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SUMMATION OF FOURIER SERIES WITH RESPECT TO WALSH–LIKE SYSTEMS AND THE DYADIC DERIVATIVE

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Dedicated to Professor Ferenc Schipp on his 70th birthday
and to Professor Péter Simon on his 60th birthday

Abstract. In this paper we present some results on summability of one- and multi-dimensional Walsh-, Walsh-Kaczmarz- and Vilenkin-Fourier series and on the dyadic and Vilenkin derivative. The Fejér and Cesàro summability methods are investigated. We will prove that the maximal operator of the summability means is bounded from the martingale Hardy space H_p to L_p $(p > p_0)$. For $p = 1$ we obtain a weak type inequality by interpolation, which ensures the a.e. convergence of the summability means. Similar results are formulated for the one- and multi-dimensional dyadic and Vilenkin derivative. The dyadic version of the classical theorem of Lebesgue is proved, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f.

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1. Introduction

In this paper we will consider summation methods for one- and multi-dimensional Walsh-, Walsh-Kaczmarz- and Vilenkin-Fourier series and the one- and multi-dimensional dyadic and Vilenkin derivative. Two types of summability methods will be investigated, the Fejér and Cesàro or (C,α) methods. The Fejér summation is a special case of the Cesàro method, $(C,1)$ is exactly the Fejér method. In the multi-dimensional case two types of convergence and maximal operators are considered, the restricted (convergence over the diagonal or over a cone), and the unrestricted (convergence over $\mathbb{N}^d$). We introduce martingale Hardy spaces H_p and prove that the maximal operators of the summability means are bounded from H_p to L_p whenever $p > p_0$ for some $p_0 < 1$. For $p = 1$ we obtain a weak type inequality by interpolation, which implies the a.e. convergence of the summability means. The a.e. convergence and the weak type inequality are proved usually with the help of a Calderon-Zygmund type decomposition lemma. However, this lemma does not work in higher dimensions. Our method, that can be applied in higher dimension, too, can be regarded as a new method to prove the a.e. convergence and weak type inequalities.

Similar results are formulated for the one- and multi-dimensional dyadic and Vilenkin derivative. We get that the maximal operators are bounded from H_p to L_p if $p > p_0$ ($p_0 < 1$) and a weak type inequality if $p = 1$. This implies the dyadic version of the classical theorem of Lebesgue, more exactly, the dyadic derivative of the dyadic integral of a function f is a.e. f. In this survey paper we summarize the results appeared in this topic in the last 10–20 years.

2. One-dimensional Fourier series

The well known Carleson's theorem [7] says, that the partial sums $s_n f$ of the trigonometric Fourier series of a one-dimensional function $f \in L_2(\mathbb{T})$ converge a.e. to f as $n \to \infty$. Later Hunt [30] extended this result to all $f \in$ $\in L_p(\mathbb{T})$ spaces, $1 < p < \infty$. This theorem does not hold, if $p = 1$. However, if we take some summability methods, we can obtain convergence for L_1 functions, too.

In 1904 Fejér [13] investigated the arithmetic means of the partial sums, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+$

$+f(x+0))/2$. One year later Lebesgue [31] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus a.e. The Cesàro means or (C,α) $(\alpha>0)$ means are generalizations of the Fejér means; if $\alpha=1$ then the two types of means are the same. M. Riesz [41] proved that the (C,α) $(\alpha>0)$ means $\sigma_n^\alpha f$ of a function $f\in L_1(\mathbb{T})$ converge a.e. to f as $n\to\infty$ (see also Zygmund [84, Vol. I, p. 94]). Moreover, it is known that the maximal operator of the (C,α) means $\sigma_*^\alpha := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha|$ is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_* f>\rho) \leq C\|f\|_1 \qquad (f\in L_1(\mathbb{T})).$$

This result can be found implicitly in Zygmund [84 Vol. I, pp. 154-156].

For the Fejér means Móricz [34] and Weisz [71] verified that σ_*^1 is bounded from $H_1(\mathbb{T})$ to $L_1(\mathbb{T})$. The author [75] extended this result to the Cesàro summation, i.e. to σ_*^α, $\alpha>0$ and $1/(\alpha+1)<p<\infty$.

In the next subsections analogous results will be given for Walsh-, Walsh-Kaczmarz- and Vilenkin-Fourier series.

2.1. Orthonormal systems

In this section we introduce the Walsh, Walsh-Kaczmarz and Vilenkin systems.

2.1.1. Walsh functions

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1 & \text{if } x\in[0,\frac{1}{2}); \\ -1 & \text{if } x\in[\frac{1}{2},1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x\in[0,1),\ n\in\mathbb{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k},$$

where

$$n=\sum_{k=0}^{\infty} n_k 2^k \qquad (0\leq n_k<2).$$

2.1.2. Walsh-Kaczmarz system

Here we consider the *Kaczmarz rearrangement* of the Walsh system. For $n \in \mathbb{N}$ there is a unique s such that $n = 2^s + \sum_{k=0}^{s-1} n_k 2^k$ $(0 \le n_k < 2)$. Define

$$\kappa_n(x) := r_s(x) \prod_{k=0}^{s-1} r_{s-k-1}(x)^{n_k} \qquad (x \in [0,1),\ n \in \mathbb{N})$$

and $\kappa_0 := 1$. It is easy to see that $\kappa_{2^n} = w_{2^n} = r_n$ $(n \in \mathbb{N})$ and

$$\{\kappa_k : k = 2^n, \dots, 2^{n+1}-1\} = \{w_k : k = 2^n, \dots, 2^{n+1}-1\}.$$

In what follows we will use the notation w_n instead of κ_n.

