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ON THE PAIRS OF MULTIPLICATIVE FUNCTIONS WITH A SPECIAL RELATION

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Dedicated to Professor János Galambos on his 70th anniversary

Abstract. It is proved that if f and g are complex-valued multiplicative functions satisfying $g(An+1) - Cf(n) = o(1)$ as $n \to \infty$ with some positive integer A and non-zero complex constant C, then either $f(n) = o(1)$, $g(An+1) = o(1)$ as $n \to \infty$ or there exist a complex number s and multiplicative functions F, G such that $f(n) = n^s F(n)$, $g(n) = n^s G(n)$, $(0 \leq \text{Re } s < 1)$ and $G(An+1) = \frac{1}{F(2)} F(n)$ are satisfied for all $n \in \mathbb{N}$. All solutions of $G(An+1) = \frac{1}{F(2)} F(n)$ are given.

1. Introduction

Let $\mathbb{N}$, $\mathcal{P}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all positive integers, prime numbers, real and complex numbers, respectively. We denote by (m,n) the greatest common divisor of the integers m and n. For each positive integer k, let $\mathbb{N}_k$ be the set of the natural numbers coprime to k and let $\mathcal{L}_k$ be the set of those arithmetical functions f for which $f(n) = o(1)$ as $n \to \infty$, $n \in \mathbb{N}_k$. Let $\mathcal{M}_k$ ($\mathcal{M}_k^*$) be the set of complex-valued multiplicative (completely multiplicative) functions $f : \mathbb{N}_k \to \mathbb{C}$. In the case $k = 1$, let

$$\mathcal{L} := \mathcal{L}_1, \ \mathcal{M} := \mathcal{M}_1 \ \text{ and } \ \mathcal{M}^* := \mathcal{M}_1^*.$$

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For positive integers n and k let $n = D_k(n)E_k(n)$, where $D_k(n)$ is the product of prime power divisors p^α of n for which $p|k$ and $(E_k(n),\ k) = 1$.

P. Erdős proved in 1946 [2] that if $f : \mathbb{N} \to \mathbb{R}$ is an additive function such that $\Delta f(n) := f(n+1) - f(n) = o(1)$ as $n \to \infty$, then $f(n)$ is a constant multiple of $\log n$. This assertion has been generalized in several directions (e.g. see [1, 5]). The characterization of multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ under suitable regularity conditions even in the simplest case $\Delta f(n) = o(1)$ is much harder.

In 1984, I. Kátai stated as a conjecture that $f \in \mathcal{M}$, $\Delta f(n) = o(1)$ as $n \to \to \infty$ imply that either $f \in \mathcal{L}$ or $f(n) = n^s$ $(n \in \mathbb{N}), 0 \le \text{Re } s < 1$. This was proved by E. Wirsing in a letter to Kátai and later in a paper [16]. It is not hard to deduce from Wirsing's theorem that if

$$f,\ g \in \mathcal{M},\ g(n+1) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then either $g \in \mathcal{L}, f \in \mathcal{L}$ or

$$f(n) = g(n) = n^s \quad (n \in \mathbb{N}),\ 0 \le \text{Re } s < 1.$$

More than 10 years ago, improving the above results, in the joint paper with I. Kátai, we proved in [10] that if $k \in \mathbb{N}$ is given and $f,\ g \in \mathcal{M}$ satisfy the condition

$$g(n+k) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then either $f \in \mathcal{L},\ g \in \mathcal{L}$ or there are $F,\ G \in \mathcal{M}$ and a complex constant s such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 \le \text{Re } s < 1$$

and

$$G(n+k) = F(n)$$

are satisfied for all $n \in \mathbb{N}$. In [3, 7, 8, 9, 13, 14], by using the result of [4], the equation $G(n+k) = F(n)$ is solved completely.

The general case concerning the characterization of those $f,\ g \in \mathcal{M}$ for which

$$g(an+b) - Cf(An+B) = o(1) \quad \text{as} \quad n \to \infty,$$

where $a > 0,\ b,\ A > 0,\ B$ are fixed integers and C is a non-zero complex constant, seems to be a hard problem. This question was solved in [11, 12] for $B = 0$ under the conditions $|f(n)| = |g(n)| = 1$ $(n \in \mathbb{N})$. A similar result was obtained in [15] under the conditions

$$f = g,\ f(n+b) - f(n) = o(1) \quad \text{as} \quad n \to \infty, n \in \mathbb{N}_b.$$

N.L. Bassily and I. Kátai [6] showed that if $f,\ g \in \mathcal{M}$ satisfy

$$g(2n+1) - Cf(n) = o(1) \quad (n \to \infty)$$

with some non-zero constant C, then either $f \in \mathcal{L},\ g \in \mathcal{L}_2$ or

$$C = f(2),\ f(n) = n^s,\ 0 \le \text{Re } s < 1,\ \text{and}\ \ f(m) = g(m)$$

for all $n \in \mathbb{N},\ m \in \mathbb{N}_2$.

The main purpose of this paper is to improve this result of N.L. Bassily and I. Kátai. We prove

Theorem 1. *Assume that $A \in \mathbb{N}$, $C \in \mathbb{C} \setminus \{0\}$ and $f,\ g \in \mathcal{M}$ satisfy the condition*

$$(1) \qquad g(An+1) - Cf(n) = o(1) \quad \text{as} \quad n \to \infty.$$

Then either $f(n) = o(1)$ and $g(An+1) = o(1)$ as $n \to \infty$ or there exist a complex number s and functions $F,\ G \in \mathcal{M}$ such that

$$f(n) = n^s F(n),\ g(m) = n^s G(n),\ (0 \le \text{Re } s < 1)$$

and

$$G(An+1) = \frac{1}{F(2)} F(n)$$

are satisfied for all $n \in \mathbb{N}$.

