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THE VOICE TRANSFORM GENERATED BY A REPRESENTATION OF THE BLASCHKE GROUP ON THE WEIGHTED BERGMAN SPACES

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Dedicated to Professor Ferenc Schipp on his 70th birthday and to Professor Péter Simon on his 60th birthday

Abstract. In this paper are presented results connected to the voice transform of the Blaschke group generated by a representation of the group on the weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$. These results are generalizations of the results obtained in [15] which are in connection to the Bergman space. Sections 1 and 2 contain the basic notations, definitions and results. In Section 3 we give a representation of Blaschke group on the weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$, we compute the matrix elements of the representation. It is proved that the representation is irreducible on $\mathcal{H}^m(\mathbb{D})$. Using the representation U_a we construct a rational orthonormal wavelet system and we prove that the weighted Bergman projection operator can be expressed using the voice transform and the wavelet system. The analogue of the Plancherel formula is proved and the square integrability of the representation is studied. Sections 4 contains the proofs.

1. The voice transform

In this section are included the basic notations, definitions and results connected to the general theory of the voice transform (see [5], [6], [7], [8], [10], [16]).

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In signal processing and image reconstruction the wavelet and Gábor transforms play an important role. H. Feichtinger and K. Gröchening unified the Gábor and wavelet transforms into a single theory. The common generalization of these transforms is the so-called *voice-transform* (see [5], [6], [7], [8]).

In the construction of the voice transform the starting point will be a representation of a locally compact topological group $(G, \cdot)$ on the Hilbert space H. It is known that every locally compact topological group has nontrivial left- and right-translation invariant Borel measures, called left invariant and right invariant Haar measures. Let m be a left-invariant Haar measure of G. Let $f : G \to \mathbb{C}$ be a Borel measurable function which is integrable with respect to the left invariant Haar measure m, the integral of f will be denoted by $\int\limits_G f\,dm = \int\limits_G f(x)\,dm(x)$. Because of left-translation invariance of the measure m it follows that

$$\int\limits_G f(x)\,dm(x) = \int\limits_G f(a^{-1}\cdot x)\,dm(x) \quad (a \in G).$$

There exist groups whose left invariant Haar measure is not right invariant. If the left invariant Haar measure of G is in the same time right invariant then we say that G is *unimodular*. Such measure will be called Haar measure of G. On a given group, Haar measure is unique only up to constant multiples. It is trivial that the commutative groups are unimodular. Furthermore it can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is unimodular.

In the definition of the voice transform a *unitary representation of the group* $(G, \cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group the neutral element of which is I, the identity operator on H and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1}, which is equal to the adjoint of U : $U^{-1} = U^*$. The homomorphism U of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$i)\ \ U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G),$$

$$ii)\ \ G \ni x \to U_x f \in H \ \text{ is continuous for all } f \in H \tag{1.1}$$

is called a unitary representation of $(G, \cdot)$ on H.

The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x \rho \rangle \quad (x \in G, f, \rho \in H). \tag{1.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G.

The set of continuous bounded functions defined on the group G with the supremum norm form a Banach space and $V_\rho : H \to C(G)$ is a bounded linear operator. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$

$$|(V_\rho f)(x)| = |\langle f, U_x \rho \rangle| \leq \|f\| \|U_x \rho\| = \|f\| \|\rho\|,$$

consequently $\|V_\rho\| \leq \|\rho\|$.

The invertibility of V_ρ is connected to the irreducibility of the representation U.

A representation U is called *irreducible* if the only closed invariant subspaces of H, i.e. closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$, are $\{0\}$ and H. Since the closure of the linear span of the set

$$\{U_x \rho : x \in G\} \tag{1.3}$$

is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection (1.3) is a closed system for any $\rho \in H, \rho \neq 0$.

The property of irreducibility gives a simple criterion for deciding when a voice transform is one to one:

Theorem 1 [10], [16]. *A voice transform V_ρ generated by a unitary representation U is one to one for all $\rho \in H \setminus \{0\}$ if and only if U is irreducible.*

The function $V_\rho f$ is continuous on G, but in general is not square integrable. If there exists $\rho \in H, \rho \neq 0$ such that $V_\rho \rho \in L^2_m(G)$, then the representation U is called *square integrable* and ρ is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by H^*. Choosing a convenient $\rho \in H^*$ the voice transform $V_\rho : H \to L^2_m(G)$ will be unitary. This is a consequence of the following theorem:

Theorem 2 [10], [16]. *Let be $U_x, (x \in G)$ an irreducible square integrable representation of G in H. Then the collection of admissible elements H^* is a linear subspace of H and for every $\rho \in H^*$ the voice transform of the function f is square integrable on G, namely $V_\rho f \in L^2_m(G)$, if $f \in H$. Moreover there is a symmetric, positive bilinear map $B : H^* \times H^* \to \mathbb{R}$ such that*

$$[V_{\rho_1} f, V_{\rho_2} g] = B(\rho_1, \rho_2) \langle f, g \rangle \quad (\rho_1, \rho_2 \in H^*, f, g \in H)$$

for all $f, g \in H$ *and* $\rho_1, \rho_2 \in H^*$*, where the inner product* $[\cdot,\cdot]$ *is the usual inner product in* $L^2_m(G)$*. If the group* G *is unimodular then* $B(\rho,\rho) = c\|\rho\|^2$ $(\rho \in H^*)$*, where* $c > 0$ *is a constant. In this case if we choose* ρ *so that* $\langle \rho,\rho\rangle = 1/c$*, then*

$$[V_\rho f, V_\rho g] = \langle f, g\rangle \quad (f, g \in H).$$

In the next sections we will construct a voice transform using so called multiplier representations generated by a collection of multiplier functions defined in the following way: $F_a : G \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ $(a \in G)$ is a *collection of multiplier functions* if

$$F_e = 1, \quad F_{a_1 \cdot a_2}(x) = F_{a_1}(a_2 \cdot x) F_{a_2}(x) \quad (a_1, a_2, x \in G),$$

where e is the neutral element of G. It can be proved that

$$(U_a f)(x) := F_{a^{-1}}(x) f(a^{-1} \cdot x) \quad (a, x \in G)$$

satisfies

$$U_{a_1} \circ U_{a_2} = U_{a_1 \cdot a_2} \quad (a_1, a_2 \in G),$$

so is a representation of G on the space of all complex valued functions on G.