2.1.3. Vilenkin system

The Walsh system is generalized as follows. We need a sequence $(p_n, n \in \in \mathbb{N})$ of natural numbers whose terms are at least 2. We suppose always that this sequence is **bounded**. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k \qquad (n \in \mathbb{N}).$$

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \le x_k < p_k,\ x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called *generalized Rademacher functions.*

The *Vilenkin system* is given by

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$. Recall that the functions corresponding to the sequence $(2, 2, \ldots)$ are the Rademacher and Walsh functions (see Vilenkin [60] or Schipp, Wade, Simon and Pál [50]).

2.2. Hardy spaces

For a set $\mathbb{X} \ne \emptyset$ let $\mathbb{X}^j$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself j-times. We briefly write $L_p[0,1)^j$ instead of the space $L_p([0,1)^j, \lambda)$ $(j \ge 1)$ where λ is the Lebesgue measure.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \le k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x. If we replace 2^{-n} by P_n^{-1} then the intervals are called *Vilenkin intervals*. The σ-algebra generated by the dyadic or Vilenkin intervals $\{I_n(x) : x \in [0,1)\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbb{N})$. For the Walsh and Walsh-Kaczmarz system we will use dyadic intervals and for the Vilenkin system Vilenkin intervals.

We investigate the class of *martingales* $f = (f_n,\ n \in \mathbb{N})$ with respect to $(\mathcal{F}_n, n \in \mathbb{N})$. The *maximal function* of a martingale f is defined by

$$f^* := \sup_{n \in \mathbb{N}} |f_n|.$$

For $0 < p \le \infty$ the *martingale Hardy space* $H_p[0,1)$ consists of all one-parameter martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

Recall that the Hardy and L_p spaces are equivalent, if $p > 1$, in other words,

$$H_p[0,1) \sim L_p[0,1) \qquad (1 < p \le \infty).$$

Moreover, the martingale maximal function is of weak type $(1,1)$:

$$\|f\|_{H_{1,\infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \le C \|f\|_1 \qquad (f \in L_1[0,1))$$

(see Neveu [36] or Weisz [66]) and $H_1[0,1) \subset L_1[0,1)$.

A first version of the *atomic decomposition* was introduced by Coifman and Weiss [9] in the classical case and by Herz [29] in the martingale case. The proof of the next theorem can be found in Weisz [66].

A function $a \in L_\infty$ is called a *p-atom* if

(a) supp $a \subset I$, $I \subset [0,1)$ is a Vilenkin interval,

(b) $\|a\|_\infty \le |I|^{-1/p}$,

(c) $\int_I a(x)\,dx = 0$.

The basic result of atomic decomposition is the following one.

Theorem 1. *A martingale f is in $H_p[0,1)$ $(0 < p \le 1\})$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\begin{aligned} &\sum_{k=0}^{\infty} \mu_k a^k = f \quad \textit{in the sense of martingales}, \\ &\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \end{aligned} \tag{1}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}, \tag{2}$$

where the infimum is taken over all decompositions of f of the form (1).

If I is a dyadic interval then let $I^r = 2^r I$ be a dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r |I|$ $(r \in \mathbb{N})$. If I is a Vilenkin interval of length P_n^{-1} then let I^r be the Vilenkin interval which contains I and has length P_{n-r}^{-1} $(r \in \mathbb{N})$.

The following result gives a sufficient condition for V to be bounded from $H_p[0,1)$ to $L_p[0,1)$. For $p_0 = 1$ it can be found in Schipp, Wade, Simon and Pál [50] and in Móricz, Schipp and Wade [35], for $p_0 < 1$ see Weisz [71].

Theorem 2. *Suppose that*

$$\int_{[0,1)\setminus I^r} |Va|^{p_0}\, d\lambda \le C_{p_0}$$

for all p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$. If the sublinear operator V is bounded from $L_{p_1}[0,1)$ to $L_{p_1}[0,1)$ $(1 < p_1 \le \infty)$ then

$$\|Vf\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)) \tag{3}$$

for all $p_0 \le p \le p_1$*. Moreover, if* $p_0 < 1$ *then the operator* V *is of weak type* $(1,1)$*, i.e. if* $f \in L_1[0,1)$ *then*

$$\sup_{\rho>0} \rho\lambda(|Vf| > \rho) \le C\|f\|_1. \tag{4}$$

Note that (4) can be obtained from (3) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [2] and Bennett and Sharpley [1] or Weisz [66, 81]. The interpolation of martingale Hardy spaces was worked out in [66]. Theorem 2 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

We formulate also a weak version of this theorem.

Theorem 3. *Suppose that*

$$\sup_{\rho>0} \rho^p \ \lambda\Big(\{|Va| > \rho\} \cap \{[0,1) \setminus I^r\}\Big) \le C_p$$

for all p*-atoms* a *and for some fixed* $r \in \mathbb{N}$ *and* $0 < p < 1$*. If the sublinear operator* V *is bounded from* L_{p_1} *to* L_{p_1} $(1 < p_1 \le \infty)$*, then*

$$\|Vf\|_{p,\infty} \le C_p\|f\|_{H_p} \quad (f \in H_p[0,1)).$$

2.3. Partial sums of Fourier series

If $f \in L_1[0,1)$ then the number

$$\hat{f}(n) := \int_{[0,1)} f w_n \, d\lambda \qquad (n \in \mathbb{N})$$

is said to be the nth *Fourier coefficient* of f, where w_n denotes the Walsh, Walsh-Kaczmarz or Vilenkin system. We can extend this definition to martingales as well in the usual way (see Weisz [67]). Denote by $s_n f$ the nth *partial sum* of the Walsh-Fourier series of a martingale f, namely,

$$s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k.$$

It is known that $s_{P_n} f = f_n$ $(n \in \mathbb{N})$ and

$$s_{P_n} f \to f \qquad \text{in } L_p\text{−norm and a.e. as } n \to \infty,$$

if $f \in L_p[0,1)$ $(1 \le p < \infty)$.