In the proof of Theorem 1, we get

$$(2) \qquad F(2n) = F(2),\quad G(m) = \chi_{2A}(m) \ \text{ for all } \ n \in \mathbb{N},\ m \in \mathbb{N}_{2A}.$$

We shall prove

Theorem 2. *Assume that $A \in \mathbb{N}$, $D \in \mathbb{C} \setminus \{0\}$ and $F,\ G \in \mathcal{M},\ F \notin \mathcal{L}$ satisfy the equation*

$$(3) \qquad G(An+1) = DF(n).$$

Let

$$I(n) = 1 \quad \text{and} \quad \Psi(n) = (-1)^{n-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Then the following assertions hold:
(a) *If A is even, then all solutions $(D,\ F,\ G)$ of* (3) *have the form*

$$(D,\ F,\ G) = (1,\ I,\ \chi_A) \quad \text{and} \quad (D,\ F,\ G) = (-1,\ \Psi,\ \chi_{2A}),$$

where χ_{2A} is an arbitrary nonprincipal character (mod $2A$).
(b) *If A is odd, then all solutions of* (3) *have the form*

$$(D,\ F,\ G) = (1,\ I,\ \chi_A)$$

and

$$(D,\ F,\ G)=(z,\ \mathcal{B}_{(1,\ \frac{1}{z})},\ \mathcal{B}_{(A,\ z)}\chi_A),$$

where z is an arbitrary non-zero complex number, χ_A is an arbitrary character $(\mathrm{mod}\ A)$ *and multiplicative functions $\mathcal{B}_{(k,\ \ell)}$ are defined as follows:*

$$\mathcal{B}_{(k,\ \ell)}\in\mathcal{M}_k,\ \mathcal{B}_{(k,\ \ell)}(2n)=\ell\ \ \textit{for all}\ \ n\in\mathbb{N}_k$$

2. Lemmas

Lemma 1. *Assume that $A\in\mathbb{N},\ D\in\mathbb{C}\setminus\{0\}$ and $F,\ G\in\mathcal{M}$ satisfy the condition*

$$G(An+1)=DF(n)\quad(n\in\mathbb{N}).$$

Let

$$\mathcal{S}_F:=\{n\in\mathbb{N}\mid F(n)\neq 0\}\quad\textit{and}\quad\mathcal{S}_G:=\{n\in\mathbb{N}\mid (n,A)=1,\ G(n)\neq 0\}.$$

Then either the set $\mathcal{S}_F$ is finite or

$$\mathcal{S}_F=\mathbb{N}\quad\textit{and}\quad\mathcal{S}_G=\{n\in\mathbb{N}\mid (n,A)=1\}.$$

Proof. Lemma 1 is a consequence of Theorem 1 in [14]. ∎

Lemma 2. *Assume that $k_0,\ K\in\mathbb{N}$ and $\Psi\in\mathcal{M}$ satisfy the condition*

$$\Psi(k_0m+1)\to 0\quad\textit{as}\quad m\to\infty,\quad (k_0m+1,K)=1. \tag{4}$$

Then there is a positive integer k such that $\Psi\in\mathcal{L}_k$.

Proof. Assume that (4) holds for some positive integers k_0,K. We shall prove that there is a $k\in\mathbb{N}$ such that $\Psi\in\mathcal{L}_k$. For every reduced residue class $l\ (\mathrm{mod}\ k_0K)$ let $E_1^{(l)},\dots,E_{\varphi(k_0K)-1}^{(l)}$ be coprime integers belonging to $l\ (\mathrm{mod}\ k_0K)$, and satisfying $\Psi(E_j^{(l)})\neq 0\ \ (j=1,\dots,\varphi(k_0K)-1)$, if there exist so many $E_j^{(l)}$. Let $E^{(l)}:=E_1^{(l)}\dots E_{\varphi(k_0K)-1}^{(l)}$. Then $\Psi(E^{(l)})\neq 0$ and for all $x\in\mathbb{N},\ \ x\equiv l\ (\mathrm{mod}\ k_0K),\ \ (x,E^{(l)})=1$, we have $xE^{(l)}\equiv 1\ (\mathrm{mod}\ k_0K)$, and so by (9) and our assumptions, we get

$$\Psi(x)\to 0\ \text{ as }\ x\to\infty,\ \ x\equiv l\pmod{k_0K},\ \ (x,E^{(l)})=1.$$

If for some l the maximal size t of the set $E_1^{(l)},\dots,E_t^{(l)}$ constructed above is less than $\varphi(k_0K)-1$, then $\Psi(x)=0$ if $x\equiv l \pmod{k_0K}$ and $(x,E^{(l)})=1$, where $E^{(l)}:=E_1^{(l)}\dots E_t^{(l)}$. Hence $\Psi\in\mathcal{L}_k$ follows, where

$$k:=k_0K\prod_{\substack{1\le l\le k_0K\\(l,k_0K)=1}}E^{(l)}.$$

Lemma 2 is proved. ■

In the following we assume that the functions $f,\ g\in\mathcal{M}$ satisfy the condition (1), i.e.

$$g(An+1)-Cf(n)=o(1)\quad\text{as}\quad n\to\infty$$

with some fixed positive integer A and a non-zero complex constant C.

We say that a function $\Psi\in\mathcal{M}$ is of a finite support if there is a finite set $\mathcal{A}$ of distinct primes $p_1<p_2<\ldots<p_r$ such that

$$\Psi(p^\alpha)=0\ \ (\alpha=1,2\ldots)\ \text{ if }\ p\notin\mathcal{A}.$$

Lemma 3. *If f* or *g is of a finite support, then $f\in\mathcal{L}$ and $g\in\mathcal{L}_D$ hold for some $D\in\mathbb{N}$.*

Proof. Let

$$\mathcal{S}_f=\{n\in\mathbb{N}\mid f(n)\ne0\}\ \text{ and }\ \mathcal{S}_g:=\{n\in\mathbb{N}\mid(n,A)=1,\ g(n)\ne0\}.$$

Assume first that f is of a finite support, that is $f(p^\alpha)=0$ $(\alpha=1,2\ldots)$ if $p\notin$ $\notin\mathcal{A}:=\{p_1,p_2,\dots,p_r\}$. Let $\Delta=p_1\cdots p_r$. For an arbitrary positive integer n let $n=D_\Delta(n)E_\Delta(n)$, where $D_\Delta(n)$ is the product of those prime power divisors p^α of n for which $p|\Delta$, and $E_\Delta(n)$ is coprime to Δ. Then $g(Am+1)\to0$ as $m\to\infty$ and $E_\Delta(m)>1$.

