# LAUDATION TO

# Professor Janos Galambos

by I. Simonelli

Janos Galambos was born in Zirc, Hungary, on September 1, 1940. He entered L. Eötvös University in Budapest in 1958, and graduated with a Ph.D. in 1963, under the supervision of Alfréd Rényi. In 1964, Galambos was married to Éva, who also graduated from L. Eötvös University with majors in mathematics and physics.

He was Assistant Professor at Eötvös University, from 1964-1965, and a Lecturer at the University of Ghana, Legon, from 1965-69, and at the University of Ibadan, Nigeria, from 1969-70. In 1970 he joined the faculty at Temple University, Philadelphia, and has remained there ever since.

Early in his career, Janos had strong ties with distinguished Hungarian mathematicians such as Alfréd Rényi, Lajos Takács, and Paul Erdős, with whom he later had joint works. In addition to writing over 130 papers and 8 books, two of which have been translated into Russian and Chinese, he has edited several books. The importance of his work has been recognized throughout the world; he has been a frequent speaker at international conferences and has traveled widely as a guest of universities and scientific institutions. Indeed he has worked/lectured on every continent except Antarctica. He is an elected member of the Hungarian Academy of Science, the International Statistical Institute, the Spanish Royal Academy of Engineers, and a Fellow of the Institute of Mathematical Statistics.

He has had 12 Ph.D. students, who now work in the USA, China, and Korea.

## The Work of Janos Galambos

Galambos's work spans a variety of topics in mathematics. He has contributed to the theoretical development and applications of number theory, probability, and statistics. We shall now describe some of his most interesting results in these areas.

## 1. Number Theory

*1.1 Limiting distributions of arithmetical functions.* Galambos's initial work dealt with problems in number theory, and in particular, with the investigation of the limiting distributions of arithmetical functions. If one considers the probability space $S_N = (\Omega_N, \mathcal{A}_N, P_N)$, where $\Omega_N = \{1, 2, \cdots, N\}$, $\mathcal{A}_N$ the set of all subsets of $\Omega_N$, and $P_N$ the probability measure on $(\Omega_N, \mathcal{A}_N)$ which assigns mass $1/N$ to every element in $\Omega_N$, then every arithmetical function restricted to $\Omega_N$ is a random variable on $S_N$. In this setting, the number theoretical definitions of asymptotic density $D(A)$ of a set of integers $A$,

$$D(A) = \lim_{N\to+\infty} \frac{1}{N} \sum_{\substack{a_i\in A,\\ a_i\leq N}} 1,$$

and the limiting distribution $F(x)$ of an arithmetical function $f(n)$,

$$F(x) = \lim_{N\to+\infty} \frac{1}{N} \sum_{\substack{n\leq N\\ f(n)\leq x}} 1$$

have a natural probabilistic interpretation. Galambos fully exploited this connection by developing a probabilistic framework from which results on arithmetical functions can be obtained. He derived a three series theorem for Bernoulli random variables defined on a sequence of arbitrary probability spaces [7, 11] [1], which, if one assumes independence, would reduce to the Kolmogorov Three Series Theorem for binary random variables. As an application of these results, one obtains simple, probabilistic proofs of theorems of Erdős (V), Kubilius (X), and Delange (IV), whose validity now immediately extends to the case when $\Omega_N$ is replaced by an arbitrary collection of natural numbers. He also developed a general framework in which arbitrary arithmetical functions can be approximated by additive ones [31], leading to limit theorems with and without normalization constants [31, 39].

Results dealing with symmetry of the limiting distribution of a strongly multiplicative function $g$ have a long tradition in number theory. Under the assumption that

$$\sum_{g(p)<0} \frac{1}{p} < +\infty,$$

Bakštys (I) derived necessary and sufficient conditions for the existence and symmetry of a limiting distribution continuous at zero. Galambos extended these results to the case when the above sum diverges [16]. He proved the following theorem:

*Let $g(n)$ be a strongly multiplicative, real valued function and assume that $g(2) \neq -1$. Then $g(n)$ has a symmetric limit distribution function continuous at zero if, and only if, there exists a real number $c > 1$ such that each of the series*

$$\text{(i)} \sum_{|\ln|g(p)||>c} \frac{1}{p}, \quad \text{(ii)} \sum_{|\ln|g(p)||<c} \frac{\ln|g(p)|}{p}, \quad \text{(iii)} \sum_{|\ln|g(p)||<c} \frac{\ln^2|g(p)|}{p}$$

[1] For the references see the next paper of this journal.

