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UNIFORM AND L–CONVERGENCE OF THE NÖRLUND LOGARITHMIC MEANS OF WALSH–KACZMARZ–FOURIER SERIES

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This paper is dedicated
to Professor Ferenc Schipp on the occasion of his 70th birthday
and to Professor Péter Simon on the occasion of his 60th birthday

Abstract. The main aim of this paper is to investigate the convergence and divergence properties of one- and two-dimensional Nörlund logarithmic means of Walsh-Kaczmarz-Fourier series of functions in the uniform and in the L Lebesgue norm. We give necessary and sufficient conditions for the convergence regarding the modulus of continuity of the functions.

1. Introduction

The n-th Riesz's logarithmic mean of a Fourier series is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}, \quad l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

The Riesz's logarithmic mean with respect to the trigonometric system was studied by a lot of authors, e.g. Szász [20] and Yabuta [21], with respect to Walsh, Vilenkin system by Simon [15] and Gát [4].

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Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund mean of an integrable function f is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. This Nörlund mean of Walsh-Fourier series was investigated by Móricz and Siddiqi [13]. The case, when $q_k = \frac{1}{k}$ is excluded, since the method of Móricz and Siddiqi does not work in this case.

If $q_k := \frac{1}{k}$, then we get the Nörlund logarithmic means

$$t_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{S_k(f)}{n-k},$$

where $l_n := \sum_{k=1}^{n-1} \frac{1}{k}$. From now, we write simply logarithmic means $t_n(f)$. Recently, Gát and Goginava [5, 7, 8] proved some convergence and divergence properties of these logarithmic means of functions in the class of continuous functions, and in the Lebesgue space with respect to the Walsh-Paley system. Moreover, they proved that the maximal norm convergence function space of these logarithmic means is $L\log^+ L$.

The main aim of this article is to investigate the convergence and divergence properties of one- and two-dimensional Nörlund logarithmic means of Walsh-Kaczmarz-Fourier series of functions in the uniform, and in the L Lebesgue norm. We give necessary and sufficient conditions for the convergence regarding the modulus of continuity of the functions.

The a.e. convergence of a subsequence of logarithmic means of Walsh-Fourier series of integrable functions was discussed by Gát and Goginava [9, 6]. More results on these logarithmic means with respect to unbounded Vilenkin system can be found in the paper [2] written by Blahota and Gát.

2. Definitions and notations

Let $I := [0, 1)$ denote the unit interval in $\mathbb{R}$. The Rademacher functions are defined by

$$r_n(x) := r_0(2^n x), \quad n \geq 1 \text{ and } x \in I, \text{ where } r_0(x) := \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ -1 & \text{if } x \in [1/2, 1), \end{cases}$$

and $r_0(x+1) := r_0(x)$. Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i,\ n_i \in \{0,1\}\ (i \in \mathbb{N})$, where only a finite number of n_i's are different from zero. Let the order of $n \geq 1$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. That is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley functions are defined by

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

Set $w := (w_n : n \in \mathbb{N})$ and $\kappa := (\kappa_n : n \in \mathbb{N})$. Each $x \in I = [0,1)$ can be expressed as $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$, where $x_j \in \{0,1\}\ (j \in \mathbb{N})$. This expression is unique if x is not a dyadic rational. In other words, if x is not of the form $j/2^n$, where j, n are nonnegative integers. If x is a dyadic rational, then we choose the expansion which terminates in zeros. In this way we have the unicity of this expression for all x.

For $A \in \mathbb{N}$ define the transformation $\tau_A : I \to I$ by

$$\tau_A(x) := \frac{x_{A-1}}{2^1} + \frac{x_{A-2}}{2^2} + \ldots + \frac{x_0}{2^{A-1}} + \sum_{j=A}^{\infty} \frac{x_j}{2^{j+1}}.$$

In other words, if the coordinates of x are $x_0, x_1, \ldots, x_{A-1}, x_A, \ldots$, then the coordinates of $\tau_A(x)$ are $x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, \ldots$. By the definition of τ_A (see [17]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N},\ x \in [0,1)).$$

Suppose that f is a Lebesgue integrable function on I and 1-periodic. We define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér kernels and the Nörlund logarithmic kernels by

$$\hat{f}^{\alpha}(k) := \int_0^1 f(t)\alpha_k(t)dt, \qquad S_n^{\alpha}(f) := \sum_{k=0}^{n-1} \hat{f}^{\alpha}(k)\alpha_k,$$