Assume that $f\notin\mathcal{L}$. Then $g(Am+1)\ne0$ holds for infinitely many integers m. It is obvious that there are an infinite sequence of primes $q_1<q_2<q_3<$ $<\cdots$ and suitable exponents α_j such that

$$\{q_1^{\alpha_1},q_2^{\alpha_2},q_3^{\alpha_3},\dots\}\subseteq\mathcal{S}_g.$$

In this case there are a positive integer ℓ, $(\ell,A)=1$ and an infinite sequence of prime powers

$$\{Q_1,Q_2,Q_3,\dots\}\subseteq\{q_1^{\alpha_1},q_2^{\alpha_2},q_3^{\alpha_3},\dots\}\subseteq\mathcal{S}_g,$$

for which $Q_j\equiv\ell\pmod A$. This shows that there exist a positive Q for which $Q\in\mathcal{S}_g$, $Q\equiv1\pmod A$ and an infinite sequence of positive integers $m_1<$

$< m_2 < \dots$ for which $(Q, Am_\nu + 1) = 1$ and $\liminf |g(Am_\nu + 1)| > 0$. Then $\liminf |g\left(Q(Am_\nu + 1)\right)| > 0$ and so

$$E_\Delta\left(Qm_\nu + \frac{Q-1}{A}\right) = 1, \quad E_\Delta(m_\nu) = 1$$

hold for every larger ν.

This contradicts Thue's theorem. Consequently $f \in \mathcal{L}$, and so it follows by Lemma 3 that $g \in \mathcal{L}_D$ holds for some $D \in \mathbb{N}$. Lemma 3 is proved. ■

The case, when g is of a finite support can be treated similarly.

Lemma 4. *If there are positive integers Δ and D such that $f \in \mathcal{L}_\Delta$ and $g \in \mathcal{L}_D$, then $f \in \mathcal{L}$.*

Proof. By using Lemma 3 we can assume that f, g are not of finite supports. Let $\Delta = \pi_1^{\delta_1} \dots \pi_r^{\delta_r}$ and $D = q_1^{d_1} \dots q_s^{d_s}$, where $\{\pi_1, \dots, \pi_r\} \subseteq \mathcal{P}$ and $\{q_1, \dots, q_s\} \subseteq \mathcal{P}$.

We may assume that for each π_j there exists at least one l_j (≥ 1) such that $f(\pi_j^{l_j}) \neq 0$. Let

$$E(t_1, \dots, t_r) := \pi_1^{t_1} \dots \pi_r^{t_r}.$$

Assume that $Q_1, \dots, Q_s$ are positive integers for which $(Q_i, Q_j) = 1$ $(1 \leq i < j \leq s)$ and $f(Q_i) \neq 0$, $(Q_i, \Delta) = 1$. For $u, v, j \in \mathbb{N}$, $u \neq v$ let $q_j^{\beta_{u,v,j}} \parallel Q_u - Q_v$ and

$$T := \max_{\substack{u,v,j \\ u \neq v}} \beta_{u,v,j}.$$

Then there is a $j_0 \in \{1, \dots, s\}$ for which

$$q_j^{T+1} \not| \, AE(t_1, \dots, t_r)Q_{j_0} + 1 \quad (j = 1, \dots, s).$$

Let now j be fixed, $l_1, \dots, l_{j-1}, l_{j+1}, \dots, l_r$ be so chosen that $f(\pi_i^{l_i}) \neq 0$ $(i = 1, \dots, j-1, j+1, \dots r)$. Let $t_j \to \infty$. Then

$$f\left(E(l_1, \dots, t_j, \dots, l_r)\right) \to 0 \quad \text{as} \quad l_j \to \infty,$$

consequently $f(\pi_j^{t_j}) \to 0$ $(t_j \to \infty)$. Thus $f \in \mathcal{L}$ and so Lemma 4 is proved. ■

Lemma 5. *If there are positive integers Δ and D such that $f \in \mathcal{L}_\Delta$ or $g \in \mathcal{L}_D$, then $f \in \mathcal{L}$.*

Proof. By using Lemma 3 we can assume that f, g are not of finite supports. We shall prove only the first assertion. Assume that that $f \in \mathcal{L}_\Delta$

holds for some positive integer Δ, $\Delta = \pi_1^{\delta_1} \dots \pi_r^{\delta_r}$. It is obvious that given an arbitrary constant c, $f(n) \to 0$ as $n \to \infty$ under the condition $D_\Delta(n) \le c$, where

$$D_\Delta(n) := \prod_{\substack{p^\alpha \| n \\ p \,| \Delta}} p^\alpha.$$

Since g is not of a finite support, there are coprime integers $Q_1, \dots, Q_t$ for which $Q_j \equiv 1 \pmod A$ and $g(Q_j) \neq 0$ $(j = 1, \dots, t)$. Let $\pi_j^{\beta_{u,v,j}} \| Q_u - Q_v$ and $T := \max_{\substack{u,v,j \\ u \neq v}} \beta_{u,v,j}$. Let m run over the set of those integers for which $(Am + 1, Q_1 \dots Q_t\Delta) = 1$. Then

$$g\left(Q_j(Am+1)\right) - Cf\left(Q_j m + \frac{Q_j - 1}{A}\right) = o(1) \tag{5}$$

as $m \to \infty$, $(Am + 1, Q_1 \dots Q_t\Delta) = 1$. For each fixed π_l no more than one j exists for which $\pi_l^{T+1} | Q_j m + \frac{Q_j-1}{A}$. Thus, if $t > r$, then for each m there is a j such that

$$D_\Delta\left(Q_j m + \frac{Q_j - 1}{A}\right) | (\pi_1 \dots \pi_r)^T$$

on the set of those integers for which $(Am + 1, Q_1 \dots Q_t\Delta) = 1$. Hence, by (5) we get that $g(Am + 1) \to 0$ as $m \to \infty$ and $(Am + 1, Q_1 \dots Q_t\Delta) = 1$. By Lemma 2 there is a $D \in \mathbb{N}$ such that $g \in \mathcal{L}_D$. Lemma 4 completes the proof of Lemma 5. ∎

Lemma 6. *If there is a positive integer n_0 or a positive integer m_0 such that $f(n_0) = 0$ or $g(Am_0 + 1) = 0$, then $f \in \mathcal{L}$.*

Proof. Applying (1) with $n = N[N(AN + 1)^2 m + 1]$, we have

$$g(AN+1)g\left(N^2(AN+1)m+1\right) - Cf(N)f\left((AN+1)^2 Nm + 1\right) = o(1)$$

as $m \to \infty$. If $g(AN+1) \neq 0$ and $f(N) = 0$, then one can deduce from the last relation and Lemma 2–5 that $f \in \mathcal{L}$. If there is an $N \in \mathbb{N}$ such that $f(N) \neq 0$ and $g(AN + 1) = 0$, then we also have $f \in \mathcal{L}$.