*converges, and that*

$$\sum_{g(p)<0} \frac{1}{p} = +\infty.$$

Let $g(n)$ be a strongly multiplicative function, and let $F(x)$ be its limit distribution. The quantity $\Delta(x) = |1 - F(x) - F(-x)|$ is a measure of how far $F(x)$ is from being symmetric at $x$. Galambos and Szüsz [85] derived a simple estimate for $\Delta(x)$, which led to the most elementary proof of Wirsing's theorem (in the case $g(n) = 1$ or $-1$).

A closely related problem dealt with the convergence of arithmetical functions with normalization constants, $f^*(n) = \dfrac{f(n) - A_N}{B_N}$, where $A_N$ and $B_N$ are given sequences of real numbers. The first result on this topic was obtained by Erdős and Kac (VII), who proved that for $f(n) = v(n)$, the number of distinct prime divisors of $n$,

$$\frac{v(n) - \log\log N}{\sqrt{\log\log N}}$$

is asymptotically normally distributed. For a class H of Kubilius, where

$$A_N = \sum_{p \le N} \frac{f(p)}{p}, \qquad B_N^2 = \sum_{p \le N} \frac{f(p)^2}{p},$$

Kubilius (IX) derived necessary and sufficient conditions for the existence of a limit distribution. Levin and Fainleib (XI) obtained results in the case where $A_N$ and $B_N^2$ are multiples of $\log\log N$. Galambos considered the above limits without assuming any specific form for $A_N$ and $B_N$ or restriction on their magnitude and derived sufficient conditions on $A_N^*$ and $B_N$ under which

$$\left(\frac{f(n) - c\log N - A_N^*}{B_N}\right)$$

has a limit distribution [28].

*1.2 Metric number theory.* During his stay at the University of Ghana, Galambos attended seminar lectures given by Sir A. Oppenheim on representation of real numbers by infinite series. He became interested in probabilistic aspects of these series, and in 1970 he wrote his first paper on the subject [9]. This was the first of several articles in which Galambos developed a general metric theory which culminated in a monograph [53]. This monograph ended with a collection of 10 problems that have influenced the work of several mathematicians (for example Wang and Wu (XVII) and Shen, Liu and Zhou (XIV)).

Galambos considered the following expansion, which he called the $(\alpha, \beta)$-expansion of a real number $x$:

*For each $j$, two sequences of real numbers $\alpha_j(x)$ and $\gamma_j(x)$ are given, with $\alpha_j(n)$ strictly decreasing with $n$, and $\alpha_j(n-1)-\alpha_j(n) \leq \gamma_j(n)$. Then an auxiliary sequence $d_j = d_j(x)$ of integers is defined such that the infinite series*

$$\gamma(x) = \alpha_1(d_1) + \gamma_1(d_1)\alpha_2(d_2) + \gamma_1(d_1)\gamma_2(d_2)\alpha_3(d_3) + \cdots$$

*is always convergent.*

A general criterion for $\gamma(x) = x$ was given [53], and Galambos showed that the classical series expansions, i.e., Cantor, q-adic, Lüroth, Engel, and Oppenheim series, fit into this pattern. He derived general metric results on the digits of these expansions, viewed as random variables on the probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = (0, 1]$ and $P$ is Lebesgue measure. There are a few cases in which the digits are stochastically independent, i.e., the Cantor series representation and the Lüroth expansions [53] and [20] (see also [108]), but usually there is a strong dependence between the digits of these expansions.

We describe one of Galambos's results by considering the $\beta$-expansion,

$$\gamma(x) = \sum_{k=0}^{+\infty} \frac{\epsilon_k(x)}{\beta^k}, \qquad 1 < \beta < 2, \quad \epsilon_k(x) = 0 \text{ or } 1,$$

where the $\epsilon_k(x)$ are strongly dependent. By eliminating the zero terms in the above series, one obtains

$$\gamma(x) = \sum_{k=1}^{+\infty} \frac{1}{\beta^{n_k}}.$$

Galambos derived the cumulative and joint distributions of the random variables $n_1$, $n_2, \cdots$ and proved that the differences $n_1, n_2 - n_1, n_3 - n_2, \cdots$ are stochastically independent if, and only if, $\beta^n - \beta^{n-1} = 1$ for some integer $n$ [29]. For an arbitrary $(\alpha, \beta)$-expansion, Galambos developed a method for reducing the case where the digits are dependent to the case of independence of the denominators in the Lüroth expansion. This led to asymptotic properties of the digits in $(\alpha, \beta)$-expansions [34, 53], including laws of large numbers and results dealing with the rate of growth of denominators of Oppenheim series [50, 53].