Carleson's theorem was extended to Walsh-Fourier series by Billard [3] and Sjölin [59], to Walsh-Kaczmarz series by Young [83] and Schipp [45] and to Vilenkin-Fourier series by Gosselin [28] (see also Schipp [45, 47]):

$$s_n f \to f \qquad \text{a.e. as } n \to \infty,$$

whenever $f \in L_p[0,1)$ $(1 < p < \infty)$. If

$$s_* f := \sup_{n \in \mathbb{N}} |s_n f|$$

denotes the *maximal partial sum operator*, then

$$\|s_* f\|_p \le C_p \|f\|_p \qquad (f \in L_p[0,1),\ 1 < p < \infty).$$

This implies besides the a.e. convergence (5) also the L_p-norm convergence of $s_n f$ to f $(1 < p < \infty)$ (see Schipp [44], Simon [51]). These theorems do not hold, if $p = 1$, however, in the next section we generalize them for $p = 1$ with the help of some summability methods.

2.4. Cesàro-summability of one-dimensional Fourier series

The *Fejér* and *Cesàro* or (C, α) *means* of a martingale f are given by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} s_k f = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(k) w_k$$

and

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha \hat{f}(k) w_k,$$

respectively, where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}.$$

If $\alpha = 1$ then $\sigma_n^\alpha f = \sigma_n f$, and so the $(C, 1)$ means are the Fejér means.

The *maximal operator* of the Cesàro means are defined by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|.$$

The next result generalizes (6) for the maximal operator of the summability means (see Zygmund [84] and Paley [40]).

Theorem 4. *If* $0 < \alpha \leq 1$ *and* $1 < p \leq \infty$ *then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1)).$$

Moreover, for all $f \in L_p[0,1)$ $(1 < p < \infty)$*,*

$$\sigma_n^\alpha f \to f \qquad \textit{a.e. and in } L_p - \textit{norm as } n \to \infty.$$

The L_p-norm convergence holds also, if $p = 1$. Applying Theorems 2 and 3, we extended the previous result to $p < 1$ in [67, 80, 81, 58]:

Theorem 5. *If* $0 < \alpha \leq 1$ *and* $1/(\alpha+1) < p \leq \infty$ *then*

$$\|\sigma_*^\alpha f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1))$$

and for $f \in H_{1/(\alpha+1)}[0,1)$*,*

$$\|\sigma_*^\alpha f\|_{1/(\alpha+1),\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^\alpha f > \rho)^{\alpha+1} \leq C \|f\|_{H_{1/(\alpha+1)}}.$$

The first inequality was proved by Fujii [17] in the Walsh case (see also Schipp, Simon [48]), by Simon [52] for the Vilenkin system, in both cases for $\alpha = p = 1$, and by Simon [54, 55] for the Walsh-Kaczmarz system and all parameters.

The critical index is $p = 1/(\alpha+1)$, if p is smaller than or equal to this critical index, then σ_*^α is not bounded anymore (see Simon and Weisz [58], Simon [53] and Gát and Goginava [23]):

Theorem 6. *The operator* σ_*^α $(0 < \alpha \leq 1)$ *is not bounded from* $H_p[0,1)$ *to* $L_p[0,1)$ *if* $0 < p \leq 1/(\alpha+1)$*.*

We get the next weak type $(1,1)$ inequality from Theorem 5 by interpolation (Weisz [67, 80, 81], for $\alpha = 1$ Schipp [43] (Walsh), Gát [19] (Walsh-Kaczmarz system), Simon [52] (Vilenkin system)).

Corollary 1. *If* $0 < \alpha \le 1$ *and* $f \in L_1[0,1)$ *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1.$$

Since the set of the Walsh polynomials is dense in $L_1[0,1)$, Corollary 1 and the usual density argument (see Marcinkievicz, Zygmund [32]) imply

Corollary 2. *If* $0 < \alpha \le 1$ *and* $f \in L_1[0,1)$ *then*

$$\sigma_n^\alpha f \to f \qquad \text{a.e. as } n \to \infty.$$

Recall that this convergence result was proved first by Fine [14] for Walsh-Fourier series. With the help of the conjugate functions we ([73]) proved also

Theorem 7. *If* $0 < \alpha \le 1$ *and* $1/(\alpha+1) < p \le \infty$ *then*

$$\|\sigma_n^\alpha f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)).$$

Corollary 3. *If* $0 < \alpha \le 1$, $1/(\alpha+1) < p < \infty$ *and* $f \in H_p[0,1)$ *then*

$$\sigma_n^\alpha f \to f \qquad \textit{in } H_p - \textit{norm as } n \to \infty.$$

Note that for $\alpha > 1$ the results can be reduced to the $\alpha = 1$ case.

3. The dyadic and Vilenkin derivative

The one-dimensional differentiation theorem due to Lebesgue

$$f(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \qquad \text{a.e.} \qquad (f \in L_1[0,1))$$

is well known (see e.g. Zygmund [84]).

In this section the dyadic and Vilenkin analogue of this result will be formulated. Gibbs [25], Butzer and Wagner [5, 6] introduced the concept of the *dyadic derivative* as follows. For each function f defined on $[0,1)$ set

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \dot{+} 2^{-j-1})), \qquad (x \in [0,1)).$$

The generalization for Vilenkin analysis is due to Onneweer [37]:

$$(\mathbf{d}_n f)(x) := \sum_{j=0}^{n-1} P_j \sum_{k=0}^{p_j-1} k p_j^{-1} \sum_{l=0}^{p_j-1} r_j(l/P_{j+1})^{p_j-k} f(x \dot{+} l/P_{j+1}), \qquad (x \in [0,1)).$$

Then f is said to be *dyadically or Vilenkin differentiable* at $x \in [0,1)$ if $(\mathbf{d}_n f)(x)$ converges as $n \to \infty$. It was verified by Butzer and Wagner [6] and Onneweer [37] that every Walsh and Vilenkin function is differentiable and

$$\lim_{n\to\infty} (\mathbf{d}_n w_k)(x) = k w_k(x) \qquad (x \in [0,1), k \in \mathbb{N}).$$

Let W be the function whose Vilenkin-Fourier coefficients satisfy

$$\hat{W}(k) := \begin{cases} 1 & \text{if } k = 0, \\ 1/k & \text{if } k \in \mathbb{N}, k \neq 0. \end{cases}$$

The *dyadic integral* of $f \in L_1[0,1)$ is introduced by

$$\mathbf{I}f(x) := f * W(x) := \int_0^1 f(t) W(x \dot{-} t)\, dt.$$

Notice that $W \in L_2[0,1) \subset L_1[0,1)$, so $\mathbf{I}$ is well defined on $L_1[0,1)$.