Finally, assume that for every positive integer N either $f(N) = g(AN+1) = 0$, or $f(N)g(AN + 1) \neq 0$. Let

$$F(n) = \begin{cases} 1, & \text{if } f(n) \neq 0 \\ 0, & \text{if } f(n) = 0 \end{cases} \quad \text{and} \quad G(m) = \begin{cases} 1, & \text{if } g(m) \neq 0 \\ 0, & \text{if } g(m) = 0. \end{cases}$$

Then $G(An + 1) = F(n)$ holds for all $n \in \mathbb{N}$ and $F, G \in \mathcal{M}$. Let

$$S_F := \{n \in \mathbb{N} \mid F(n) \neq 0\} \quad \text{and} \quad S_G := \{n \in \mathbb{N} \mid G(n) \neq 0\}.$$

Since $f(n_0) = 0$ (or $g(Am_0 + 1) = 0$), it follows from Lemma 1 that $|S_F| < \infty$, thus $f(n) \to 0$ and $g(An + 1) \to 0$ as $n \to \infty$.

The proof of Lemma 6 is thus complete. ■

Lemma 7. *Assume that $A \in \mathbb{N}$, $C \in \mathbb{C} \setminus \{0\}$ and $f, g \in \mathcal{M}$ satisfy the relation*

$$g(An + 1) - Cf(n) = o(1) \quad as \quad n \to \infty.$$

Let P, Q and N be positive integers satisfying the conditions

$$Q \equiv 1 \pmod{A} \quad and \quad N = P(Q - 1) + 1. \tag{6}$$

If $f \notin \mathcal{L}$, then

$$g(N)\,f(PQd) = f(P)g(Q)f(N)\,f(d), \tag{7}$$

where

$$d = d(P, Q) := \begin{cases} 2, & \text{if } (P-1)\frac{Q-1}{A} \text{ is odd} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let P, Q and N be positive integers satisfying (6). Let

$$N_1 := E_Q(N),\ P_1 := E_Q(P),\ Q_1 := E_N(Q),$$

where $E_k(n)$ is the product of all prime power divisors of n which are prime to k and $\left(\frac{n}{E_k(n)}, E_k(n)\right) = 1$.

First we prove that there is a positive integer n_0 such that

$$\begin{cases} (N_1,\ APQn_0 + 1) = 1, \\ \left(P_1,\ NQn_0 + \frac{Q-1}{A}\right) = 1, \\ (Q_1,\ ANn_0 + 1) = 1, \\ (N_1P_1Q_1,\ n_0) = d(P, Q). \end{cases} \tag{8}$$

We infer from the facts $N = P(Q - 1) + 1$ and $(N_1, Q) = 1$ that $N_1 = E_Q(N)$ is an odd positive integer. By (6), we have $(N_1, PQ) = (N_1, A) = 1$ and so an application of the Chinese Remainder Theorem shows that there exists an n_1 for which

$$(N_1,\ APQn_1 + 1) = (N_1,\ n_1) = 1.$$

It is clear to check from the definition of $P_1 = E_Q(P)$ that if P_1 is odd, then there is an $n_2' \in \mathbb{N}$ such that

$$(P_1,\ ANQn_2' + 1) = (P_1,\ n_2') = 1.$$

Assume that $P_1 = E_Q(P)$ is even. Then Q and $N = P(Q-1)+1$ are odd numbers. In this case there is an n_2'' for which

$$\left(P_1,\ NQn_2'' + \frac{Q-1}{A}\right) = 1 \ \text{ and } \ (P_1,\ n_2'') = d_2(P,Q),$$

where

$$d_2(P,Q) := \begin{cases} 2, & \text{if } (P-1)Q\frac{Q-1}{A} \text{ is odd} \\ 1, & \text{otherwise.} \end{cases}$$

Consequently, we can find an n_2 for which

$$\left(P_1,\ NQn_2 + \frac{Q-1}{A}\right) = 1 \ \text{ and } \ (P_1,\ n_2) = d_2(P,Q).$$

Finally, we infer from the definition of $Q_1 = E_N(Q)$ and $A|Q-1$ that $(Q_1, AN) = 1$. Hence, if Q_1 is odd, then there exists an n_3' for which

$$(Q_1,\ ANn_3' + 1) = 1 \ \text{ and } \ (Q_1,\ n_3') = 1.$$

Assume that Q_1 is even. Then $(AN, 2) = 1$ and so there is a n_3'' for which

$$(Q_1,\ ANn_3'' + 1) = 1 \ \text{ and } \ (Q_1,\ n_3'') = d_3(P,Q),$$

where

$$d_3(P,Q) := \begin{cases} 2, & \text{if } (P-1)(Q-1) \text{ is odd} \\ 1, & \text{otherwise.} \end{cases}$$

Consequently, we can find an n_3 for which

$$(Q_1,\ ANn_3 + 1) = 1 \ \text{ and } \ (Q_1,\ n_3) = d_3(P,Q).$$

We can check from definitions of $d(P,Q), d_2(P,Q)$ and $d_3(P,Q)$ that

$$d_2(P,Q)d_3(P,Q) = d(P,Q).$$

Since $(N_1, P_1) = (N_1, Q_1) = (P_1, Q_1) = 1$ it follows from the Chinese Remainder Theorem that there is an n_0 such that

$$n_0 \equiv n_1 \pmod{N_1},\ n_0 \equiv n_2 \pmod{P_1},\ n_0 \equiv n_3 \pmod{Q_1}.$$

Hence, the proof of (8) is finished.

Next, we shall prove (7). Let n_0 be a positive integer which satisfies (8). By considering $n = N_1P_1Q_1m + n_0$, it is clear to check from (8)

$$\text{(9)} \qquad \begin{cases} (N,\ APQn+1) = 1, \\ \left(P,\ NQn + \frac{Q-1}{A}\right) = 1, \\ (Q,\ ANn+1) = 1. \end{cases}$$

By using the multiplicativity of f and g, we get from (9) the following relations:

$$\begin{aligned}
Cg(N)f\,(PQn) &= g(N)g\,(APQn+1)+o(1)= \\
&= g\left[A\left(NPQn+\frac{N-1}{A}\right)+1\right]+o(1)= \\
&= Cf\left(NPQn+\frac{N-1}{A}\right)+o(1)= \\
&= Cf(P)f\left(NQn+\frac{Q-1}{A}\right)+o(1)= \\
&= f(P)g\left[A\left(NQn+\frac{Q-1}{A}\right)+1\right]+o(1)= \\
&= f(P)g(Q)g\,[ANn+1]+o(1)= \\
&= Cf(P)g(Q)f\,(Nn)+o(1),
\end{aligned}$$

which imply

$$g(N)f\,(PQn)-f(P)g(Q)f\,(Nn)=o(1) \tag{10}$$

as $m\to\infty,\;\; n=N_1P_1Q_1m+n_0\to\infty$.