If F_a is continuous and bounded for every $a \in G$, then $L^2_m(G)$ is an invariant subspace and $U_a,\ a \in G$ is a representation on $L^2_m(G)$. The representations obtained as below are named *multiplier representations* (see [18]).

Taking as starting point (not necessarily commutative) locally compact groups we can construct in this way important transformations in signal processing and control theory. For example the affine wavelet transform and the Gábor transform are all special voice transforms (see [5], [6], [7], [8], [10], [16]).

In this paper results connected to the voice transform generated by a representation of the Blaschke group on the weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$ are presented. These results are generalizations of the results obtained in [15] which are in connection to the Bergman space.

2. The Blaschke group and the weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$

2.1. The Blaschke group

The affine wavelet transform is a voice transform of the affine group which is a subgroup of the Möbius group (i.e. the group of linear fractional transformations with the composition operation). In this section we will present another subgroup of the Möbius group, namely the Blaschke group (see [3], [9], [16]).

Let us denote by

$$B_a(z) := \epsilon\frac{z-b}{1-\bar{b}z} \quad (z\in\mathbb{C},\, a=(b,\epsilon)\in\mathbb{B}:=\mathbb{D}\times\mathbb{T},\, \bar{b}z\neq 1) \tag{2.1}$$

the so called *Blaschke functions*, where

$$\mathbb{D}_+ := \mathbb{D} := \{z\in\mathbb{C} : |z|<1\},\ \ \mathbb{T} := \{z\in\mathbb{C} : |z|=1\},\ \ \mathbb{D}_- := \{z\in\mathbb{C} : |z|>1\}. \tag{2.2}$$

If $a\in\mathbb{B}$, then B_a is a 1-1 map on $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{D}_-$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1}\circ B_{a_2})(z) := := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B}$ let us define the operation induced by the function composition in the following way: $B_{a_1}\circ B_{a_2} = = B_{a_1\circ a_2}$. The group $(\mathbb{B},\circ)$ will be isomorphic with the group $(\{B_a, a\in\mathbb{B}\},\circ)$. If we use the notations $a_j := (b_j,\epsilon_j)$, $j\in\{1,2\}$ and $a := (b,\epsilon) =: a_1\circ a_2$, then

$$b = \frac{b_1\bar{\epsilon}_2 + b_2}{1+b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2,1)}(b_1\bar{\epsilon}_2), \quad \epsilon = \epsilon_1\frac{\epsilon_2 + b_1\bar{b}_2}{1+\epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\epsilon_1)}(\epsilon_2). \tag{2.3}$$

The neutral element of the group $(\mathbb{B},\circ)$ is $e := (0,1)\in\mathbb{B}$ and the inverse element of $a=(b,\epsilon)\in\mathbb{B}$ is $a^{-1} = (-b\epsilon,\bar{\epsilon})$.

The integral of the function $f:\mathbb{B}\to\mathbb{C}$, with respect to the left invariant Haar-measure m of the group $(\mathbb{B},\circ)$, is given by

$$\int_{\mathbb{B}} f(a)\,dm(a) = \frac{1}{2\pi}\int_{\mathbb{I}}\int_{\mathbb{D}} \frac{f(b,e^{it})}{(1-|b|^2)^2}\,db_1 db_2 dt, \tag{2.4}$$

where $a=(b,e^{it}) = (b_1+ib_2, e^{it})\in\mathbb{D}\times\mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $a\to a_0\circ a$ and under the inverse transformation $a\to a^{-1}$, so this group is unimodular.

We will study the voice transform of the Blaschke group. In the construction a unitary representation of the Blaschke group on the weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$ will be used.

2.2. The weighted Bergman spaces $\mathcal{H}^m(\mathbb{D})$

In this section we summarize the basic results connected to the weighted Bergman spaces (see [3], [9]). Let us denote by $\mathcal{A}$ the set of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic in $\mathbb{D}$. Let us denote by

$$dA_m(z) := \frac{m-1}{\pi}(1-|z|^2)^{m-2}\, dxdy, \quad z = x+iy$$

the weighted area measure on $\mathbb{D}$. For all $m \in \mathbb{N}$, $m \geq 2$ let us consider the following subset of analytic functions:

$$\mathcal{H}^m(\mathbb{D}) := \left\{ f \in \mathcal{A} : \int_{\mathbb{D}} |f(z)|^2 dA_m(z) < \infty \right\}. \tag{2.5}$$

The set $\mathcal{H}^m(\mathbb{D})$ is the weighted Bergman space. This space with the scalar product

$$\langle f, g \rangle_m := \int_{\mathbb{D}} f(z)\overline{g(z)} dA_m(z) \tag{2.6}$$

is a Hilbert space. In the special case when $m = 2$, $\mathcal{H}^2(\mathbb{D})$ is the so called *Bergman space* (see [3], [5]). It can be proved that the function

$$f(z) := \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathbb{D}) \tag{2.7}$$

from $\mathcal{A}$ belongs to the set $\mathcal{H}^m(\mathbb{D})$ if and only if the coefficients satisfy

$$\sum_{n=0}^{\infty} |c_n|^2 \lambda_n^{[m]} < \infty,$$

where

$$\lambda_n^{[m]} := \int_0^1 (1-r^2)^{m-2} r^{2n+1}\, dr \quad (m \geq 2, n \in \mathbb{N}).$$