Let the *maximal operator* be defined by

$$\mathbf{I}_* f := \sup_{n\in\mathbb{N}} |\mathbf{d}_n(\mathbf{I}f)|.$$

The boundedness of $\mathbf{I}_*$ from $L_p[0,1)$ to $L_p[0,1)$ $(1 < p \leq \infty)$ is due to Schipp [42] and Pál and Simon [38, 39]:

Theorem 8. *If* $1 < p < \infty$ *then*

$$\|\mathbf{I}_* f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1)).$$

Schipp and Simon [48] verified that $\mathbf{I}_*$ is bounded from $L\log L[0,1)$ to $L_1[0,1)$. Recall that $L\log L[0,1) \subset H_1[0,1)$. These results are extended to $H_p[0,1)$ spaces in the next theorem (see Weisz [74] and Simon and Weisz [57]).

Theorem 9. *Suppose that* $f \in H_p[0,1) \cap L_1[0,1)$ *and*

$$\int_0^1 f(x)\,dx = 0.$$

Then

$$\|\mathbf{I}_* f\|_p \le C_p \|f\|_{H_p}$$

for all $1/2 < p < \infty$.

We get by interpolation

Corollary 4. *If* $f \in L_1[0,1)$ *satisfies (7), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_* f > \rho) \le C\|f\|_1.$$

The dyadic analogue of the Lebesgue's differentiation theorem follows easily from the preceding weak type inequality:

Corollary 5. *If* $f \in L_1[0,1)$ *satisfies (7), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \quad a.e. \quad as \quad n \to \infty.$$

Corollaries 4 and 5 are due to Schipp [42] (see also Weisz [69]) and Pál and Simon [38, 39].

4. More-dimensional Fourier series

The analogue of the Carleson's theorem does not hold in higher dimensions. However, the summability results above can be generalized for the more-dimensional case. For multi-dimensional trigonometric Fourier series Zygmund [84] verified that if $f \in L(\log L)^{d-1}(\mathbb{T}^d)$ then the Cesàro means $\sigma_n^\alpha f$ converge to f a.e. and if $f \in L_p[0,1)^d$ $(1 \le p < \infty)$ then $\sigma_n^\alpha f \to f$ in $L_p[0,1)^d$ norm as $\min(n_1,\dots,n_d) \to \infty$. Moreover, if n must be in a cone then the a.e. convergence holds for all $f \in L_1(\mathbb{T}^d)$. More exactly, Marcinkievicz and Zygmund [32] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbb{T}^d)$ converge a.e. to f as $\min(n_1, \dots, n_d) \to \infty$ provided that n is in a positive cone, i.e. provided that $2^{-\tau} \le n_i/n_j \le 2^\tau$ for every $i, j = 1, \dots, d$ and for some $\tau \ge 0$ ($n = (n_1, \dots, n_d) \in \mathbb{N}^d$).

4.1. d-dimensional Hardy spaces

By a *Vilenkin rectangle* we mean a Cartesian product of d Vilenkin intervals. For $n \in \mathbb{N}^d$ and $x \in [0,1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$, where $n = (n_1, \dots, n_d)$ and $x = (x_1, \dots, x_d)$. The σ-algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0,1)^d\}$ will be denoted again by $\mathcal{F}_n$ ($n \in \mathbb{N}^d$).

For d-parameter martingales $f = (f_n, n \in \mathbb{N}^d)$ with respect to ($\mathcal{F}_n$, $n \in$ $\in \mathcal{N}^d$) we introduce three kinds of maximal functions and Hardy spaces. The *maximal functions* are defined by

$$f^\diamond := \sup_{n \in \mathbb{N}} |f_{\mathbf{n}}|, \qquad f^* := \sup_{n \in \mathbb{N}^d} |f_n|,$$

where $\mathbf{n} := (n, \dots, n) \in \mathbb{N}^d$ for $n \in \mathbb{N}$. In the first maximal function we have taken the supremum over the diagonal, in the second one over $\mathbb{N}^d$. Let E_n denote the conditional expectation operator with respect to $\mathcal{F}_n$. Obviously, if $f \in L_1[0,1)^d$ then $(E_n f, n \in \mathbb{N}^d)$ is a martingale. In the third maximal function the supremum is taken over $d-1$ indices: for fixed x_i we define

$$f^i(x) := \sup_{n_k \in \mathbb{N}, k=1,\dots,d; k \ne i} |E_{n_1} \dots E_{n_{i-1}} E_{n_{i+1}} \dots E_{n_d} f(x)|.$$