It follows from (8) that we can choose a positive integer m_0 such that

$$t_0:=\frac{N_1P_1Q_1}{d}m_0+\frac{n_0}{d}\quad\text{and}\quad (t_0,\;dPQN)=1.$$

Taking $m=D_Q(N)D_Q(P)D_N(Q)d^2t+m_0$, from (10) we have $n=N_1P_1Q_1m+$ $+n_0=d(NPQdt+t_0)$, consequently

$$\Big(g(N)f\,(PQd)-f(P)g(Q)f\,(Nd)\Big)f(NPQdt+t_0)=o(1)$$

as $t\to\infty$. It is obvious that $g(N)f\,(PQd)-f(P)g(Q)f\,(Nd)=0$, because in the other case, we have $f(NPQdt+t_0)=o(1)$ as $t\to\infty$, therefore we get from Lemma 2 and Lemma 5 that $f\in\mathcal{L}$. Since $(N,d)=1$, we have

$$g(N)f\,(PQd)=f(P)g(Q)f\,(Nd)=f(P)g(Q)f\,(N)\,f\,(d)\,,$$

which completes the proof of (7).

Lemma 7 is proved. ■

Lemma 8. *Assume that* $A\in\mathbb{N}$*,* $C\in\mathbb{C}\setminus\{0\}$ *and* $f,\;g\in\mathcal{M}$ *satisfy the relation*

$$g(An+1)-Cf(n)=o(1)\quad\textit{as}\quad n\to\infty.$$

If $f \notin \mathcal{L}$, then

(11) $$f \in \mathcal{M}_{2A}^*,\ g \in \mathcal{M}_{2A}^* \quad \text{and} \quad H(n) := \frac{g(n)}{f(n)} = \chi_{2A}(n) \quad (n \in \mathbb{N}_{2A}).$$

Hence χ_k denotes the character $(\bmod\, k)$.

Proof. First we prove

$$H(n) := \frac{g(n)}{f(n)} = \chi_{2A}(n)\ (n \in \mathbb{N}_{2A}).$$

Let $Q \in \mathbb{N}$ be a positive integer such that $Q \equiv 1 \pmod A$ and let $P = 2Qm+1$, $(m \in \mathbb{N})$. Then $d = d(P,Q) = 1$, $(P,Q) = 1$, $N = 2(Q-1)Qm + Q$ and by (7), we have

(12) $$H\left[2(Q-1)Qm + Q\right] = \frac{f(P)g(Q)}{f(PQ)} = H(Q).$$

Thus, we infer from Lemma 19.3 of [1] that

(13) $$H(n) = \chi_{2Q(Q-1)}(n) \quad \text{on the set} \quad \Big(n, 2Q(Q-1)\Big) = 1.$$

It is clear to see that there is a number $M \in \mathbb{N}$ for which

$$\Big(M(AM+1), A+1\Big) \in \{1,\ 2\}.$$

Then by applying (12) and (13) for the cases when $Q = A+1$, and $Q = AM+1$, respectively, we infer that

$$H \in \mathcal{M}_{2A(A+1)}^* \quad \text{and} \quad H \in \mathcal{M}_{2AM(AM+1)}^*.$$

Since

$$\Big(2A(A+1), 2AM(AM+1)\Big) = 2A\Big(A+1, M(AM+1)\Big) \in \{2A,\ 4A\},$$

we get from the above relations that

$$H \in \mathcal{M}_{2A}^*.$$

On the other hand, we have $(2(Q-1)m+1,\ Q,\ 2A) = 1$, consequently

$$H\Big[2(Q-1)Qm + Q\Big] = H(Q)H\Big[2(Q-1)m + 1)\Big].$$

Thus, (12) gives $H\Big[2(Q-1)m+1\Big] = 1$ and

$$H(n) = \chi_{2A}(n) \quad \text{and} \quad g(n) = \chi_{2A}(n)f(n)\ (n \in \mathbb{N}_{2A}).$$

Now we prove that

$$f,\ g \in \mathcal{M}^*_{2A}. \tag{14}$$

Let $Q = 2Ax + 1$, $P = 2Ay + 1$, $N = (2A)^2 xy + Q$. It is obvious that $d(P, Q) = 1$. From (7), (11) and (14) we have

$$H(N) = H(Q) = H(P) = H(PQ) = 1 \tag{15}$$

and

$$g(N)f(PQ) = f(P)g(Q)f(N). \tag{16}$$

From (15) and (16) we have

$$f(PQ) = f(P)g(Q) = f(P)f(Q) \text{ and } g(PQ) = f(PQ) = g(P)g(Q). \tag{17}$$

Now let $(nm, 2A) = 1$, $n, m \in \mathbb{N}$. We can choose two positive integers z, t such that

$$nz \equiv 1 \pmod{2A},\ (z, nm) = 1,\ mt \equiv 1 \pmod{2A},\ (t, nmz) = 1.$$

We infer from (17) that

$$f(nzmt) = f(nz)f(mt) = f(n)f(z)f(m)f(t),$$

$$g(nzmt) = g(nz)g(mt) = g(n)g(z)g(m)g(t)$$

and

$$f(nzmt) = f(nm)f(z)f(t), \quad g(nzmt) = g(nm)g(z)g(t).$$

Hence

$$f(nm) = f(n)f(m) \quad \text{and} \quad g(nm) = g(n)g(m),$$

and so (14) and (11) are proved. Lemma 8 is proved. ∎

Lemma 9. *Assume that $a,\ b \in \mathbb{N}$, $D \in \mathbb{C} \setminus \{0\}$ and $T \in \mathcal{M}^*$, $T \notin \mathcal{L}$ satisfy the relations*

$$T(n) \neq 0 \ \ (\forall n \in \mathbb{N}), \ \ T(an + b) - DT(n) = o(1) \quad \text{as} \quad n \to \infty. \tag{18}$$

Then $T(a) = D$ and there is a complex number s such that

$$T(n) = n^s, \ \ (0 \leq \text{Re } s < 1)$$

holds for all $n \in \mathbb{N}$.