The weighted Bergman space $\mathcal{H}^m(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA_m)$. For each $z \in \mathbb{D}$ the point-evaluation map

$$\tau_z : \mathcal{H}^m(\mathbb{D}) \to \mathbb{C}, \quad \tau_z(f) = f(z)$$

is a bounded linear functional on $\mathcal{H}^m(\mathbb{D})$. Each function $f \in \mathcal{H}^m(\mathbb{D})$ has the property

$$|f(z)| \leq C\|f\|_{\mathcal{H}^m(\mathbb{D})} \quad (z \in \mathbb{D}).$$

From this inequality it follows that the norm convergence in $\mathcal{H}^m(\mathbb{D})$ implies the locally uniform convergence on $\mathbb{D}$. Therefore, by the Riesz Representation Theorem there is a unique element in $\mathcal{H}^m(\mathbb{D})$, denoted by $K(.,z)$, such that

$$f(z) = \tau_z(f) = \langle f, K(.,z)\rangle_m = \int_{\mathbb{D}} f(\xi)\overline{K(\xi,z)}dA_m(\xi),$$

$$(f \in \mathcal{H}^m(\mathbb{D}),\, \xi = \xi_1 + i\xi_2,\, z \in \mathbb{D}).$$

The function

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \quad \text{with} \quad K(.,z) \in \mathcal{H}^2(\mathbb{D})$$

is called the weighted Bergman kernel for $\mathbb{D}$.

For any orthonormal basis $\{\varphi_n,\ n = 0,1,2,...\}$ in $\mathcal{H}^m(\mathbb{D})$ the kernel function has the representation

$$K(\xi,z) = \sum_{n=1}^{\infty} \varphi_n(\xi)\overline{\varphi_n(z)}, \quad (\xi,z) \in \mathbb{D}\times\mathbb{D}, \tag{2.8}$$

with uniform convergence on compact subsets of $\mathbb{D}\times\mathbb{D}$. The functions

$$\varphi_n(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}}z^n, \quad (z \in \mathbb{D},\, n = 0,1,2,...)$$

form an orthonormal basis in $\mathcal{H}^m(\mathbb{D})$, consequently

$$K(\xi,z) = \frac{1}{(1-\overline{z}\xi)^m}. \tag{2.9}$$

The explicit formula for the kernel function shows that

$$f(z) = \frac{m-1}{\pi}\int_{\mathbb{D}} f(\xi)\frac{1}{(1-\overline{\xi}z)^m}(1-|\xi|^2)^{m-2}d\xi_1 d\xi_2, \quad (f \in \mathcal{H}^m(\mathbb{D}),\, z \in \mathbb{D}).$$

Since $\mathcal{H}^m(\mathbb{D})$ is closed subspace of $L^2(\mathbb{D}, dA_m)$ there is an orthogonal projection operator

$$P_m : L^2(\mathbb{D}, dA_m) \to \mathcal{H}^m(\mathbb{D}).$$

P_m is a bounded Hermitian operator of the norm 1 which satisfies $P_m f = f$ for $f \in \mathcal{H}^m(\mathbb{D})$, this is the so called *weighted Bergman projection*. Thus the weighted Bergman projection can be expressed by integration with respect to the Bergman kernel in the following way:

$$(2.10)\quad (P_m f)(z) = \langle f, K(.,z)\rangle = \frac{m-1}{\pi}\int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^m}(1-|\xi|^2)^{m-2}d\xi_1 d\xi_2$$

$$(f \in L^2(\mathbb{D}, dA_m), \quad z,\ \xi \in \mathbb{D}, \quad \xi = \xi_1 + i\xi_2).$$

The projection operator can be extended on $L^1(\mathbb{D}, dA_m)$ mapping each $f \in$ $\in L^1(\mathbb{D}, dA_m)$ to a function analytic in $\mathbb{D}$. Since (2.10) is a pointwise formula and $\mathcal{H}^m(\mathbb{D})$ is dense in $\mathcal{H}_1(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C},\ f \in \mathcal{A}(\mathbb{D})\ \int_{\mathbb{D}} |f(z)|dA_m(z) < \infty\}$ it follows that

$$(2.11)\qquad f(z) = \frac{m-1}{\pi}\int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^m}(1-|\xi|^2)^{m-2}d\xi_1 d\xi_2$$

$$(f \in \mathcal{H}_1(\mathbb{D}), \quad z, \quad \xi \in \mathbb{D},\ \xi = \xi_1 + i\xi_2),$$

and the integral converges uniformly in z in every compact subset of $\mathbb{D}$ (see [5], p. 6).

3. New results

3.1. The representation of Blaschke group on the Hilbert space $\mathcal{H}^m(\mathbb{D})$

In this section we will extend the results obtained in [15] to the case when the representation of the Blaschke group is on the weighted Bergman spaces. Let us consider the following set of functions

$$(3.1)\qquad F_a(z) := \frac{\sqrt{\epsilon(1-|b|^2)}}{1-\bar{b}z} \quad (a = (b,\epsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}).$$

For every power m ($m \geq 2, m \in \mathbb{N}$) F_a induce a unitary representation of Blaschke group on the space $\mathcal{H}^m(\mathbb{D})$. Namely, let us define

$$(3.2)\qquad U_a^m f := [F_{a^{-1}}]^m f \circ B_a^{-1} \quad (a \in \mathbb{B}, \quad m \in \mathbb{N}, \quad m \geq 2, \quad f \in \mathcal{H}^m(\mathbb{D})).$$

It can be proved the following

Theorem 3. *For all $m \in \mathbb{N}, m \geq 2$ $U_a^m (a \in \mathbb{B})$ is a unitary representation of the group $\mathbb{B}$ on the Hilbert space $\mathcal{H}^m(\mathbb{D})$.*

3.2. The properties of the voice transform induced by representation U_a^m

In what follows we will compute the matrix elements of the representation (3.2). We will prove that the representation $U_a := U_a^m$ is irreducible, using this we prove the analogue of the Placherel formula. Using the properties of this matrix elements we will give an addition formula and we will give classes of admissible elements for the representation U_a.