For $0 < p \le \infty$ the *martingale Hardy spaces* $H_p^\diamond[0,1)^d$, $H_p[0,1)^d$ and $H_p^i[0,1)^d$ consists of all d-parameter martingales for which

$$\|f\|_{H_p^\diamond} := \|f^\diamond\|_p < \infty, \qquad \|f\|_{H_p} := \|f^*\|_p < \infty, \qquad \|f\|_{H_p^i} := \|f^i\|_p < \infty,$$

respectively. One can show (see Weisz [66]) that $L(\log L)^{d-1}[0,1)^d \subset$ $\subset H_1^i[0,1)^d \subset H_{1,\infty}[0,1)^d$ ($i = 1, \dots, d$), more exactly,

$$\|f\|_{H_{1,\infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \le C \|f\|_{H_1^i} \qquad (f \in H_1^i[0,1)^d)$$

and

$$\|f\|_{H_1^i} \le C + C \| |f| (\log^+ |f|)^{d-1} \|_1 \qquad (f \in L(\log L)^{d-1}[0,1)^d),$$

where $\log^+ u = 1_{\{u>1\}} \log u$. Moreover, it is known that

$$H_p^\diamond[0,1)^d \sim H_p[0,1)^d \sim H_p^i[0,1)^d \sim L_p[0,1)^d \qquad (1<p<\infty).$$

4.1.1. The Hardy spaces $H_p^\diamond[0,1)^d$

To obtain some convergence results of the summability means over the diagonal we consider the Hardy space $H_p^\diamond[0,1)^d$. Now the situation is similar to the one-dimensional case.

A function $a \in L_\infty[0,1)^d$ is a *cube p-atom* if

(a) supp $a \subset I$, $I \subset [0,1)^d$ is a Vilenkin cube,

(b) $\|a\|_\infty \le |I|^{-1/p}$,

(c) $\int_I a(x)\,dx = 0$.

The basic result of atomic decomposition is the following one (see Weisz [66, 81]).

Theorem 10. *A d-parameter martingale f is in $H_p^\diamond[0,1)^d$ $(0<p\le 1)$ if and only if there exist a sequence $(a^k, k\in\mathbb{N})$ of cube p-atoms and a sequence $(\mu_k, k\in\mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^\infty \mu_k a^k = f \quad \textit{in the sense of martingales},$$
$$\sum_{k=0}^\infty |\mu_k|^p < \infty. \tag{8}$$

Moreover,

$$\|f\|_{H_p^\diamond} \sim \inf \left(\sum_{k=0}^\infty |\mu_k|^p\right)^{1/p}, \tag{9}$$

where the infimum is taken over all decompositions of f of the form (8).

For a rectangle $R = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$ let $R^r := I_1^r \times \ldots \times I_d^r$ $(r\in\mathbb{N})$. The following result generalizes Theorem 2.

Theorem 11. *Suppose that*

$$\int_{[0,1)^d \setminus I^r} |Va|^{p_0}\,d\lambda \le C_{p_0}$$

for all cube p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$. If the sublinear operator V is bounded from $L_{p_1}[0,1)^d$ to $L_{p_1}[0,1)^d$ $(1 < p_1 \le \infty)$ then

$$\|Vf\|_p \le C_p \|f\|_{H_p^\diamond} \qquad (f \in H_p^\diamond[0,1)^d) \tag{10}$$

for all $p_0 \le p \le p_1$. Moreover, if $p_0 < 1$ then the operator V is of weak type $(1,1)$, i.e. if $f \in L_1[0,1)^d$ then

$$\sup_{\rho>0} \rho\lambda(|Vf| > \rho) \le C\|f\|_1. \tag{11}$$

4.1.2. The Hardy spaces $H_p[0,1)^d$

In the investigation of the convergence in the Prighheim's sense (i.e. over all n) we use the Hardy spaces $H_p[0,1)^d$. The atomic decomposition for $H_p[0,1)^d$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2[0,1)^d$ instead of $L_\infty[0,1)^d$. This atomic decomposition was proved by Chang and Fefferman [8, 12] and Weisz [77, 81]. For an open set $F \subset [0,1)^d$ denote by $\mathcal{M}(F)$ the maximal Vilenkin subrectangles of F.

A function $a \in L_2[0,1)^d$ is a *p-atom* if

(a) supp $a \subset F$ for some open set $F \subset [0,1)^d$,

(b) $\|a\|_2 \le |F|^{1/2-1/p}$,

(c) a can be further decomposed into the sum of "elementary particles" $a_R \in$ $\in L_2$, $a = \sum_{R\in\mathcal{M}(F)} a_R$ in L_2, satisfying

(d) supp $a_R \subset R \subset F$,

(e) for all $i = 1, \dots, d$ and $R \in \mathcal{M}(F)$ we have

$$\int_{[0,1)} a_R(x)\, dx_i = 0,$$

(f) for every disjoint partition $\mathcal{P}_l$ $(l = 1, 2, \dots)$ of $\mathcal{M}(F)$,

$$\left(\sum_l \left\| \sum_{R\in\mathcal{P}_l} a_R \right\|_2^2 \right)^{1/2} \le |F|^{1/2-1/p}.$$

Theorem 12. *A d-parameter martingale f is in $H_p[0,1)^d$ $(0 < p \le 1)$ if and only if there exist a sequence $(a^k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a^k = f \quad \text{in the sense of martingales,}$$
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{12}$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (12).

The corresponding results to Theorems 2 and 11 for the $H_p[0,1)^d$ space are much more complicated. Since the definition of the p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms. A function $a \in L_2[0,1)^d$ is called a *simple p-atom*, if there exist Vilenkin intervals $I_i \subset [0,1)$, $i = 1, \ldots, j$ for some $1 \le j \le d-1$ such that

(a) supp $a \subset I_1 \times \ldots I_j \times A$ for some measurable set $A \subset [0,1)^{d-j}$,
(b) $\|a\|_2 \le (|I_1| \cdots |I_j||A|)^{1/2-1/p}$,
(c) $\int_{I_i} a(x) x_i \, dx_i = \int_A a \, d\lambda = 0$ for $i = 1, \ldots, j$.

Of course if $a \in L_2[0,1)^d$ satisfies these conditions for another subset of $\{1, \ldots, d\}$ than $\{1, \ldots, j\}$, then it is also called simple p-atom.