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k, \qquad K_n^{\alpha} := \frac{1}{n}\sum_{k=0}^{n} D_k^{\alpha},$$

$$F_n^{\alpha} := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{D_k^{\alpha}}{n-k},$$

where $\alpha = w$ or κ. Recall that

$$D_{2^n}(x) := D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n & \text{if } x \in [0, 1/2^n), \\ 0 & \text{if } x \in [1/2^n, 1). \end{cases} \tag{1}$$

Set $K = I$ or I^2. Denote by $L(K)$ the set of measurable functions f defined on K for which

$$\|f\|_L = \int_K |f| < \infty$$

and by $C(K)$ the space of continuous functions on K, with the supremum norm

$$\|f\|_C = \sup_{x\in K} |f(x)|.$$

Let $f \in C(I)$. The expression

$$\omega(\delta, f)_C := \sup_{|h|\le\delta} \|f(. \oplus h) - f(.)\|_C$$

is called the modulus of continuity of f, and for $f \in L(I)$

$$\omega(\delta, f)_L := \sup_{|h|\le\delta} \|f(. \oplus h) - f(.)\|_L$$

is called the integral modulus of continuity, where $\oplus$ denotes the dyadic addition (see [14]).

On the unit square $I^2 = [0,1) \times [0,1)$ we consider the two-dimensional systems as $\{\alpha_n(x) \times \alpha_m(y) : n, m \in \mathbb{N}\}$. The two-dimensional Fourier coefficients, the rectangular partial sums of Fourier series and the Dirichlet kernels are defined by

$$\hat{f}^{\alpha}(i,j) := \int_0^1 \int_0^1 f(t,s)\alpha_i(t)\alpha_j(s)dtds,$$

$$S_{k,l}^{\alpha}(f) := \sum_{i=0}^{k-1}\sum_{j=0}^{l-1} \hat{f}^{\alpha}(i,j)\alpha_i\alpha_j \quad \text{and} \quad D_{k,l}^{\alpha} := \sum_{i=0}^{k-1}\sum_{j=0}^{l-1} \alpha_i\alpha_j = D_k^{\alpha,1} D_l^{\alpha,2},$$

where $\alpha = w$ or κ. Let $X = L(I^2)$ or $C(I^2)$. The total modulus of continuity in case $X = C(I^2)$, and the total integrated modulus of continuity in case $X = L(I^2)$ are defined by

$$\omega(\delta, f)_X := \sup\{\|f(. \oplus u, . \oplus v) - f(.,.)\|_X : u^2 + v^2 \leq \delta^2\}.$$

The partial modulus of continuity in case $X = C(I^2)$, and the partial integrated modulus of continuity in case $X = L(I^2)$ are defined by

$$\omega_1(\delta, f)_X := \sup\{\|f(. \oplus u, .) - f(.,.)\|_X : |u| \leq \delta\},$$

$$\omega_2(\delta, f)_X := \sup\{\|f(., . \oplus v) - f(.,.)\|_X : |v| \leq \delta\}.$$

The mixed modulus of continuity in case $X = C(I)$, and the mixed integrated modulus of continuity in case $X = L(I)$ are given by

$$\omega_{1,2}(\delta_1, \delta_2, f)_X :=$$

$$:= \sup\{\|f(. \oplus u, . \oplus v) - f(. \oplus u, .) - f(., . \oplus v) + f(.,.)\|_X : |u| \leq \delta_1, |v| \leq \delta_2\}.$$

3. On the one-dimensional Nörlund logarithmic means

During the proofs of Theorems and Lemmas c, C will denote constants which may vary at different occurrences. In order to prove our main theorems we need the following lemma of Gát and Goginava in the paper [5]:

Lemma 1. *Let* $p_A := 2^{2A} + \cdots + 2^2 + 2^0$, *then* $\|F_{p_A}^w\|_L \geq c \log p_A$.

By the help of this lemma we prove the following

Lemma 2. *Let* $p_A := 2^{2A} + \cdots + 2^2 + 2^0$, *then there exists an* $n_0 \in \mathbb{N}$ *such that*

$$\|F^{\kappa}_{p_A}\|_L \geq c \log p_A$$

for $A > n_0$.