The representation has the following form

$$(U_{a^{-1}}^m f)(z) := e^{i\frac{m\psi}{2}} \frac{(1-|b|^2)^{\frac{m}{2}}}{(1-\bar{b}z)^m} f\left(e^{i\psi}\frac{z-b}{1-\bar{b}z}\right) \quad (a=(b,e^{i\psi}) \in \mathbb{B}) \tag{3.3}$$

and is unitary with respect to the scalar product

$$\langle f,g\rangle = \langle f,g\rangle_m := \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_m(z). \tag{3.4}$$

The voice transform induced by representation U_a by definition is

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle_m \quad (a=(b,e^{i\psi}) \in \mathbb{B},\ f,\rho \in \mathcal{H}^m(\mathbb{D})). \tag{3.5}$$

The matrix elements of the representation in the base $h_n(z) := z^n \ \ (n \in \mathbb{N}, z \in \in \mathbb{D})$ are defined by the following formulae

$$v_{kn}(a^{-1}) := \langle h_k, U_{a^{-1}}^m h_n\rangle_m = \tag{3.6}$$

$$= e^{-i(n+\frac{m}{2})\psi}(1-|b|^2)^{\frac{m}{2}}\frac{m-1}{\pi}\int_{\mathbb{D}} \overline{\left(\frac{(z-b)^n}{(1-\bar{b}z)^{n+m}}\right)} z^k (1-|z|^2)^{m-2}\, dxdy.$$

Theorem 4. *The matrix elements of the representation (3.3) have the following form*

$$v_{kn}(a^{-1}) = (1-r^2)^{\frac{m}{2}} e^{-i(n+\frac{m}{2})\psi} e^{-i(n-k)\varphi}\alpha_{kn}(r) \quad (k,n \in \mathbb{N}), \tag{3.8}$$

where $\alpha_{kn}(r)$ *can be expressed using the Jacobi polynomials in the following way:*

$$\alpha_{kn}(r) := C(k,m,n)r^{n-k}\left[(1-u)^n u^{k+m-1}\right]^{(n+m-1)}_{u=r^2}, \tag{3.9}$$

and

$$C(k,m,n) = \frac{2^{m-1}(m-1)!}{(n+m-1)!(2k+2)(2k+4)...(2k+2m-2)}.$$

The radial part $\alpha_{kn}(r)$ *satisfies the following relations*

$$\alpha_{kn}(r) = (-1)^{n-k}\alpha_{nk}(r). \tag{3.10}$$

It is known that the matrix elements of the representations in general satisfy the following so called addition formula (see [18]):

$$v_{kn}(a_1 \circ a_2) = \sum_{\ell} v_{k\ell}(a_1)v_{\ell n}(a_2).$$

From this relation we obtain the following addition formula:

$$(1-r^2)^{\frac{m}{2}} e^{-i(n+\frac{m}{2})\psi} e^{-i(n-k)\varphi}\alpha_{kn}(r) = \tag{3.11}$$

$$\sum_{\ell}(1-r_1^2)^{\frac{m}{2}} e^{-i(\ell+\frac{m}{2})\psi_1} e^{-i(\ell-k)\varphi_1}\alpha_{k\ell}(r_1)(1-r_2^2)^{\frac{m}{2}} e^{-i(n+\frac{m}{2})\psi_2} e^{-i(n-\ell)\varphi_2}\alpha_{\ell n}(r_2),$$

where $a_j := (r_j e^{i\varphi_j}, e^{i\psi_j})$, $j \in \{1,2\}$ and $a := (re^{i\varphi}, e^{i\psi}) = a_1 \circ a_2$.

Theorem 5. *The representation* U_a $(a \in \mathbb{B})$ *is irreducible on the space* $\mathcal{H}^m(\mathbb{D})$*.*

From Theorem 1, Theorem 2 (see [10], [16]), Theorem 3 and Theorem 5 it follows

Consequence 1. The voice transform generated by representation U_a, $a \in \mathbb{B}$ is one to one.

Consequence 2. If $\mathcal{H}^m(\mathbb{D})^*$ denotes the set of admissible elements from $\mathcal{H}^m(\mathbb{D})$, then there is a symmetric positive bilinear map

$$B : \mathcal{H}^m(\mathbb{D})^* \times \mathcal{H}^m(\mathbb{D})^* \to \mathbb{R}$$

such that
(3.12)|

$$[V_{\rho_1} f_1, V_{\rho_2} f_2] = B(\rho_1,\rho_2)\langle f_1, f_2\rangle_m \quad (f_1, f_2 \in \mathcal{H}^m(\mathbb{D}), \quad \rho_1,\rho_2 \in \mathcal{H}^2(\mathbb{D})^*),$$

where

$$[F,G] := \int_{\mathbb{B}} F(a)\overline{G(a)}\, dm(a)$$

and $dm(a)$ is the Haar measure of the group $\mathbb{B}$.

For the special case $m = 2$ in paper [15] we gave a direct proof of this result, from which it turns out that every $\rho \in \mathcal{H}^2(\mathbb{D})$ is admissible and the voice transform induced by $U_a = U_a^2$ satisfies

$$(3.13) \qquad [V_{\rho_1} f, V_{\rho_2} g] = 4\pi\langle \rho_1, \rho_2\rangle \, \langle f, g\rangle \quad (f, g, \rho_1, \rho_2 \in \mathcal{H}^2(\mathbb{D})).$$

Theorem 6. *Every $\rho_n = z^n$ $(n \in \mathbb{N})$ is admissible, namely*

$$\int_{\mathbb{B}} |V_{\rho_n} \rho_n(a)|^2 dm(a) < \infty.$$

Theorem 7. *Every element $\rho \in \mathcal{H}^\infty(\mathbb{D})$ is admissible, namely*

$$\int_{\mathbb{B}} |V_\rho \rho(a)|^2 dm(a) < \infty.$$

3.3. Construction of orthogonal rational wavelets in the weighted Bergman spaces

In this section we give an orthogonal rational wavelet system, and we show that the Bergman projection operator can be expressed with this system and the voice transforms with the parameters of the functions of the system. Let us consider the shift operator

$$(3.13) \qquad (S\varphi)(z) = z\varphi(z) \quad (\varphi \in \mathcal{H}^m(\mathbb{D})).$$

Denote by

$$(3.14) \qquad \varphi_{a,n}(z) := \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}} (U_{a^{-1}} S^n \varphi)(z)$$

$$(a = (b, \epsilon) \in \mathbb{B}, \quad m \in \mathbb{N}, \quad m \geq 2, \quad \varphi \in \mathcal{H}^m(\mathbb{D}), \quad n \in \mathbb{N}).$$

If we consider as *mother wavelet* $\varphi = 1 \in \mathcal{H}^m(\mathbb{D})$, then the corresponding rational wavelets are

$$\varphi_{a,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}} \frac{[\epsilon(1-|b|^2)]^{\frac{m}{2}}}{(1-\bar{b}z)^m} \left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^n, \quad n \in \mathbb{N}. \tag{3.15}$$

Taking into account the unitarity of the representation U_a it follows that they form an orthonormal system in $\mathcal{H}^m(\mathbb{D})$, for every $a \in \mathbb{B}$.