Note that $H_p[0,1)^d$ cannot be decomposed into simple p-atoms, a counterexample can be found in Weisz [66]. However, the following result, which is due to the author [77, 81], says that for an operator V to be bounded from $H_p[0,1)^d$ to $L_p[0,1)^d$ $(0 < p \le 1)$ it is enough to check V on simple p-atoms and the boundedness of V on $L_2[0,1)^d$. Let H^c denote the complement of the set H.

Theorem 13. *Suppose that the operators V_n are linear for every $n \in \mathbb{N}^d$ and*

$$V := \sup_{n \in \mathbb{N}^d} |V_n|$$

is bounded on $L_2[0,1)^d$*. Suppose that there exist* $\eta_1,\ldots,\eta_d>0$*, such that for all simple* p_0*-atoms* a *and for all* $r_1,\ldots,r_d\geq 1$

$$\int_{(I_1^{r_1})^c\times\ldots\times(I_j^{r_j})^c}\int_A |Va|^{p_0}\,d\lambda \leq C_{p_0}2^{-\eta_1 r_1}\cdots 2^{-\eta_j r_j}.$$

If $j=d-1$ *and* $A=I_d\subset[0,1)$ *is a Vilenkin interval, then we assume also that*

$$\int_{(I_1^{r_1})^c\times\ldots\times(I_{d-1}^{r_{d-1}})^c}\int_{(I_d)^c} |Va|^{p_0}\,d\lambda \leq C_{p_0}2^{-\eta_1 r_1}\cdots 2^{-\eta_{d-1} r_{d-1}}.$$

Then

$$\|Vf\|_p\leq C_p\|f\|_{H_p}\qquad (f\in H_p[0,1)^d)$$

for all $p_0\leq p\leq 2$*. In particular, if* $p_0<1$ *and* $f\in H_1^i[0,1)^d$ *for some* $i=1,\ldots,d$ *then*

$$\sup_{\rho>0}\rho\lambda(|Vf|>\rho)\leq C\|f\|_{H_1^i}. \tag{13}$$

In some sense the space $H_1^i[0,1)^d$ plays the role of the one-dimensional $L_1[0,1)$ space.

4.2. Partial sums of more-dimensional Fourier series

The Kronecker product $(w_n, n\in\mathbb{N}^d)$ of d Walsh-, Walsh-Kaczmarz- or Vilenkin systems is said to be a *d-dimensional system.* Thus

$$w_n(x):=w_{n_1}(x_1)\cdots w_{n_d}(x_d),$$

where $n=(n_1,\ldots,n_d)\in\mathbb{N}^d$, $x=(x_1,\ldots,x_d)\in[0,1)^d$. For Vilenkin systems the sequences $(p_n^{(j)}, n\in\mathbb{N})$ can be different, but bounded sequences.

The nth Fourier coefficient of $f\in L_1[0,1)^d$ is introduced by

$$\hat{f}(n):=\int_{[0,1)^d} f w_n\,d\lambda\qquad (n\in\mathbb{N}^d).$$

With the usual extension of Fourier coefficients to martingales we can define the nth partial sum of the Walsh-Fourier series of a martingale f by

$$s_nf:=\sum_{j=1}^{d}\sum_{k_j=0}^{n_j-1}\hat{f}(k)w_k\qquad (n\in\mathbb{N}^d).$$

Under $\sum_{j=1}^{d}\sum_{k_j=0}^{n_j-1}$ we mean the sum $\sum_{k_1=0}^{n_1-1}\dots\sum_{k_d=0}^{n_d-1}$.

It is known that $s_{P_{n_1}^{(1)},\dots,P_{n_d}^{(d)}}f=f_n$ $(n\in\mathbb{N}^d)$ and

$$s_{P_{n_1}^{(1)},\dots,P_{n_d}^{(d)}}f\to f \qquad \text{in } L_p\text{−norm as } n\to\infty,$$

if $f\in L_p[0,1)^d$ $(1\le p<\infty)$. If $p>1$ then the convergence holds also a.e. Moreover,

$$s_nf\to f \qquad \text{in } L_p\text{−norm as } n\to\infty,$$

whenever $f\in L_p[0,1)^d$ $(1<p<\infty)$ (see e.g. Schipp, Wade, Simon and Pál [50]). The a.e. convergence of s_nf is not true (Fefferman [10, 11]). However, investigating the partial sums over the diagonal, only, we have the following results (Móricz [33] or Schipp, Wade, Simon and Pál [50]):

$$\left\|\sup_{n\in\mathbb{N}}|s_{\mathbf{n}}f|\right\|_2\le C\|f\|_2 \qquad (f\in L_2[0,1)^d)$$

and for $f\in L_2[0,1)^d$

$$s_{\mathbf{n}}f\to f \qquad \text{a.e. as } n\to\infty \quad (n\in\mathbb{N}). \tag{14}$$

In contrary to the trigonometric case, it is unknown whether this result holds for functions in $L_p[0,1)^d$, $1<p<2$.

4.3. Summability of d-dimensional Fourier series

The *Fejér* and *Cesàro means* of a martingale f are defined by

$$\sigma_nf:=\frac{1}{\prod_{i=1}^d n_i}\sum_{j=1}^{d}\sum_{k_j=1}^{n_j}s_kf=\sum_{j=1}^{d}\sum_{k_j=0}^{n_j-1}\prod_{i=1}^{d}\Big(1-\frac{k_i}{n_i}\Big)\hat f(k)w_k,$$

and

$$\sigma_n^\alpha f:=\frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}}\sum_{j=1}^{d}\sum_{k_j=1}^{n_j}A_{n_j-k_j}^{\alpha_j-1}s_kf=$$

$$=\frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}}\sum_{j=1}^{d}\sum_{k_j=0}^{n_j-1}\left(\prod_{i=1}^{d}A_{n_i-k_i-1}^{\alpha_i}\right)\hat f(k)w_k,$$

respectively. We define a cone by

$$\mathbb{N}^d_\tau := \{n \in \mathbb{N}^d : 2^{-\tau} \le n_i/n_j \le 2^\tau, i,j = 1,\ldots,d\}.$$

For a given $\tau \ge 0$ the *restricted and non-restricted maximal operators* are defined by

$$\sigma^\alpha_\diamond f := \sup_{n \in \mathbb{N}^d_\tau} |\sigma^\alpha_n f|, \qquad \sigma^\alpha_* f := \sup_{n \in \mathbb{N}^d} |\sigma^\alpha_n f|.$$

The next result follows easily from Theorem 4 by iteration.