Proof. During the proof of Lemma 2 we will use the following equation:

$$D^{\kappa}_{2^A+j}(x) = D_{2^A}(x) + r_A(x) D^{w}_{j}(\tau_A(x)), \quad j = 0, 1, ..., 2^A - 1. \tag{2}$$

Let $|m| = A$, then

$$l_m F^{\kappa}_m(x) = \sum_{j=1}^{2^A} \frac{D^{\kappa}_j(x)}{m-j} + \sum_{j=2^A+1}^{m-1} \frac{D^{\kappa}_j(x)}{m-j} =: I + II.$$

First, we discuss II by the help of the equation (2).

$$II = \sum_{j=1}^{m-2^A-1} \frac{D^{\kappa}_{2^A+j}(x)}{m-2^A-j} = l_{m-2^A} D_{2^A}(x) + r_A(x) \sum_{j=1}^{m-2^A-1} \frac{D^{w}_j(\tau_A(x))}{m-2^A-j} =$$
$$= l_{m-2^A} D_{2^A}(x) + r_A(x) l_{m-2^A} F^{w}_{m-2^A}(\tau_A(x)).$$

Now, we investigate I. By the help of Abel's transformation we could write

$$I = \sum_{j=0}^{2^A} \frac{D^{\kappa}_j(x)}{m-j} =$$
$$= \sum_{j=0}^{2^A-1} \left(\frac{1}{m-j} - \frac{1}{m-j-1} \right) j K^{\kappa}_j(x) + \frac{2^A}{m-2^A} K^{\kappa}_{2^A}(x).$$

Now, we choose $m = p_A$ (we note that $|m| = 2A$) and we write

$$\|F^{\kappa}_{p_A}\|_L \geq \left\| \frac{l_{p_{A-1}}}{l_{p_A}} r_{2A} F^{w}_{p_{A-1}} \circ \tau_{2A} \right\|_L - \left\| \frac{l_{p_{A-1}}}{l_{p_A}} D_{2^{2A}} \right\|_L -$$
$$- \left\| \frac{1}{l_{p_A}} \sum_{j=0}^{2^{2A}-1} \left(\frac{1}{p_A - j} - \frac{1}{p_A - j - 1} \right) j K^{\kappa}_j \right\|_L -$$
$$- \left\| \frac{1}{l_{p_A}} \frac{2^{2A}}{p_{A-1}} K^{\kappa}_{2^{2A}} \right\|_L .$$

By Lemma 1 we have

$$\left\|\frac{l_{p_{A-1}}}{l_{p_A}} r_{2A} F^w_{p_{A-1}} \circ \tau_{2A}\right\|_L \geq c\|F^w_{p_{A-1}}\|_L \geq c \log p_{A-1}.$$

It is evident that

$$\left\|\frac{l_{p_{A-1}}}{l_{p_A}} D_{2^{2A}}\right\|_L \leq c.$$

For the Walsh-Kaczmarz system it was proved [16] that

$$\sup_n \|K^\kappa_n\|_1 < \infty.$$

This immediately gives that

$$\left\|\frac{1}{l_{p_A}} \sum_{j=0}^{2^{2A}-1} \left(\frac{1}{p_A - j} - \frac{1}{p_A - j - 1}\right) j K^\kappa_j\right\|_L \leq c \frac{1}{l_{p_A}} \sum_{j=0}^{2^{2A}-1} \frac{1}{(p_A - j)} \leq c$$

and

$$\left\|\frac{1}{l_{p_A}} \frac{2^{2A}}{p_{A-1}} K^\kappa_{2^{2A}}\right\|_L \leq c.$$

Summarizing our results, we get

$$\|F^\kappa_{p_A}\|_L \geq c \log p_{A-1} - c \geq C \log p_A$$

for A big enough.

This completes the proof of this lemma.

It is well known that the following are true [17, 18]:

Theorem A. *Let either $X = C(I)$ or $X = L(I)$. Let $f \in X$ and*

$$\omega(\delta, f)_X = o\left(\frac{1}{\log(1/\delta)}\right),$$

then $\|S^\kappa_n(f) - f\|_X \to 0$ as $n \to \infty$.

Since,

$$\|t^\kappa_n(f) - f\|_X \leq \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{\|S^\kappa_k(f) - f\|_X}{n - k}$$

and the fact that the logarithmic summability method is regular from Theorem A we conclude that the followings are true.

Theorem 1. *Let $f \in C(I)$ and*

$$\omega(\delta, f)_C = o\left(\frac{1}{\log(1/\delta)}\right),$$

then $\|t_n^\kappa(f) - f\|_C \to 0$ as $n \to \infty$.