Proof. Assume that $a,\ b\in\mathbb{N},\ D\in\mathbb{C}\backslash\{0\}$ and $T\in\mathcal{M}^*$ satisfy the relations (18). Since $T\in\mathcal{M}^*$ and

$$\Big(a^2m+b\Big)(a+1)=a\Big[a(a+1)m+b\Big]+b,$$

we get from (18) that

$$\begin{aligned}DT(am)T(a+1)&=T\Big(a^2m+b\Big)T(a+1)+o(1)=\\&=T\Big[a\Big(a(a+1)m+b\Big)+b\Big]+o(1)=\\&=DT\Big(a(a+1)m+b\Big)+o(1)=\\&=D^2T(a+1)T(m)+o(1)\end{aligned}$$

which from the fact $T\notin\mathcal{L}$ implies $T(a)=D$.

In the following we denote by J the set of those pairs (Q,R) of positive integers for which

$$T(Qn+R)-T(Qn)=o(1)\quad\text{as}\quad n\to\infty.$$

By using the same method that was applied in [11] and [12], we prove that the following assertions hold:

(a) $(Q,1)\in J$ if $(q,1)\in J$ and $Q\geq q$

(b) $(Q,R)\in J$ if $(q,1)\in J$, $q\geq 2$ and $0<R<Q/(q-1)$

(c) $(h,1)\in J$ if $(h+1,1)\in J$ and $h\geq 2$.

Assume that $(k,1)\in J$. By using $T\in\mathcal{M}^*$, we have

$$\begin{aligned}T(k)T\big((k+1)n+1\big)&=T\Big[k\Big((k+1)n+1\Big)+1\Big]+o(1)=\\&=T(k+1)T(kn+1)+o(1)=T(k)T(k+1)T(n),\end{aligned}$$

and so, we deduce that $(k+1,1)\in J$. By using induction, we have proved that (a) holds.

Assume again that $(k,1)\in J$ and $k\geq 2$. We shall prove (b) by induction on r. From (a) it is clear that (b) is satisfied for $r=1$. Assume that $(q,r)\in J$ holds for all integers q and r satisfying $0<r<q/(k-1)$ and $r<r_0$, where $r_0\geq 1$ is an integer. Let q_0 be an integer such that

$$0<r_0<\frac{q_0}{k-1}. \tag{19}$$

In order to show (b) it suffices to prove that $(q_0,r_0)\in J$. Without loss of generality we may assume that q_0 and r_0 are coprimes.

Let q and r be positive integers such that

$$r_0 q = q_0 r + 1 \quad \text{and} \qquad r < r_0. \tag{20}$$

It follows by (19) and (20) that

$$0 < r < (q_0 r + 1)/q_0 = r_0 q/q_0 < q/(k-1).$$

Thus, by using our assumption and the fact $r < r_0$, we have $(q, r) \in J$.

On the other hand, by (20), we infer from the facts $(q_0, 1) \in J$ and $(q, r) \in J$ that

$$\begin{aligned} T(q)T(q_0 n + r_0) &= T\Big(q_0(qn + r) + 1\Big) = T\Big(q_0(qn + r)\Big) + o(1) = \\ &= T(q_0)T(qn + r) + o(1) = \\ &= T(q_0)T(qn) + o(1), \end{aligned}$$

which shows that $(q_0, r_0) \in J$. Thus, we have proved (b).

Finally, we prove (c). Assume that $(h+1, 1) \in J$ and $h \geq 2$. For each $\ell \in \mathbb{N}, 0 \leq \ell \leq h-1$ let

$$\mathcal{A}_\ell := \{n \in \mathbb{N} \mid n \equiv \ell \pmod{h}\}$$

and we can choose positive integers $q = q(\ell)$ and $r = r(\ell)$ such that

$$(h\ell + 1)q = h^2 r + 1. \tag{21}$$

We shall prove that

$$T(hn + 1) - T(hn) = o(1) \ \text{ as } \ n \to \infty, \ n \in \mathcal{A}_\ell. \tag{22}$$

Let $n = hm + \ell \in \mathcal{A}_\ell$. Since $(h+1, 1) \in J$ and $h \geq 2$, by (a) we have $(h^2, 1) \in J$. Thus

$$\begin{aligned} T(q)T(hn + 1) &= T(qhn + q) = T\Big(qh^2 m + q(h\ell + 1)\Big) = \\ &= T\Big(h^2(qm + r) + 1\Big) = \\ &= T(h^2(qm + r)) + o(1) = \\ &= T(h)T\Big(q(hm + \ell) + hr - q\ell\Big) + o(1) = \\ &= T(h)T(q(hm + \ell)) + o(1) = \\ &= T(h)T(q)T(n) + o(1). \end{aligned}$$

In the last step, the assertion is true if $hr - q\ell = 0$. If $hr - q\ell \neq 0$, then we get from (21) that

$$0 < hr - q\ell = \frac{(q-1)}{h} < \frac{q}{h},$$

which, by applying (b) with $k = h+1$, implies that $(q, hr - q\ell) \in J$. This, with $(h^2, 1) \in J$ shows that (22) is true. This completes the proof of (c).

By (18) and using $T(a) = D$, one can deduce that $(a,\ b) \in J$ and $(a, 1) \in J$. If $a = 1$, then Lemma 9 follows from the Wirsing's theorem. If $a \geq 2$, then by using (c) one can deduce that $(2, 1) \in J$, and so

$$T(2n+1) - T(2n) = o(1) \quad \text{as} \quad n \to \infty.$$

By using the result of Bassily and Kátai [6], it follows that there is a complex number s such that $0 \leq \text{Re } s < 1$ and $T(n) = n^s$ for all $n \in \mathbb{N}$.

Lemma 9 is proved. ■

3. Proof of Theorem 1

Assume that $A \in \mathbb{N}$, $C \in \mathbb{C} \setminus \{0\}$ and the functions $f, g \in \mathcal{M}$ satisfy (1). Then from Lemma 8 we have

$$f,\ g,\ H \in \mathcal{M}^*_{2A},\ H = \chi_{2A}. \tag{23}$$

Let $a := 2A + 1$. From (1) and (23) we obtain

$$\begin{aligned} Cg(a)f(n) &= g(a)g(An+1) + o(1) = \\ &= g\Big[A\Big(an+2\Big)+1\Big] + o(1) = Cf\Big(an+2\Big) + o(1), \end{aligned}$$

therefore

$$f\Big(an+2\Big) - g(a)f(n) = o(1) \quad \text{as} \quad n \to \infty. \tag{24}$$

Next, we prove that

$$f\left(2p^k\right) = \frac{f(2p)^k}{f(2)^{k-1}} \tag{25}$$

holds for all $p \in \mathcal{P}$ and $k \in \mathbb{N}$.

