special equality problem led to the original Matkowski–Sutó problem and some of its generalizations.

2. Preliminary results

We need the following definitions.

Definition 1. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in I$$

then we say that φ is equivalent to ψ on I and denote it by $\varphi(x) \sim \psi(x)$ if $x \in I$ or in short $\varphi \sim \psi$ on I .

We define the following sets:

$$\begin{aligned} T_+(I) &:= \{t \in \mathbb{R} \mid I + t \subset \mathbb{R}_+\}, \\ T_-(I) &:= \{t \in \mathbb{R} \mid -I + t \subset \mathbb{R}_+\}. \end{aligned}$$

The following comparison theorem was given by Maksa–Páles in [3]:

Theorem 2. *Let $\varphi, \psi : I \rightarrow \mathbb{R}$ be continuous strictly monotonic functions, $2 \leq n \in \mathbb{N}$, and let $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in (0, 1)$ with $\sum_{k=1}^n \lambda_k = \sum_{k=1}^n \mu_k = 1$. Then the inequality*

$$\varphi^{-1} \left(\sum_{k=1}^n \lambda_k \varphi(x_k) \right) \leq \psi^{-1} \left(\sum_{k=1}^n \mu_k \psi(x_k) \right)$$

holds for all $(x_1, \dots, x_n) \in I^n$ if and only if $\psi \circ \varphi^{-1}$ is convex (concave) if ψ is increasing (decreasing) and $\lambda_k = \mu_k$ for all $k \in \{1, \dots, n\}$.

Theorem 3 (J. Jarczyk, [5]). *Let $I \subset \mathbb{R}$ be a non-trivial interval, $\kappa, \lambda \in \mathbb{R} \setminus \{0, 1\}$ and $\mu, \nu \in (0, 1)$, and let $\varphi, \psi : I \rightarrow \mathbb{R}$ be continuous strictly monotonic functions such that (φ, ψ) satisfies the following equation:*

$$\begin{aligned} &\kappa x + (1 - \kappa)y = \\ &= \lambda \varphi^{-1}(\mu \varphi(x) + (1 - \mu)\varphi(y)) + (1 - \lambda)\psi^{-1}(\nu \psi(x) + (1 - \nu)\psi(y)). \end{aligned}$$

Then

$$(2) \quad \kappa = \lambda(\mu - \nu) + \nu$$

and one of the following conditions holds in I :

- (a) $\varphi(x) \sim x, \psi(x) \sim x$;
- (b) $\kappa = \lambda = \frac{1}{2}, \mu + \nu = 1$ and $\varphi(x) \sim e^{tx}, \psi(x) \sim e^{-tx}$ with some $t \in \mathbb{R} \setminus \{0\}$;
- (c) $\kappa = \frac{\mu^2}{\mu^2 + (1-\mu)^2}, \lambda = \frac{1}{\mu^2 + (1-\mu)^2}, \nu = \frac{1}{2}$ and either $\varphi(x) \sim \sqrt{x+t}$ and $\psi(x) \sim \log(x+t)$ if $t \in T_+(I)$ or $\varphi(x) \sim \sqrt{-x+t}$ and $\psi(x) \sim \log(-x+t)$ if $t \in T_-(I)$;
- (d) $\kappa = \frac{\nu^2}{\nu^2 + (1-\nu)^2}, \lambda = \frac{-2\nu(1-\nu)}{\nu^2 + (1-\nu)^2}, \mu = \frac{1}{2}$ and either $\varphi(x) \sim \log(x+t)$ and $\psi(x) \sim \sqrt{x+t}$ if $t \in T_+(I)$ or $\varphi(x) \sim \log(-x+t)$ and $\psi(x) \sim \sqrt{-x+t}$ if $t \in T_-(I)$;

- (e) $\mu \neq \nu$, $\mu + \nu \neq 1$, $\kappa = \frac{\mu\nu}{\mu+\nu-1}$, $\lambda = \frac{\kappa-\nu}{\mu-\nu}$ and either
 $\varphi(x) \sim \sqrt{x+t}$ and $\psi(x) \sim \sqrt{x+t}$ if $t \in T_+(I)$ or
 $\varphi(x) \sim \sqrt{-x+t}$ and $\psi(x) \sim \sqrt{-x+t}$ if $t \in T_-(I)$.

Conversely, all the pairs listed above satisfy the functional equation.

Observe that if $I = \mathbb{R}$ then only cases (a) and (b) are possible.

3. Main result

Theorem 4. Let $n \geq 3$ be a fixed natural number, $(p_1, p_2, \dots, p_n) \in K_n$, $p_i > 0$ and $q_1 + \dots + q_n = 1$, $q_i > 0$ ($i = 1, \dots, n$). If the functions $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of the functional equation

$$(3) \quad \varphi^{-1} \left(\sum_{i=1}^n p_i \varphi(x_i) + \left(1 - \sum_{i=1}^n p_i \right) \varphi(M(x_1, \dots, x_n)) \right) = \psi^{-1} \left(\sum_{i=1}^n q_i \psi(x_i) \right)$$

($x_1, \dots, x_n \in I$) then

$$q_j - p_j = \frac{1}{n} \left(1 - \sum_{i=1}^n p_i \right) \quad (j = 1, \dots, n)$$

and the following cases are possible

- if $\sum_{i=1}^n p_i \neq 1$ then $\varphi(x) \sim x$, $\psi(x) \sim x$ ($x, y \in I$);
- if $\sum_{i=1}^n p_i = 1$ then $\varphi(x) \sim \psi(x)$ ($x, y \in I$).

Conversely, the functions given in the above cases are solutions of equation (3).