Theorem 2. *Let $f \in L(I)$ and*

$$\omega(\delta, f)_L = o\left(\frac{1}{\log(1/\delta)}\right),$$

then $\|t_n^\kappa(f) - f\|_L \to 0$ as $n \to \infty$.

In this paper we prove the sharpness of these results. Namely, we prove the following theorems:

Theorem 3. *There exists a function $f \in C(I)$ such that*

$$\omega(\delta, f)_C = O\left(\frac{1}{\log(1/\delta)}\right)$$

and $t_n^\kappa(f, 0)$ diverges.

Theorem 4. *There exists a function $g \in L(I)$ such that*

$$\omega(\delta, g)_L = O\left(\frac{1}{\log(1/\delta)}\right)$$

and $t_n^\kappa(g)$ does not converge to g in L-norm.

To prove Theorem 3 we modify the counterexample function of Gát and Goginava defined in [5] and show that this modified function is really a counterexample function for the Walsh-Kaczmarz logarithmic means.

The construction: Choose a monotonically increasing sequence of positive integers $\{n_k : k \geq 1\}$ such that

(3) $$n_k^2 \leq n_{k+1},$$

(4) $$\sum_{l=1}^{k-1} \frac{2^{2n_l}}{n_l} < \frac{2^{2n_k}}{n_k}.$$

Set

$$\psi_{n_k}(x) := \begin{cases} 2^{2n_k+2}x & \text{if } 0 \le x < 2^{-2n_k-2}, \\ -2^{2n_k+2}(x - 2^{-2n_k-1}) & \text{if } 2^{-2n_k-2} \le x < 2^{-2n_k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the functions φ_{n_k} periodically by

$$\varphi_{n_k}(x) := \sum_{j=0}^{2^{2n_k+1}-1} \psi_{n_k}\left(x - \frac{j}{2^{2n_k+1}}\right), \quad \varphi_{n_k}(x+1) := \varphi_{n_k}(x).$$

The counterexample function f is defined by

$$f(x) := \sum_{k=1}^{\infty} \frac{f_{n_k}(x)}{n_k},$$

where $f_{n_k}(x) := \varphi_{n_k}(x)\ \mathrm{sgn} F^{\kappa}_{p_{n_k}}(x)$ and p_{n_k} is defined in Lemma 2. The method of the article [5] immediately gives that

$$\omega(1/2^{2n_k}, f_{n_i})_C = O\left(\frac{2^{2n_i}}{2^{2n_k}}\right) \quad (i = 1, 2, ..., k-1) \tag{5}$$

and

$$\omega(\delta, f)_C = O\left(\frac{1}{\log(1/\delta)}\right).$$

Proof of Theorem 3. The only fact we have to prove is that $t^{\kappa}_{p_{n_k}}(f, 0)$ diverges. To do this we follow the method of Gát and Goginava [5] and write

(6)

$$|t^{\kappa}_{p_{n_k}}(f,0) - f(0)| = |t^{\kappa}_{p_{n_k}}(f,0)| = \left|\int_0^1 f(t) F^{\kappa}_{p_{n_k}}(t)dt\right| \ge$$

$$\ge \frac{c}{n_k}\left|\int_0^1 f_{n_k}(t) F^{\kappa}_{p_{n_k}}(t)dt\right| - \sum_{i=0}^{k-1} \frac{c}{n_i}\left|\int_0^1 f_{n_i}(t) F^{\kappa}_{p_{n_k}}(t)dt\right| -$$

$$- \sum_{i=k+1}^{\infty} \frac{c}{n_i}\left|\int_0^1 f_{n_i}(t) F^{\kappa}_{p_{n_k}}(t)dt\right| =: I - II - III.$$

By Lemma 2 we write
(7)

$$I=\frac{1}{n_k}\int\limits_0^1\varphi_{n_k}(t)|F^{\kappa}_{p_{n_k}}(t)|dt=\frac{1}{n_k}\sum_{j=0}^{2^{2n_k+1}-1}\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\varphi_{n_k}(t)|F^{\kappa}_{p_{n_k}}(t)|dt=$$

$$=\frac{1}{n_k}\sum_{j=0}^{2^{2n_k+1}-1}\left|F^{\kappa}_{p_{n_k}}\left(j2^{-2n_k-1}\right)\right|\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\varphi_{n_k}(t)dt=$$

$$=\frac{c}{n_k}\sum_{j=0}^{2^{2n_k+1}-1}\left|F^{\kappa}_{p_{n_k}}\left(j2^{-2n_k-1}\right)\right|\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}1dt\geq$$

$$\geq\frac{c}{n_k}\|F^{\kappa}_{p_{n_k}}\|_L\geq c>0$$

for k big enough.