For each $p \in \mathcal{P}$ and $k \in \mathbb{N}$, we define the sequence $T_k(n, p)$ by the formula

$$T_k(n,p) := (ap)^k D_p(2)n + 2\frac{(ap)^k-1}{ap-1},$$

where $D_p(2) = (p,\ 2)$ and $E_p(2) = \frac{2}{D_p(2)}$. Since

$$T_k(n,p) = ap\Big[(ap)^{k-1}D_p(2)n + 2\frac{(ap)^{k-1}-1}{ap-1}\Big]+2 = apT_{k-1}(n,p)+2$$

and

$$\Big(pD_p(2),\ \frac{T_{k-1}(n,p)}{D_p(2)}\Big) = 1,$$

we obtain from (24) that

$$\begin{aligned} f\Big(T_k(n,p)\Big) &= f\Big(apT_{k-1}(n,p)+2\Big)= \\ &= g(a)f\Big(pT_{k-1}(n,p)\Big)+o(1) = \\ &= \frac{g(a)f(pD_p(2))}{f(D_p(2))}f\Big(T_{k-1}(n,p)\Big)+o(1) \\ &= \frac{g(a)f(2p)}{f(2)}f\Big(T_{k-1}(n,p)\Big)+o(1), \end{aligned}$$

because

$$\frac{g(a)f(pD_p(2))}{f(D_p(2))} = \frac{g(a)f(2p)}{f(2)}.$$

This implies

$$(26)\quad f\Big[(ap)^k D_p(2)n + 2\frac{(ap)^k-1}{ap-1}\Big] = g(a)\Big(\frac{g(a)f(2p)}{f(2)}\Big)^{k-1} f\Big(pD_p(2)n\Big)+o(1).$$

as $n\to\infty$.

On the other hand, since $(a,\ 2)=1$ and $p\in\mathcal{P}$, we can find some $m_0\in\mathbb{N}$ such that

$$\Big((ap)^k m_0 + E_p(2),\ 2A\Big) = (m_0,\ 2A) = 1.$$

Choosing the subset of n's of the form

$$n = \frac{(ap)^k-1}{ap-1}\Big(2Am+m_0\Big),$$

then

$$\Big((ap)^k(2Am+m_0)+E_p(2),\ 2A\Big) = (2Am+m_0,\ 2A) = 1,$$

which with (23) and (26) implies

$$\Big[f\left(2p^k\right)-\frac{f(2p)^k}{f(2)^{k-1}}\Big]g(a)^k f\Big[\frac{(ap)^k-1}{ap-1}\Big]f\left(2Am+m_0\right)=o(1).$$

This completes the proof of (25).

We define $f^*\in\mathcal{M}^*$ as

$$f^*(p)=\frac{f(2p)}{f(2)}\quad(\forall p\in\mathcal{P}).$$

Let

$$f(n):=f^*(n)F(n)\quad\text{ for all }\ n\in\mathbb{N}. \tag{27}$$

Then one can check from (25) that

$$F(2p^k)=F(2)\ \text{ for all }\ p\in\mathcal{P},\ k\in\mathbb{N},$$

Consequently

$$F(n)=1\ \text{ for all }\ n\in\mathbb{N},\ (n,2)=1$$

and

$$F(2^\alpha)=F(2)\ \text{ for all }\ \alpha\in\mathbb{N}.$$

Hence

$$F(2n)=F(2)\ \text{ for all }\ n\in\mathbb{N}. \tag{28}$$

Now we prove the theorem.

From (1) and (23), we have

$$g(2An+1)=\chi_{2A}(2An+1)f(2An+1)=f(2An+1)$$

and

$$f(2An+1)-Cf(2n)=o(1)\ \text{ as }\ n\to\infty.$$

This with (28) gives

$$f^*(2An+1)-Cf^*(2)F(2)f^*(n)=o(1)\ \text{ as }\ n\to\infty.$$

By using Lemma 9, the last relation implies that there is a complex number s such that

$$f^*(n)=n^s\ (0\le\ \text{Re s }\ <1),\ \text{ and }\ f(n)=n^sF(n)\ (n\in\mathbb{N}).$$

Now let

$$g(n) = n^s G(n) \quad (n \in \mathbb{N}).$$

It is clear to see from the fact $f^*(2A) = Cf^*(2)F(2)$ that $A^s = CF(2)$. Thus, by using (1), we have

$$(An+1)^s G(An+1) = g(An+1) = Cf(n) + o(1) = Cn^s F(n) + o(1),$$

which gives

$$G(An+1) - \frac{1}{F(2)}F(n) = o(1) \quad \text{as} \quad n \to \infty.$$

Finally, by (11), (23) and (28) we have

$$G(2An+1) = \frac{g(2An+1)}{(2An+1)^s} = \frac{H(2An+1)f(2An+1)}{(2An+1)^s} = F(2An+1) = 1$$

and

$$G(m) = \chi_{2A}(m) \tag{29}$$

hold for all $n \in \mathbb{N},\ m \in \mathbb{N}_{2A}$. Hence

$$\begin{aligned} G(AN+1) &= G(AN+1)G(2ANn+1) = \\ &= G\Big[A\Big(2(AN+1)Nn+N\Big)+1\Big] = \\ &= \frac{1}{F(2)}F\Big(2(AN+1)Nn+N\Big) + o(1) = \\ &= \frac{1}{F(2)}F(N)F\Big(2(AN+1)n+1\Big) + o(1) = \\ &= \frac{1}{F(2)}F(N) + o(1) \end{aligned}$$

as $n \to \infty$. Thus $G(AN+1) = \frac{1}{F(2)}F(N)$ holds for each $N \in \mathbb{N}$.

Theorem 1 is proved.