Proof. If $\sum_{i=1}^n p_i = 1$ then we have the equality of weighted quasi-arithmetic means and by Theorem 2 we get the statement.

If $\sum_{i=1}^n p_i \neq 1$ we put $J := \varphi(I)$ and $u_i := \varphi(x_i)$ and we get

$$p_1 u_1 + \dots + p_n u_n + \left(1 - \sum_{i=1}^n p_i \right) \varphi \left(\frac{\varphi^{-1}(u_1) + \dots + \varphi^{-1}(u_n)}{n} \right) =$$

$$(4) \quad = \varphi \circ \psi^{-1} (q_1 \psi \circ \varphi^{-1}(u_1) + \dots + q_n \psi \circ \varphi^{-1}(u_n))$$

for all $u_i \in J$.

Setting $u := u_1$ and $v := u_2 = \dots = u_n$ in the equation above we get

$$\begin{aligned} p_1 u + v \sum_{i=2}^n p_i &= \left(\sum_{i=1}^n p_i - 1 \right) \varphi \left(\frac{1}{n} \varphi^{-1}(u) + \left(1 - \frac{1}{n}\right) \varphi^{-1}(v) \right) + \\ &+ \varphi \circ \psi^{-1} (q_1 \psi \circ \varphi^{-1}(u) + (1 - q_1) \psi \circ \varphi^{-1}(v)) \quad (u, v \in J). \end{aligned}$$

Dividing by $\sum_{i=1}^n p_i$ we get

$$\begin{aligned} &\frac{p_1}{\sum_{i=1}^n p_i} u + \left(1 - \frac{p_1}{\sum_{i=1}^n p_i} \right) v = \\ &= \left(1 - \frac{1}{\sum_{i=1}^n p_i} \right) \varphi \left(\frac{1}{n} \varphi^{-1}(u) + \left(1 - \frac{1}{n}\right) \varphi^{-1}(v) \right) + \\ &+ \frac{1}{\sum_{i=1}^n p_i} \varphi \circ \psi^{-1} (q_1 \psi \circ \varphi^{-1}(u) + (1 - q_1) \psi \circ \varphi^{-1}(v)) \quad (u, v \in J). \end{aligned}$$

Let $\frac{p_1}{\sum_{i=1}^n p_i} =: \kappa$, $\frac{1}{\sum_{i=1}^n p_i} =: \lambda$, $q_1 =: \mu$ and $\frac{1}{n} =: \nu$, then

$$\kappa u + (1 - \kappa) v = \lambda \varphi \circ \psi^{-1} (\mu \psi \circ \varphi^{-1}(u) + (1 - \mu) \psi \circ \varphi^{-1}(v)) +$$

$$(5) \quad + (1 - \lambda) \varphi (\nu \varphi^{-1}(u) + (1 - \nu) \varphi^{-1}(v)) \quad (u, v \in J).$$

$p_i > 0$ for all $i = 1, \dots, n$ implies $\kappa \in \mathbb{R} \setminus \{0, 1\}$, hence we can apply Theorem 3.

The case $\lambda = 1$ gives $\sum_{i=1}^n p_i = 1$ which we have already discussed. $\lambda = 0$ is not possible. Hence by (2) we get that

$$q_1 - p_1 = \frac{1}{n} \left(1 - \sum_{i=1}^n p_i \right).$$

Setting $u := u_2$ and $v := u_1 = u_3 = \dots = u_n$ in (4) and applying Theorem 3 again to the new equation and so on we get that

$$q_j - p_j = \frac{1}{n} \left(1 - \sum_{i=1}^n p_i \right)$$

for $j = 1, \dots, n$.

From the first case of Theorem 3 we have that $\psi \circ \varphi^{-1}$ and φ^{-1} are the identical functions, which means $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x, y \in I$.

In case (b) of Theorem 3 $\kappa = \lambda = \frac{1}{2}$ and $\mu + \nu = 1$, which gives $p_1 = 1$, $q_1 = 1 - \frac{1}{n}$ and $\sum_{i=1}^n p_i = 2$. Since $(p_1, p_2, \dots, p_n) \in K_n$ we know that $p_j \geq 0$ and $\sum_{i=1}^n p_i - p_j \leq 1$ ($j = 1, 2, \dots, n$). From these relations we have that $p_j \geq 1$ for all $j = 1, 2, \dots, n$ and $\sum_{i=2}^n p_i = 1$, hence $(p_1, p_2, \dots, p_n) \in \{(1, 1, 0, \dots, 0), \dots, (1, 0, \dots, 0, 1)\}$ which contradicts $p_1, \dots, p_n > 0$.

In case (c) of Theorem 3 we have $\nu = \frac{1}{2}$, but $\nu = \frac{1}{n}$ and $n \geq 3$, so this case is not possible.

In case (d) of Theorem 3 we have $\mu = \frac{1}{2}$, but $\mu = q_1$ which implies $q_1 = \frac{1}{2}$. Furthermore, $\kappa = \frac{\nu^2}{\nu^2 + (1-\nu)^2}$ and $\lambda = \frac{-2\nu(1-\nu)}{\nu^2 + (1-\nu)^2}$, which give $p_1 = -\frac{1}{2} \cdot \frac{1}{n-1} < 0$ which is a contradiction, so this case is not possible either.

In case (e) from the relations between the parameters we get that $p_1 = \frac{q_1(nq_1-1)}{n-1}$, but $p_1 > 0$, hence $q_1 > \frac{1}{n}$. Applying Theorem 3 n times we get that $q_i > \frac{1}{n}$ ($i = 1, \dots, n$), but $q_1 + \dots + q_n = 1$, which is again a contradiction. By easy computation we can see that the functions found in the above cases are solutions of the functional equation (3) and this completes our proof. ■

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