It is known that

$$\|S^{\kappa}_n(f)-f\|_C\leq c\omega\left(\frac{1}{n},f\right)_C\log(n+1)$$

and

$$\frac{\omega(\delta,f)_C}{\delta}\leq 2\frac{\omega(\delta',f)_C}{\delta'}\quad\text{for}\quad 0<\delta'<\delta.$$

Therefore, we have

$$\|t^{\kappa}_n(f)-f\|_C\leq c\omega\left(\frac{1}{n},f\right)_C\log(n+1)$$

(for more details see [5]). This and (3), (4), (5) imply that

$$II\leq c\sum_{i=0}^{k-1}\frac{1}{n_i}\omega\left(\frac{1}{2^{2n_k}},f_{n_i}\right)_C\log(p_{n_k}+1)=O\left(\frac{n_k}{2^{2n_k}}\sum_{i=0}^{k-1}\frac{2^{2n_i}}{n_i}\right)=$$

$$=O\left(\frac{n_k}{2^{2n_k}}\frac{2^{2n_{k-1}}}{n_{k-1}}\right)=o(1)$$

as $k\to\infty$.

Since, $\|D_n^\kappa\|_L \le c\log n$, we immediately get that $\|F_n^\kappa\|_L = O(\log n)$. This and (3) yield

$$III = O\left(\sum_{i=k+1}^{\infty} \frac{\|F^\kappa_{p_{n_k}}\|_L}{n_i}\right) = O\left(\frac{n_k}{n_{k+1}}\right) = o(1) \quad \text{as} \quad k \to \infty.$$

Summarizing our results, we conclude that

$$\overline{\lim_{k\to\infty}} |t^\kappa_{p_{n_k}}(f,0) - f(0)| > 0.$$

That is, the proof is complete.

Now, we prove Theorem 4 and show that the counterexample function g given in the article [5] is really a counterexample function for the logarithmic means of Walsh-Kaczmarz-Fourier series, too.

First, we give the construction. Choose a monotonically increasing sequence of positive integers $\{m_k : k \ge 1\}$ such that

$$2m_{k-1} \le m_k, \tag{8}$$

$$\sum_{l=1}^{k-1} \frac{2^{2m_l}}{m_l} < \frac{2^{2m_k}}{m_k}. \tag{9}$$

Set

$$g(x) := \sum_{j=1}^{\infty} g_j(x), \quad \text{where} \quad g_j(x) := \frac{D_{2^{2m_j+1}}(x)}{m_j}.$$

In the article [5] it is proved that

$$\omega(\delta, g)_L = O\left(\frac{1}{\log(1/\delta)}\right) \tag{10}$$

and

$$\omega(\delta, g_l)_L = O(2^{2m_l}\delta/m_l) \quad \text{for} \quad l = 1,2,...,k-1,\ \delta > 0. \tag{11}$$

Proof of Theorem 4. During the proof of this theorem we will follow the method of Gát and Goginava in the article [5]. Simple calculation gives

$$\|t^\kappa_{p_{m_k}}(g) - g\|_L \ge$$

$$(12)\qquad \geq \left\| t^{\kappa}_{p_{m_k}}\left(\sum_{i=k}^{\infty} g_i\right)\right\|_L - \sum_{i=k}^{\infty}\|g_i\|_L - \left\| t^{\kappa}_{p_{m_k}}\left(\sum_{i=1}^{k-1} g_i\right) - \sum_{i=1}^{k-1} g_i\right\|_L =: $$
$$=: I - II - III.$$

We have the following

$$t^{\kappa}_{p_{m_k}}(g_i) = \frac{1}{m_i} S_{2^{2m_i+1}}(F^{\kappa}_{p_{m_k}}) = \frac{F^{\kappa}_{p_{m_k}}}{m_i} \quad (i = k, k+1, \ldots).$$