We observe that if we consider the neutral element of the group $a = e = (0,1) \in \mathbb{B}$ then we reobtain the classical orthonormal basis in $\mathcal{H}^m(\mathbb{D})$

$$\varphi_n(z) = \varphi_{e,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}} z^n, \quad n \in \mathbb{N}.$$

Theorem 8. *For all $z \in \mathbb{D}$ and $a \in \mathbb{B}$ the weighted Bergman projection operator $P_m : L^2(\mathbb{D}, dA_m) \to \mathcal{H}^m(\mathbb{D})$ can be written in the following way*

$$P_m f(z) = \sum_{n=0}^{\infty} V_{\varphi_n} f(a^{-1}) \varphi_{a,n}(z) \quad (a \in \mathbb{B}). \tag{3.16}$$

Consequence 3. *Every f from $\mathcal{H}^m(\mathbb{D})$ can be represented as*

$$f(z) = \sum_{n=0}^{\infty} V_{\varphi_n} f(a^{-1}) \varphi_{a,n}(z) \quad (a \in \mathbb{B},\ z \in \mathbb{D}).$$

Consequence 4. *For every $a \in \mathbb{B}$ the functions*

$$\varphi_{a,n}(z) = \sqrt{\frac{\Gamma(n+m)}{n!\Gamma(m)}} \frac{[\epsilon(1-|b|^2)]^{\frac{m}{2}}}{(1-\bar{b}z)^m} \left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^n \quad (z \in \mathbb{D},\ n \in \mathbb{N}) \tag{3.17}$$

form an orthonormal basis in $\mathcal{H}^m(\mathbb{D})$.

From Consequence 3 we can deduce the following characterization of the poles.

Consequence 5. *Let F analytic continuation of the function $f \in \mathcal{H}^m(\mathbb{D})$. Then F has n-tuple pole at $\frac{1}{b}$ outside of the unit disc if and only if for $a = (b, \epsilon) \in \mathbb{B}$*

$$V_{\varphi_n} f(a^{-1}) \neq 0, \text{ and for all } k,\ k > n,\ V_{\varphi_k} f(a^{-1}) = 0.$$

4. Proofs

Proof of Theorem 3. The set of functions $(F_a, a \in \mathbb{B})$ defined by (3.1) satisfies the following relation

$$F_{a_1} \circ B_{a_2} \cdot F_{a_2} = F_{a_1 \circ a_2} \quad (a_1, a_2 \in \mathbb{B}). \tag{4.1}$$

To prove this we will use the identity

$$|1 + b_1 \overline{b}_2 \overline{\epsilon}_2| \sqrt{1 - |b|^2} = \sqrt{|1 + b_1 \overline{b}_2 \overline{\epsilon}_2|^2 - |b_1 \overline{\epsilon}_2 + b_2|^2} = \sqrt{(1 - |b_1|^2)(1 - |b_2|^2)}.$$

Using this we obtain that

$$F_{a_1}(B_{a_2}(z)) \cdot F_{a_2}(z) = \sqrt{\epsilon_1 \epsilon_2 (1 - |b_1|^2)(1 - |b_2|^2)} \frac{1}{1 - \overline{b}_1 \epsilon_2 \frac{z - b_2}{1 - \overline{b}_2 z}} \frac{1}{1 - \overline{b}_2 z} =$$

$$= \frac{\sqrt{\epsilon_1 \epsilon_2 (1 - |b|^2)} |1 + b_1 \overline{b}_2 \overline{\epsilon}_2|}{1 + \overline{b}_1 b_2 \epsilon_2} \frac{1}{1 - \overline{b} z} =$$

$$= \sqrt{\epsilon (1 - |b|^2)} \frac{1}{1 - \overline{b} z} = F_{a_1 \circ a_2}(z).$$

From this it follows that

$$U^m_{a_1}(U^m_{a_2} f) = U^m_{a_1}([F_{a_2^{-1}}]^m \cdot f \circ B^{-1}_{a_2}) = [F_{a_1^{-1}}]^m \cdot [F_{a_2^{-1}} \circ B_{a_1^{-1}}]^m \cdot f \circ B^{-1}_{a_2} \circ B^{-1}_{a_1} =$$

$$= [F_{(a_1 \circ a_2)^{-1}}]^m \cdot (f \circ B^{-1}_{a_1 \circ a_2}) = U^m_{a_1 \circ a_2} f.$$

In what follows we will show that the restriction of linear application U^m_a on the Hilbert space $\mathcal{H}^m(\mathbb{D})$ is unitary with respect to the inner product defined by (2.6) which implies that if $f \in \mathcal{H}^m$ then $U^m_a f \in \mathcal{H}^m$. To prove this we will use the following result: if $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic bijection, then the determinant of the corresponding Jacobi matrix is equal to $|\varphi'(z)|^2$ and making the change of variables $w = \varphi(z)$ we get the following integral transformation formula

$$\int_{\mathbb{D}} F(w) \, dudv = \int_{\mathbb{D}} F(\varphi(z)) |\varphi'(z)|^2 \, dxdy \quad (w = u + iv, z = x + iy \in \mathbb{D}).$$

In the special case when $\varphi = B_a$, then

$$B'_a(z) = \epsilon \frac{1-|b|^2}{(1-\bar{b}z)^2} = [F_a(z)]^2 \quad (a = (b,\epsilon) \in \mathbb{B}).$$