Galambos's interest in number theory is not limited to the topics described above. For example, he investigated asymptotic properties of intermediate prime divisors and the asymptotic extremal property of prime divisors. He proved [54] that for intermediate primes which divide $n$, say $p_j(n)$, if $A_x = \{n : \log\log p_{j+1}(n) - \log\log p_j(n) \leq \leq x\}$, then

$$D(A_x) = 1 - e^{-x}.$$

This result was further generalized by Galambos and J.-M. De Koninck [91], proving the following remarkable result:

*Let $j = j(N)$ be a positive integer valued function tending to infinity with $N$. Assume that $j$ is such that, with perhaps the exception of a set of density zero, $p_j(n) \to \to +\infty$ with $N$, and $(\log p_j(n))/\log N \to 0$ as $N \to +\infty$, where $1 \le n \le N$. Then the points $\log\log p_{j+k}$, $k \ge 1$, form a Poisson process in limit as $N \to +\infty$.*

Galambos applied this result to investigate extremal properties associated with prime divisors [54], and he extended results of Erdös (VI) into Poisson limits.

## 2. Statistics

We limit our description of Galambos's work in this field to characterization of probability distributions and extremes.

*2.1 Characterization of probability distributions.* Characterization problems started in statistics, and the emphasis is on the identification of a property $A$ for which the collection of distribution functions which satisfy this property is a parametric family of distributions (or at most a few parametric families). Galambos considered a variety of properties, and in each case he stressed both mathematical and methodological results. His work on characterization using properties of order statistics [19, 43, 44] had a strong influence in the development of this subject. We mention the important work with S. Kotz [61], in which the emphasis is on the characterization of the exponential and logistic distributions obtained via the study of monotonic transformations, functional equations, and properties of conditional distributions. This last approach was further exploited by Galambos and Castillo to characterize bivariate distributions with given marginal distributions [90], [101], and [103]. This was one of the first systematic approaches used to characterize bivariate distributions.

*2.2 Extremes.* Problems dealing with the distribution of the maximum and minimum of random variables (extremes) stem from applications of probability in actuarial and insurance problems, some of which date back to the early eighteenth century. Much of the early development of the theory was motivated by these applied problems, and the only book available until the early 70's was a book by Gumbel (VIII), written primarily for engineers. Galambos's contribution to the subject has been important not only for its theoretical development, but also for its impact on changing the attitude of the mathematical community towards this subject. His book [60] was the first book on the subject written for mathematicians. The book was translated into Russian in 1984, and its second edition [86] was translated into Chinese in 2001.

Galambos started his long sequel of papers on extremes by characterizing the limit distribution for the maximum (under proper normalization) of a wide class of dependent random variables [21]. He constructed a model where dependence can be described in terms of edges of a graph, and then he applied graph-sieve inequalities (due to Rényi (XIII)) and their extensions [3] to derive several limit theorems. He also extended a result of Mogyoródi (XII) to a random number of dependent random variables [30, 111]. This work and its generalizations [30, 86, 97] constitute an important chapter on the theory of extremes for dependent random variables.

In the case of independent identically distributed random variables (i.i.d.), Galambos and A. Obretenov [92] obtained necessary and sufficient conditions for the existence of a limit distribution of the maximum of the $X_i$'s (under proper normalization). These conditions are expressed in terms of the hazard rate and expected residual life of the underlying population.

Galambos recognized and stressed the importance of exchangeability in the study of order statistics [58, 65, 86]. He proved that in studying the distribution of the number of occurrences in a given finite sequence of events, one can always assume that the events are exchangeable [32]; this led to the derivation of the distribution of order statistics. In the case where this property extends to infinite sequences, he obtained Poisson limits for sequences of dependent random variables, limit laws for mixtures [47], and derivations of the limit distributions of order statistics [32, 35, 47].

Galambos also contributed to the development of the theory of extremes for i.i.d. random vectors. He derived necessary and sufficient conditions for the multivariate extremes to have a limit distribution (under proper normalization), and asymptotic independence [46]. He also gave the exact distribution of all possible limits of extreme distributions.

Galambos was also interested in other aspects of statistics. His joint work with H.A. David [40] is considered to be one of the fundamental papers on the theory of concomitants. He developed nonparametric tests for extreme value distributions, [69, 100, 132], and was instrumental in the development of a methodology for the proper use of extreme value theory in applications, [119, 133].

## 3. Probability

*3.1 Inequalities.* Galambos's dissertation dealt with probabilistic inequalities, and he continued to work on this topic throughout the years. He mainly worked on Bonferroni-type inequalities, inequalities which are valid in any probability space, and used them as a unifying tool to solve problems in different branches of mathematics.