By (8) and Lemma 2 we get

$$I = \left\| \sum_{i=k}^{\infty} \frac{1}{m_i} S_{2m_i+1}(F^{\kappa}_{p_{m_k}})\right\|_L = \sum_{i=k}^{\infty} \frac{\|F^{\kappa}_{p_{m_k}}\|_L}{m_i} \geq c\frac{\|F^{\kappa}_{p_{m_k}}\|_L}{m_k} \geq c > 0$$

for k big enough and

$$II \leq \sum_{i=k}^{\infty} \frac{1}{m_i} \leq \frac{c}{m_k} = o(1) \quad \text{as} \quad k \to \infty.$$

The estimation

$$\|t^{\kappa}_n(g) - g\|_L \leq c\omega\left(\frac{1}{n}, g\right)_L \log(n+1)$$

goes analogously to the estimation $\|t^{\kappa}_n(g) - g\|_C$ (for more details see [5]). This, (8), (9) and (11) yield

$$III \leq \sum_{i=1}^{k-1} \|t^{\kappa}_{p_{m_k}}(g_i) - g_i\|_L = O\left(\sum_{i=1}^{k-1} \omega\left(\frac{1}{2^{2m_k}}, g_i\right)_L m_k\right) =$$
$$= O\left(\frac{m_k}{2^{2m_k}} \sum_{i=1}^{k-1} \frac{2^{2m_i}}{m_i}\right) = O\left(\frac{m_k}{2^{2m_k}} \frac{2^{2m_{k-1}}}{m_{k-1}}\right) = o(1) \quad \text{as} \quad k \to \infty.$$

Summarizing our results, we conclude that

$$\overline{\lim_{k\to\infty}} \|t^{\kappa}_{p_{m_k}}(g) - g\|_L > 0.$$

This completes the proof of this theorem.

4. On the logarithmic means of cubical partial sums

We define the logarithmic means and kernels (of Marcinkiewicz type) of cubical partial sums by

$$T_n^\alpha(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_{k,k}^\alpha(f)}{n-k}, \qquad \mathcal{F}_n^\alpha := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_{k,k}^\alpha}{n-k}.$$

We define the Marcinkiewicz kernels $\mathcal{K}_n$ by

$$\mathcal{K}_n := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}^\alpha,$$

where $\alpha = w$ or κ. For the Walsh system this logarithmic mean was investigated by Gát and Goginava in the article [8]. Now, we would like to discuss the behavior of this logarithmic mean of quadratical partial sums with respect to the double Walsh-Kaczmarz system. We show that the behavior of T_n^κ is very close to the behavior of T_n^w in our special sense.

The following Lemma proved by Goginava [10] will play an important role in the proof of our main theorems.

Lemma 3. *If $f \in X$, then*

$$\|S_{m,n}^\kappa(f) - f\|_X \le$$

$$\le c\left\{\omega_1\left(\frac{1}{m}, f\right)_X \log m + \omega_2\left(\frac{1}{n}, f\right)_X \log n + \omega_{1,2}\left(\frac{1}{m}, \frac{1}{n}, f\right)_X \log m \log n\right\},$$

where $X = C(I^2)$ or $L(I^2)$.

It is evident that the condition

$$\omega(\delta, f)_X \le o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

provides the convergence of $\|S_{n,n}^\kappa(f) - f\|_X \to 0$ (as $n \to \infty$) for $f \in X := C(I^2)$ or $L(I^2)$.

$$\|T_n^\kappa(f) - f\|_X \le \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{\|S_{k,k}^\kappa(f) - f\|_X}{n-k}$$

and the fact that the logarithmic summability method is regular, then by Lemma 3 we conclude that the following is true.

Theorem 5. *Let either* $X := C(I^2)$ *or* $X := L(I^2)$. *Let* $f \in X$ *and*

$$\omega(\delta, f)_X = o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Then $\|T_n^\kappa(f) - f\|_X \to 0$ *as* $n \to \infty$.

In this section we investigate the sharpness of this result. Namely, we prove the following theorems:

Theorem 6. *There exists a function* $f \in C(I^2)$ *such that*

$$\omega(\delta, f)_C = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $T_n^\kappa(f, 0, 0)$ *diverges.*

Theorem 7. *There exists a function* $g \in L(I^2)$ *such that*

$$\omega(\delta, g)_L = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $T_n^\kappa(g)$ *does not converge to* g *in* L*-norm.*

To prove our theorems we need the following lemma [8]:

Lemma 4. *Let* $p_A := 2^{2A} + \cdots + 2^2 + 2^0$, *then*

$$\|\mathcal{F}_{p_A}^w\|_L \geq c \log^2 p_A$$

for every positive integer A.