Theorem 14. *If* $0 < \alpha_j \le 1$ $(j = 1,\ldots,d)$ *and* $1 < p \le \infty$ *then*

$$\|\sigma^\alpha_* f\|_p \le C_p \|f\|_p \qquad (f \in L_p[0,1)^d).$$

Moreover, for all $f \in L_p[0,1)^d$ $(1 < p < \infty)$,

$$\sigma^\alpha_n f \to f \qquad \text{a.e. and in } L_p - \textit{norm as } n \to \infty.$$

The L_p-norm convergence holds also, if $p = 1$. Here $n \to \infty$ means that $\min(n_1,\ldots,n_d) \to \infty$ (the Pringsheim's sense of convergence).

4.3.1. Restricted summability

In this and the next subsections the results are true for Walsh- and Vilenkin series as they are formulated, in case of Walsh-Kaczmarz series only for Fejér means, i.e. for $\alpha_j = 1$ $(j = 1,\ldots,d)$. Here we investigate the operator $\sigma^\alpha_\diamond$ and the convergence of $\sigma^\alpha_n f$ over the cone $\mathbb{N}^d_\tau$, where $\tau \ge 0$ is fixed.

Theorem 15. *If* $0 < \alpha_j \le 1$ $(j = 1,\ldots,d)$ *and*

$$p_0 := \max\{1/(\alpha_j + 1), \quad j = 1,\ldots,d\} < p < \infty,$$

then

$$\|\sigma^\alpha_\diamond f\|_p \le C_p \|f\|_{H^\diamond_p} \qquad (f \in H^\diamond_p[0,1)^d).$$

This theorem for Walsh and Vilenkin systems can be found in Weisz [76, 81, 82] and for Walsh-Kaczmarz systems in Simon [53].

For the Fejér means (i.e. $\alpha_j = 1$, $j = 1,\ldots,d$) there are counterexamples for the boundedness of $\sigma^\alpha_\diamond$ if $p \le p_0 = 1/2$ (Goginava and Nagy [26, 27]).

Theorem 16. *The operator* $\sigma^1_\diamond$ $(\alpha_j = 1, j = 1,\ldots,d)$ *is not bounded from* $H^\diamond_p[0,1)^d$ *to* $L_p[0,1)^d$ *if* $0 < p \le 1/2$.

By interpolation we obtain ([76])

Corollary 6. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$ *and* $f \in L_1[0,1)^d$ *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_\diamond^\alpha f > \rho) \leq C\|f\|_1.$$

The set of the Walsh polynomials is dense in $L_1[0,1)^d$, so Corollary 6 implies the Walsh analogue of the Marcinkiewicz-Zygmund result.

Corollary 7. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$ *and* $f \in L_1[0,1)^d$ *then*

$$\sigma_n^\alpha f \to f \qquad \text{a.e. as } n \to \infty \text{ and } n \in \mathbb{N}_\tau^d.$$

Note that this corollary is due to the author [68, 76, 82] for Walsh and Vilenkin systems, to Simon [53] for Walsh-Kaczmarz systems. For Fejér means of two-dimensional Walsh-Fourier series it can also be found in Gát [18] (see also Móricz, Schipp and Wade [35]).

The following results are known ([76]) for the norm convergence of $\sigma_n f$.

Theorem 17. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$ *and* $p_0 < p < \infty$*, then*

$$\|\sigma_n^\alpha f\|_{H_p^\diamond} \leq C_p\|f\|_{H_p^\diamond} \qquad (f \in H_p^\diamond[0,1)^d)$$

whenever $n \in \mathbb{N}_\tau^d$.

Corollary 8. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$, $p_0 < p < \infty$ *and* $f \in H_p^\diamond$ *then*

$$\sigma_n^\alpha f \to f \qquad \text{in } H_p^\diamond-\text{norm as } n \to \infty \text{ and } n \in \mathbb{N}_\tau^d.$$

4.3.2. Unrestricted summability

Now we deal with the operator σ_*^α and the convergence of $\sigma_n^\alpha f$ as $n \to \infty$, i.e. $\min(n_1, \ldots, n_d) \to \infty$. The next result is due to the author ([72, 78, 77, 82]) for Walsh and Vilenkin systems and to Simon [53] for Walsh-Kaczmarz systems.

Theorem 18. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$ *and*

$$p_0 := \max\{1/(\alpha_j + 1), \quad j = 1, \ldots, d\} < p < \infty,$$

then

$$\|\sigma_*^\alpha f\|_p \leq C_p\|f\|_{H_p} \qquad (f \in H_p[0,1)^d).$$

Theorem 19. (Goginava [26]) *The operator* σ_*^1 $(\alpha_j = 1,\ j = 1, \ldots, d)$ *is not bounded from* $H_p[0,1)^d$ *to* $L_p[0,1)^d$ *if* $0 < p \leq 1/2$.

By interpolation we get here a.e. convergence for functions from the spaces $H_1^i[0,1)^d$ instead of $L_1[0,1)^d$.

Corollary 9. *If* $0 < \alpha_j \leq 1$ *and* $f \in H_1^i[0,1)^d$ $(i, j = 1, \ldots, d)$ *then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_*^\alpha f > \rho) \leq C \|f\|_{H_1^i}.$$

Recall that $H_1^i[0,1)^d \supset L(\log L)^{d-1}[0,1)^d$ for all $i = 1, \ldots, d$.