His first paper on this topic dealt with what is known as the graph-sieve inequalities. These types of results originated in number theory, and they were later applied to probability. Let $A_1, A_2, \cdots, A_n$ be arbitrary events in a given probability space $(\Omega, \mathcal{A}, P)$, and let $v_n$ be the number of those $A_j's$, $1 \leq j \leq n$, that occur. Rényi (XIII) applied the sieve method to derive upper and lower bounds for $P(v_n = 0)$, and Galambos derived upper and lower bounds for $P(v_n = k)$ and $P(v_n \geq k)$, $k \geq 1$. These results continue to be the most general results of this type and have turned out to be very important in the derivation of limit theorems in combinatorics, extremes, random subsets, etc. [10, 125]. This approach was later expanded in [26].

Galambos derived a method for proving Bonferroni-type inequalities [41], later refined in [62], which had significant implications in the derivation of inequalities for multivariate extreme distributions. We state this later result:

*Let* $S_{0,n} = 1$*, and for* $k \geq 1$*,*

$$S_{k,n} = E\left[\binom{v_n}{k}\right].$$

*Then with* $a = 1, 0$ *or* $-1$*, there exist constants* $c_k = c_k(n, r; a)$ *such that*

$$aP(v_n = r) + \sum_{k=0}^{n} c_k S_{k,n} \geq 0$$

*if and only if it holds for independent identically distributed events, and* $c_k =$ $= c_k(N, r; a)$*, where* $N$ *runs through all integers greater than or equal to* $n$*.*

Extensions to the bivariate case were obtained by Galambos and Y. Xu [123], and to $P(v_n \geq k)$ by Galambos and Simonelli [127]. These results provide a method of proof as well as a method to derive new inequalities in closed form, needed when $S_k$ is known for several $k's$, or in proving limit theorems [102].

The definition of Bonferroni-type inequality can be easily adjusted to cover multiple sequences of events. Let $A_i$, $1 \leq i \leq n$, and $B_j$, $1 \leq j \leq m$, be two arbitrary sequences of events, and let $v_n(A)$ and $v_m(B)$ denote the number of $A_i$'s and $B_j$'s which occur, respectively. Galambos and Xu [115] derived the best upper and lower bounds for $P(v_n(A) \geq 1)$ among the linear combinations of $S_{1,1}, S_{2,1}, S_{1,2}$, and $S_{2,2}$, where

$$S_{k,t} = E\left[\binom{v_n}{k}\binom{v_m}{t}\right].$$

The book on Bonferroni-type inequalities [125] written with Simonelli remains the main reference on this subject.

*3.2 Products of random variables.* Zolotarev (XVIII) and Bakšt̃ys (II) were among the first to methodically investigate problems on arbitrary products of independent random variables; the only books on the subject available at the beginning of 2004 were a book by Springer (XVI), which only deals with finite products, and a book by Bareikis and Š̃iaulys (III) (in Lithuanian). Galambos started working on products dealing with multiplicative functions [7, 8, 16], and realized the need to develop a unified theory. He suggested that Simonelli investigate limit theorems for products of independent random variables with and without normalization. Following Simonelli's work in (XV), Galambos and Simonelli extended and applied these results in a sequel of papers which culminated in a book [142]. In dealing with asymptotic results for products of independent random variables, $X_1, X_2, \cdots$, they proved the following:

*For the complete convergence of* $(\prod_{i=1}^{n})X_i e^{-a_n}$ *to a random variable* $Y$*,* $P(Y=0)=0$*, where the* $a_n$ *are constants, it is sufficient that there exists a random variable* $V$ *such that as* $n \to +\infty$

$$\left|\prod_{i=1}^{n} X_i\right| e^{-a_n} \longrightarrow V \quad \textit{completely,}\ \ P(V=0)=0,$$

*and either* $lim_{n\to+\infty} P(X_n \le 0)=0$*, or* $X_1 \cdots X_n$ *is symmetric for some* $n$*, or*

$$\sum_{i=1}^{+\infty} \min\{P(X_i<0),\, P(X_i>0)\} = +\infty.$$

They also derived necessary conditions for the limit to be symmetric and developed limit theorems for products of random variables arising from triangular arrays.

Characterization results based on finite and infinite products were developed in [138, 141, 142]. The theory of products was also applied to study the limit distribution of multiplicative functions; Galambos and Simonelli provided a pure probabilistic proof of Bakš̃tys's theorem [24], and consequently, of Erdős–Whitney's theorem (the only non-probabilistic result being the Kubilius–Turán Inequality).

**Postscript.** It has been a pleasure to highlight some of Galambos's profound work in mathematics, but he has been more than a distinguished mathematician. He has been an inspiring teacher and mentor to many of us, and we wish to convey our most sincere appreciation for all he has done.

## General References

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