We want to show that U_a^m is unitary, namely

$$\langle U_a^m f, U_a^m g\rangle_m = \langle f, g\rangle_m \qquad (f, g \in \mathcal{H}^m(\mathbb{D})).$$

Indeed, making the change of variable $w = B_a(z)$ in the integral on the left hand side we obtain that

$$\langle U_a^m f, U_a^m g\rangle_m =$$

$$= \frac{m-1}{\pi} \int_{\mathbb{D}} |F_{a^{-1}}(w)|^{2m} f(B_a^{-1}(w))\overline{g(B_a^{-1}(w))}|1-|w|^2|^{m-2}\, dudv =$$

$$= \frac{m-1}{\pi} \int_{\mathbb{D}} |F_a(z)|^4 |F_{a^{-1}}(B_a(z))|^{2m} f(z)\overline{g(z)}|1-|B_a(z)|^2|^{m-2}\, dxdy.$$

From (2.1) it follows that

$$|1-|B_a(z)|^2| = \frac{|1-\bar{b}z|^2 - |z-b|^2}{|1-\bar{b}z|^2} = \frac{(1-|b|^2)(1-|z|^2)}{|1-\bar{b}z|^2} =$$

$$= |F_a(z)|^2(1-|z|^2),$$

and using (4.1) we obtain that

$$\langle U_a^m f, U_a^m g\rangle_m =$$

$$= \frac{m-1}{\pi} \int_{\mathbb{D}} |F_a(z)|^{2m} |F_{a^{-1}}(B_a(z))|^{2m} f(z)\overline{g(z)}(1-|z|^2)^{m-2}\, dxdy = \langle f, g\rangle_m.$$

Proof of Theorem 4. The matrix elements by definition are equal by

$$v_{kn}(a^{-1}) := \langle h_k, U_{a^{-1}}^m h_n\rangle_m =$$

$$= e^{-i(n+\frac{m}{2})\psi}(1-|b|^2)^{\frac{m}{2}} \frac{m-1}{\pi} \int_{\mathbb{D}} \overline{\Big(\frac{(z-b)^n}{(1-\bar{b}z)^{n+m}}\Big)} z^k (1-|z|^2)^{m-2}\, dxdy.$$

Let us denote by $z = \rho e^{it},\ b = re^{i\varphi}\ (t, \varphi \in \mathbb{I})$ and in the last integral replacing t by $t + \varphi$ we obtain that

$$\frac{m-1}{\pi}\int\limits_{\mathbb{D}}\overline{\left(\frac{(z-b)^n}{(1-\overline{b}z)^{n+m}}\right)}z^k(1-|z|^2)^{m-2}\,dxdy =$$

$$=\frac{m-1}{\pi}\int\limits_0^1\int\limits_{-\pi}^{\pi}\frac{(\rho e^{-it}-re^{-i\varphi})^n}{(1-r\rho e^{-i(t-\varphi)})^{n+m}}\rho^{k+1}e^{ikt}(1-\rho^2)^{m-2}\,d\rho dt =$$

$$=e^{-in\varphi}\frac{m-1}{\pi}\int\limits_0^1\int\limits_{-\pi}^{\pi}\frac{(\rho e^{-i(t-\varphi)}-r)^n}{(1-r\rho e^{-i(t-\varphi)})^{n+m}}\rho^{k+1}e^{ikt}(1-\rho^2)^{m-2}\,d\rho dt =$$

$$=e^{-i(n-k)\varphi}\frac{m-1}{\pi}\int\limits_0^1\int\limits_{-\pi}^{\pi}\frac{(\rho e^{-it}-r)^n}{(1-\rho re^{-it})^{n+m}}\rho^{k+1}e^{ikt}(1-\rho^2)^{m-2}\,d\rho dt =$$

$$=e^{-i(n-k)\varphi}\frac{m-1}{\pi}\int\limits_0^1\int\limits_{-\pi}^{\pi}\frac{(\rho e^{it}-r)^n}{(1-\rho re^{it})^{n+m}}\rho^{k+1}e^{-ikt}(1-\rho^2)^{m-2}\,d\rho.$$

Let us denote by

$$\alpha_{kn}(r):=\frac{m-1}{\pi}\int\limits_0^1\int\limits_{-\pi}^{\pi}\frac{(\rho e^{it}-r)^n}{(1-r\rho e^{it})^{n+m}}\rho^{k+1}e^{-ikt}(1-\rho^2)^{m-2}\,d\rho dt.$$

We will show that $\alpha_{kn}(r)$ can be expressed using the Jacobi polynomials in the following way:

$$\alpha_{kn}(r):=C(k,m,n)r^{n-k}\left[(1-u)^n u^{k+m-1}\right]^{(n+m-1)}_{u=r^2},$$

where

$$C(k,m,n)=\frac{2^{m-1}(m-1)!}{(n+m-1)!(2k+2)(2k+4)...(2k+2m-2)}.$$

Indeed, if we substitute e^{-it} by ζ, on the base of Cauchy integral formula we obtain that

$$\alpha_{kn}(r):=\int\limits_0^1 2\rho^{k+1}(1-\rho^2)^{m-2}\frac{m-1}{2\pi i}\int\limits_{\mathbb{T}}\frac{(\rho-r\zeta)^n}{(\zeta-r\rho)^{n+m}}\zeta^{m+k-1}\,d\zeta d\rho =$$

$$= \int_0^1 \frac{2(m-1)}{(n+m-1)!} \rho^{2k+n+m} (1-\rho^2)^{m-2} r^{-k-m+1} \times$$

$$\times \left[\left(1 - \frac{r}{\rho} z\right)^n \left(\frac{r}{\rho} z\right)^{k+m-1} \right]^{(n+m-1)}_{z=r\rho} d\rho.$$

Denoting by $u = \frac{rz}{\rho}$, from this it follows that

$$\alpha_{kn}(r) = \left[(1-u)^n u^{k+m-1}\right]^{(n+m-1)}_{u=r^2} \frac{(m-1)r^{n-k}}{(n+m-1)!} \int_0^1 2\rho^{2k+1}(1-\rho^2)^{m-2} d\rho.$$