Corollary 10. *If* $0 < \alpha_j \leq 1$ *and* $f \in H_1^i[0,1)^d$ $(i, j = 1, \ldots, d)$ *then*

$$\sigma_n^\alpha f \to f \qquad \text{a.e. as } n \to \infty.$$

For the $L(\log L)[0,1)^2$ space and Walsh system see also Móricz, Schipp and Wade [35]. Gát [21, 22] proved for the Fejér means that this corollary does not hold for all integrable functions.

Theorem 20. *The a.e. convergence is not true for all* $f \in L_1[0,1)^d$.

Theorem 21. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$ *and* $p_0 < p < \infty$, *then*

$$\|\sigma_n^\alpha f\|_{H_p} \leq C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d, \quad n \in \mathbb{N}^d).$$

Corollary 11. *If* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$, $p_0 < p < \infty$ *and* $f \in H_p$ *then*

$$\sigma_n^\alpha f \to f \qquad \text{in } H_p - \text{norm as } n \to \infty\ .$$

5. More-dimensional dyadic and Vilenkin derivative

The multi-dimensional version of Lebesgue's differentiation theorem reads as follows:

$$f(x) = \lim_{h \to 0} \frac{1}{\prod_{j=1}^{d} h_j} \int_{x_1}^{x_1+h_1} \ldots \int_{x_d}^{x_d+h_d} f(t)\, dt \qquad \text{a.e.,}$$

if $f \in L(\log L)^{d-1}[0,1)^d$. If $\tau^{-1} \le |h_i/h_j| \le \tau$ for some fixed $\tau \ge 0$ and all $i,j = 1,\dots,d$, then it holds for all $f \in L_1[0,1)^d$ (see Zygmund [84]). To present the dyadic version of this result we introduce first the *multi-dimensional dyadic derivative* ([4]) by the limit of

$$(\mathbf{d}_n f)(x) :=$$

$$:= \sum_{i=1}^{d} \sum_{j_i=0}^{n_i-1} 2^{j_1+\ldots+j_d-d} \sum_{\epsilon_i=0}^{1} (-1)^{\epsilon_1+\ldots+\epsilon_d} f(x_1 \dot{+} \epsilon_1 2^{-j_1-1}, \ldots, x_d \dot{+} \epsilon_d 2^{-j_d-1}).$$

For simplicity we suppose that the sequences $(p_n^{(j)},\ n \in \mathbb{N})$ are all the same. The *multi-dimensional Vilenkin derivative* is defined by

$$(\mathbf{d}_n f)(x) := \sum_{i=1}^{d} \sum_{j_i=0}^{n_i-1} \Big(\prod_{i=1}^{d} P_{j_i} \Big) \sum_{k_i=0}^{p_{j_i}-1} \Big(\prod_{i=1}^{d} k_i/p_{j_i} \Big) \times$$

$$\times \sum_{l_i=0}^{p_j-1} \Big(\prod_{i=1}^{d} r_{j_i}(l_i/P_{j_i+1})^{p_{j_i}-k_i} f(x \dot{+} l_i/P_{j_i+1}) \Big).$$

The *d-dimensional integral* is defined by

$$\mathbf{I}f(x) := f * (W \times \ldots \times W)(x) = \int_0^1 \ldots \int_0^1 f(t) W(x_1 \dot{-} t_1) \cdots W(x_d \dot{-} t_d)\, dt$$

and for given $\tau \ge 0$ let the *maximal operators* be

$$\mathbf{I}_\diamond f := \sup_{|n_i - n_j| \le \tau, i,j=1,\ldots,d} |\mathbf{d}_n(\mathbf{I}f)|, \qquad \mathbf{I}_* f := \sup_{n \in \mathbb{N}^d} |\mathbf{d}_n(\mathbf{I}f)|.$$

Theorem 22. *Suppose that $f \in H_p^\diamond[0,1)^d \cap L_1[0,1)^d$ and*

$$\int_0^1 f(x)\, dx_i = 0 \quad (i = 1,\ldots,d). \tag{15}$$

Then

$$\|\mathbf{I}_\diamond f\|_p \le C_p \|f\|_{H_p^\diamond}$$

for all $d/(d+1) < p < \infty$.

Corollary 12. *If* $f \in L_1[0,1)^d$ *satisfies (15), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_\diamond f > \rho) \le C\|f\|_1.$$

Corollary 13. *If* $\tau \ge 0$ *is arbitrary and* $f \in L_1[0,1)^d$ *satisfies (15), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \quad a.e., \ as \ n \to \infty \ \ and \ |n_i - n_j| \le \tau.$$

Theorem 22 and Corollaries 12 and 13 are due to the author [69, 81, 56]. The two corollaries were also shown by Gát [20] and Gát and Nagy [24].

For the operator $\mathbf{I}_*$ the following results were verified in Weisz [70, 79, 81].

Theorem 23. *If (15) is satisfied and* $1/2 < p < \infty$ *then*

$$\|\mathbf{I}_* f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p[0,1)^d).$$

Corollary 14. *If* $f \in H_1^i[0,1)^d$ $(i = 1,\ldots,d)$ *satisfies (15), then*

$$\sup_{\rho>0} \rho\, \lambda(\mathbf{I}_* f > \rho) \le C\|f\|_{H_1^i}.$$

Corollary 15. *If* $f \in H_1^i[0,1)^d (\supset L(\log L)^{d-1}[0,1)^d)$ $(i = 1,\ldots,d)$ *satisfies (15), then*

$$\mathbf{d}_n(\mathbf{I}f) \to f \qquad a.e., \ as \quad n \to \infty.$$

Note that this result for $f \in L\log L$ is due to Schipp and Wade [49] in the two-dimensional case.

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