4. Proof of Theorem 2

It is obvious that the functions defined in a) and b) of Theorem 2 satisfy (3). We note that in the case when A is even, for any nonprincipal character χ_{2A} (mod $2A$), we have $(A+1, 2A) = 1$ and

$$\Big(\chi_{2A}(A+1)\Big)^2 = \chi_{2A}(A^2+2A+1) = 1,\ \chi_{2A}(A+1) = -1.$$

Now we assume that $A \in \mathbb{N}$, $D \in \mathbb{C} \setminus \{0\}$ and $F,\ G \in \mathcal{M},\ F \notin \mathcal{L}$ satisfy the equation (3), i.e.

$$G(An+1) = DF(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

From Theorem 1, we infer that F and G satisfy (28) and (29), consequently

$$(30) \qquad F(n) = F\,[(n,2)] \quad \text{and} \quad G(m) = \chi_{2A}(m) \quad \text{for all} \quad n \in \mathbb{N},\ m \in \mathbb{N}_{2A}.$$

Case I. A **is even**

In this case we have $(An+1,\ 2A) = 1$, therefore we infer from (3) and (30) that

$$\begin{aligned} DF(n+1) &= DF\Big((A+1)n+1\Big) = G\Big[A\Big((A+1)n+1\Big)+1\Big] = \\ &= G\Big[(An+1)(A+1)\Big] = G(An+1)G(A+1) = \\ &= D^2F(n)F(1) = D^2F(n) \end{aligned}$$

and so

$$F(n+1) = DF(n),\ F(n+1) = D^n$$

hold for all $n \in \mathbb{N}$. Since $F(3) = F\,[(3,\ 2)] = F(1) = 1$, we obtain from the above relation

$$1 = F(3) = F(2+1) = D^2, \quad D \in \{1,\ -1\}.$$

If $D = 1$, then $F(n) = D^{n-1} = 1$ and $G(An+1) = DF(n) = 1$ for all $n \in \mathbb{N}$. Consequently

$$G(n) = \chi_A(n) \quad \text{for all} \quad n \in \mathbb{N}_A,$$

which proves that $(D,\ F,\ G) = (1,\ I,\ \chi_A)$.

If $D = -1$, then $F(n) = D^{n-1} = (-1)^{n-1} = \Psi(n)$. From (3) we deduce that

$$G(An+1) = DF(n) = (-1)^n,$$

consequently

$$G(2An+1) = 1 \quad \text{and} \quad G(2An+A+1) = -1$$

hold for all $n \in \mathbb{N}$. These imply that $G = \chi_{2A}$, where χ_{2A} is any nonprincipal character (mod $2A$). Thus $(D,\ F,\ G) = (-1,\ \Psi,\ \chi_{2A})$, which completes the proof of the assertion (a).

Case II. A **is odd**

For each $\alpha \in \mathbb{N}$ let $n_\alpha \in \mathbb{N}$ such that $2^\alpha \| \, An_\alpha + 1$. It is obvious from the fact $(A, 2) = 1$ that n_α is odd for all $\alpha \geq 1$. We get from (3) and (30) that

$$\begin{aligned} G(An_\alpha + 1)G(An_1 + 1) &= \frac{G(2^\alpha)G(2)}{G(2^{\alpha+1})} G\Big[(An_\alpha + 1)\,(An_1 + 1)\Big] = \\ &= \frac{G(2^\alpha)G(2)}{G(2^{\alpha+1})} G\Big[A\,(An_\alpha n_1 + n_\alpha + n_1) + 1\Big] = \\ &= D\frac{G(2^\alpha)G(2)}{G(2^{\alpha+1})} F\,(An_\alpha n_1 + n_\alpha + n_1) = \\ &= D\frac{G(2^\alpha)G(2)}{G(2^{\alpha+1})}. \end{aligned}$$

On the other hand, we obtain from (3) and (30) that

$$G(An_\alpha + 1)G(An_1 + 1) = D^2 F(n_\alpha)F(n_1) = D^2,$$

from which we get

$$G(2^{\alpha+1}) = \frac{G(2)}{D} G(2^\alpha) = \Big(\frac{G(2)}{D}\Big)^\alpha G(2) \quad \text{for all} \quad \alpha \in \mathbb{N}. \tag{31}$$

Now we define $G^* \in \mathcal{M}^*$ in $\mathbb{N}_A$ as

$$G^*(p) = \begin{cases} G(p), & \text{if } (p,\ 2A) = 1 \\ \\ \frac{G(2)}{D}, & \text{if } p = 2. \end{cases}$$

Let

$$G(n) := G^*(n)\overline{G}(n) \qquad \text{for all} \quad n \in \mathbb{N}_A.$$

Then one can check from (30) and (31) that

$$\overline{G}(2n) = \overline{G}(2) \quad \text{for all} \quad n \in \mathbb{N}_A, \tag{32}$$

which with (3) implies

$$G(An + 1) = G^*(An + 1)\overline{G}(An + 1) = DF(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

By putting $n = 2m + 1$, using (3), (30) and (32), we infer that

$$G(2Am + A + 1) = G^*(2Am + A + 1)\overline{G}(2Am + A + 1) = G^*\left(Am + \frac{A+1}{2}\right) G(2)$$

and

$$G(2Am + A + 1) = DF(2m + 1) = D.$$

hold for all $m \in \mathbb{N}$. Consequently

$$G^*\left(Am + \frac{A+1}{2}\right) = \frac{D}{G(2)}$$

and so

$$G^*(n) = \chi_A(n),\ G(n) = \chi_A(n)\overline{G}(n) \ \text{ for all } \ n \in \mathbb{N}_A. \tag{33}$$

Finally, from (3) we get

$$G(An + 1) = \chi_A(An + 1)\overline{G}(An + 1) = \overline{G}(An + 1) = DF(n).$$

Since A is odd, we deduce from (32) that $\overline{G}(An + 1) = \overline{G}[(An + 1,\ 2)] = \overline{G}[(n + 1,\ 2)] = \overline{G}(n + 1)$, therefore

$$\overline{G}(n + 1) = DF(n) \text{ for all } \ n \in \mathbb{N}. \tag{34}$$

It obvious from (30), (32) and (34) that $\overline{G}(2) = D$ and $DF(2) = \overline{G}(3) = 1$, which imply

$$F(2n) = F(2) = \frac{1}{D} \ \text{ and } \ \overline{G}(2m) = \overline{G}(2) = D \ \text{ for all } \ n \in \mathbb{N},\ m \in \mathbb{N}_A.$$

Therefore, we proved that $F = \mathcal{B}_{(1,\ \frac{1}{D})}$, $\overline{G} = \mathcal{B}_{(A,\ D)}$ and $G = \mathcal{B}_{(A,\ D)}\chi_A$.

The assertion (b) and so Theorem 2 is proved. ■

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