Let us denote by

$$C(k,m,n) := \frac{2(m-1)}{(n+m-1)!} \int_0^1 \rho^{2k+1}(1-\rho^2)^{m-2} d\rho.$$

Using partial integration $m-2$ times we obtain that

$$C(k,m,n) = \frac{2^{m-1}(m-1)!}{(n+m-1)!(2k+2)(2k+4)...(2k+2m-2)}.$$

To show i) let us use the following relation

$$v_{kn}(a^{-1}) = \langle h_k, U_{a^{-1}} h_n \rangle = \overline{\langle U_{a^{-1}} h_n, h_k \rangle} = \overline{\langle h_n, U_a h_k \rangle} = \overline{v_{nk}(a)} \quad (m, n \in \mathbb{N}).$$

Taking into account that $a^{-1} = (be^{i(\varphi+\pi)}, e^{-i\psi})$, then for $a = (r, 1)$, namely when $\varphi = \psi = 0$ we obtain that

$$(1-r^2)\alpha_{kn}(r) = v_{kn}(a) = \overline{v_{nk}(a^{-1})} = (-1)^{k+n}(1-r^2)\alpha_{nk}(r)$$

which implies that i) is true.

Proof of Theorem 5. It can be proved that this representation is irreducible on the Hilbert space $\mathcal{H}^m(\mathbb{D})$. Let us consider the power functions $h_n(z) := z^n \quad (z \in \mathbb{C}, n \in \mathbb{N})$. First we take the images of these functions under the representation

$$b \to (U_a h_n)(z) := \epsilon^{(n+\frac{m}{2})\psi} (1-|b|^2)^{\frac{m}{2}} \frac{(z-b)^n}{(1-\overline{b}z)^{n+m}}$$

$$(a = (b, e^{i\psi}) \in \mathbb{B},\ b = b_1 + ib_2 \in \mathbb{D}),$$

and we compute the partial derivatives with respect to the variables b_1 and b_2, and we take the values of these partial derivatives in $e = (0,1) \in \mathbb{B}$. An easy computation gives that

$$\frac{\partial}{\partial b_1} U_e h_n = -nh_{n-1} + (n+m)h_{n+1},$$

$$\frac{\partial}{\partial b_2} U_e h_n = -inh_{n-1} - i(n+m)h_{n+1}.$$

From this we obtain that

$$\left(\frac{\partial}{\partial b_1} + i\frac{\partial}{\partial b_2}\right) U_e h_n = 2(n+m)h_{n+1}, \quad \left(\frac{\partial}{\partial b_1} - i\frac{\partial}{\partial b_2}\right) U_e h_n = 2nh_{n-1}.$$

From the definition of U_a follows that for the one parameter subgroup $\alpha(t) = (0, e^{it})$ $(t \in \mathbb{R})$ of $\mathbb{B}$

$$U_{\alpha(t)} h_n = e^{i(n+\frac{m}{2})t} h_n \quad (n \in \mathbb{N},\ t \in \mathbb{R}),$$

which means that the subspace wrapped by the function h_n is an invariant one dimensional subspace of the representation. It is known that any invariant subspace of the representation can be written as the direct sum of this kind of subspaces. From these it follows that the representation $U_a\,(a \in \mathbb{B})$ is irreducible. Indeed let H be at least one dimensional closed invariant subspace of the representation. This is also invariant subspace of the representation U_a, for $a = (0, e^{it})$ $(t \in \mathbb{R})$, consequently it contains one of the power functions h_n. On the base of the definition of invariant subspaces it is evident that

$$\frac{1}{b_1}(U_{(b_1,1)} h_n - U_{(0,1)} h_n) \in H, \quad \frac{1}{b_2}(U_{(ib_2,1)} h_n - U_{(0,1)} h_n) \in H \quad (b_1, b_2 \in \mathbb{R}).$$

From the closeness of the subspace it follows that the limit of this expression when $b_1 \to 0, b_2 \to 0$ is also in H, namely

$$\left(\frac{\partial}{\partial b_1} + i\frac{\partial}{\partial b_2}\right) U_e h_n = 2(n+m)h_{n+1} \in H,$$
$$\left(\frac{\partial}{\partial b_1} - i\frac{\partial}{\partial b_2}\right) U_e h_n = 2nh_{n-1} \in H.$$

From this evidently follows that $h_k \in H$, if $k \geq n$, and $h_k \in H$, if $k < n$ and $k \geq 0$. This implies that $H = \mathcal{H}^m(\mathbb{D})$, and the irreducibility of the representation U_a $(a \in \mathbb{B})$ is proved.

Proof of Theorem 6. To prove the admissibility of $\rho_n = z^n$ we have to show that

$$\int_{\mathbb{B}} |V_{\rho_n}\rho_n(a)|^2 dm(a) < \infty.$$

Using (3.8) we obtain that

$$\int_{\mathbb{B}} |V_{\rho_n}\rho_n(a)|^2 dm(a) =$$

$$= \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{I}} \int_0^1 |(1-r^2)^{\frac{m}{2}} e^{-i(n+\frac{m}{2})\psi} e^{-i(n-n)\varphi} \alpha_{nn}(r)|^2 \frac{r}{(1-r^2)^2} d\psi d\varphi dr =$$

$$= 2\pi \int_0^1 (1-r^2)^{m-2} r |\alpha_{nn}(r)|^2 dr < \infty \quad \text{for} \quad m \geq 2.$$

Proof of Theorem 7. Using the definition of the voice transform and the unitarity of the representation U we obtain the following estimation for the absolute value of the voice transform

$$|(V_\rho \rho)(a)| = |\langle U_{a^{-1}}\rho, \rho\rangle| \leq (1-|b|^2)^{\frac{m}{2}} \|\rho\|^2_{\mathcal{H}^\infty(\mathbb{D})} \frac{m-1}{\pi} \int_{\mathbb{D}} \frac{(1-|z|^2)^{m-2} dxdy}{|1-\overline{b}z|^m}.$$

The integral which appears in this estimation is a special case of the integral operators which involve the power of the Bergman kernel, see [9] p. 7, where it is showed that

$$I_{m-2,0}(|b|) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{(1-|z|^2)^{m-2} dxdy}{|1-\overline{b}z|^m} \sim \log \frac{1}{1-|b|^2} \quad \text{as} \quad |b| \to 1^-.$$

Using this we obtain that

$$\int_{\mathbb{B}} |(V_\rho \rho)(a)|^2 dm(a) \leq$$

$$\leq (m-1)^2 \frac{1}{2\pi} \int_{\mathbb{I}} \int_{D} (1-|b|^2)^m \|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})} |I_{m-2,0}(|b|)|^2 \frac{1}{(1-|b|^2)^2} db_1 db_2 d\psi \leq$$

$$\leq (m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_D (1-|b|^2)^{m-2}|I_{m-2,0}(|b|)|^2 db_1 db_2 =$$

$$=(m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_{U(0,1-\epsilon)} (1-|b|^2)^{m-2}|I_{m-2,0}(|b|)|^2 db_1 db_2 +$$

$$+ (m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_{U(0,1-\epsilon,1)} (1-|b|^2)^{m-2}|I_{m-2,0}(|b|)|^2 db_1 db_2 \leq$$

$$\leq (m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_{U(0,1-\epsilon)} (1-|b|^2)^{m-2}|I_{m-2,0}(|b|)|^2 db_1 db_2 +$$

$$+ C(m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_D (1-|b|^2)^{m-2}|\log\frac{1}{1-|b|^2}|^2 db_1 db_2.$$

Let us denote by

$$J_1 = (m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_{U(0,1-\epsilon)} (1-|b|^2)^{m-2}|I_{m-2,0}(|b|)|^2 db_1 db_2,$$

and

$$J_2 = C(m-1)^2\|\rho\|^4_{\mathcal{H}^\infty(\mathbb{D})}\int\limits_D (1-|b|^2)^{m-2}|\log\frac{1}{1-|b|^2}|^2 db_1 db_2.$$

Using the inequality

$$\frac{1}{|1-\bar{b}z|} \leq \frac{2}{1-|b|^2} \quad (b, z \in D)$$

we obtain that J_1 is finite. To prove that J_2 is finite first we use twice the partial integration, then the L'Hospital rule and we obtain that

$$\int\limits_D (1-|b|^2)^{m-2}|\log\frac{1}{1-|b|^2}|^2 db_1 db_2 = 2\pi\int\limits_0^1 (1-r^2)^{m-2}\log^2(1-r^2) r dr =$$

$$= \pi\int\limits_0^1 (1-x)^{m-2}\log^2(1-x) dx = \frac{2}{(m-1)^3} < \infty.$$

Consequently

$$\int_{\mathbb{B}} |(V_\rho \rho)(a)|^2 dm(a) \leq J_1 + J_2 \leq \infty,$$

which means that every element from $\mathcal{H}^\infty(\mathbb{D})$ is admissible.

Proof of Theorem 8. Let consider the following infinite series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+m)}{n!\Gamma(m)} \overline{\varphi_{a,n}(y)} \varphi_{a,m}(z).$$

Since

$$|\overline{\varphi_{a,n}(y)} \varphi_{a,n}(z)| \leq \frac{(1-r^2)^m}{(1-r_1 r)^m (1-r_2 r)^m} \left(\frac{r+r_1}{1+r_1 r} \frac{r+r_2}{1+r_2 r} \right)^n$$

$$(z = r_1 e^{it}, \quad y = r_2 e^{it} \in \mathbb{D}, \quad a = (re^{i\varphi}, e^{i\psi}) \in \mathbb{B}),$$

and $\left|\frac{r+r_1}{1+r_1 r}\right| < 1$, $\left|\frac{r+r_2}{1+r_2 r}\right| < 1$ we obtain that for a fixed $z \in \mathbb{D}$ and for a fixed $a = (re^{i\varphi}, e^{i\psi}) \in \mathbb{B}$ the series converges absolutely and uniformly in $y \in \mathbb{D}$. This permits the interchange of summation and integration in the following expression

$$\sum_{n=0}^{\infty} (V_{\varphi_n} f)(a^{-1}) \varphi_{a,n}(z) =$$

$$= \sum_{n=0}^{\infty} < f, U_{a^{-1}} \varphi_n > \varphi_{a,n}(z) =$$

$$= \int_{\mathbb{D}} f(y) \sum_{n=0}^{\infty} \frac{\Gamma(n+m)}{n!\Gamma(m)} \overline{\varphi_{a,n}(y)} \varphi_{a,m}(z) dA_m(y).$$

Using that

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+m)}{n!\Gamma(m)} \overline{y^n} z^n = \frac{1}{(1-\overline{y}z)^m} \quad (z, y \in \mathbb{D}),$$

we obtain that

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+m)}{n!\Gamma(m)} \overline{\varphi_{a,n}(y)} \varphi_{a,m}(z) =$$

$$= \frac{(1-|b|^2)^m}{(1-b\overline{y})^m (1-\overline{b}z)^m} \sum_{n=0}^{\infty} \frac{\Gamma(n+m)}{n!\Gamma(m)} \overline{\left(\frac{\epsilon(y-b)}{1-\overline{b}y} \right)^n} \left(\frac{\epsilon(z-b)}{1-\overline{b}z} \right)^n =$$

$$= \frac{(1-|b|^2)^m}{(1-b\overline{y})^m(1-\overline{b}z)^m}\frac{1}{\left(1-\overline{\left(\frac{y-b}{1-\overline{b}y}\right)}\frac{z-b}{1-\overline{b}z}\right)^m} =$$

$$= \frac{1}{(1-\overline{y}z)^m} = K(y,z).$$

Consequently

$$\sum_{n=0}^{\infty}(V_{\varphi_n}f)(a^{-1})\varphi_{a,n}(z) = \int_{\mathbb{D}} f(y)\frac{2}{(1-\overline{y}z)^m}dA_m(y) = (P_m f)(z).$$

Theorem 3, Theorem 5 and Theorem 7 are valid also when we suppose just that $m \geq 2$.

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