By the help of this lemma we prove the following lemma for Walsh-Kaczmarz system.

Lemma 5. *Let* $p_A := 2^{2A} + \cdots + 2^2 + 2^0$, *then there exists an* $n_0 \in \mathbb{N}$ *such that*

$$\|\mathcal{F}_{p_A}^\kappa\|_L \geq c \log^2 p_A$$

for $A > n_0$.

Proof. Let $m = p_A$.

$$l_m \mathcal{F}_m^\kappa(x^1, x^2) = \sum_{j=1}^{2^{2A}} \frac{D_j^\kappa(x^1) D_j^\kappa(x^2)}{m-j} + \sum_{j=2^{2A}+1}^{m-1} \frac{D_j^\kappa(x^1) D_j^\kappa(x^2)}{m-j} =: I + II.$$

We will use the notation $D_j^{\kappa,i}(x^1,x^2) := D_j^{\kappa}(x^i)$, $r_A^i(x^1,x^2) := r_A(x^i)$ and $F_n^{\kappa,i}(x^1,x^2) := F_n^{\kappa}(x^i)$ for $i=1,2$.

To discuss II, we use the equation (2), which immediately yields

$$\begin{aligned} II = & \sum_{j=1}^{m-2^{2A}-1} \frac{D_{2^{2A}+j}^{\kappa,1} D_{2^{2A}+j}^{\kappa,2}}{m-2^{2A}-j} = \\ = & \sum_{j=1}^{p_{A-1}-1} \frac{D_{2^{2A}}^{1} D_{2^{2A}}^{2}}{p_{A-1}-j} + r_{2A}^2 D_{2^{2A}}^1 \sum_{j=1}^{p_{A-1}-1} \frac{D_j^{w,2}\circ\tau_{2A}}{p_{A-1}-j} + \\ & + r_{2A}^1 D_{2^{2A}}^2 \sum_{j=1}^{p_{A-1}-1} \frac{D_j^{w,1}\circ\tau_{2A}}{p_{A-1}-j} + r_{2A}^1 r_{2A}^2 \sum_{j=1}^{p_{A-1}-1} \frac{D_{j,j}^{w}\circ(\tau_{2A}\times\tau_{2A})}{p_{A-1}-j} = \\ = & \, l_{p_{A-1}} D_{2^{2A}}^1 D_{2^{2A}}^2 + r_{2A}^2 D_{2^{2A}}^1 l_{p_{A-1}} F_{p_{A-1}}^{w,2}\circ\tau_{2A} + r_{2A}^1 D_{2^{2A}}^2 l_{p_{A-1}} F_{p_{A-1}}^{w,1}\circ\tau_{2A} + \\ & + r_{2A}^1 r_{2A}^2 l_{p_{A-1}} \mathcal{F}_{p_{A-1}}^{w}\circ(\tau_{2A}\times\tau_{2A}) =: II_1 + II_2 + II_3 + II_4. \end{aligned}$$

By the equation

$$\|F_{p_{A-1}}^{w,1}\circ\tau_{2A}\|_L = \|F_{p_{A-1}}^{w,1}\|_L \le \frac{1}{l_{p_{A-1}}} \sum_{j=1}^{p_{A-1}-1} \frac{\|D_j^w\|_L}{p_{A-1}-j} \le c\log p_{A-1},$$

we have that

$$\left\|\frac{1}{l_{p_A}} II_2\right\|_L \le c\log p_{A-1} \quad \text{and} \quad \left\|\frac{1}{l_{p_A}} II_3\right\|_L \le c\log p_{A-1}.$$

The equation (1) implies that

$$\left\|\frac{1}{l_{p_A}} II_1\right\|_L \le c.$$

Now, we discuss I. Abel's transformation gives that

$$I = \sum_{j=1}^{2^{2A}-1} \left(\frac{1}{p_A - j} - \frac{1}{p_A - j - 1}\right) j\mathcal{K}_j^{\kappa} + \frac{2^{2A}}{p_{A-1}} \mathcal{K}_{2^{2A}}^{\kappa} =: I_1 + I_2.$$

For the Walsh-Kaczmarz-Marcinkiewicz kernels [11] holds that

$$\sup_n \|\mathcal{K}_n^{\kappa}\|_L < \infty.$$

This implies that

$$\left\|\frac{1}{l_{p_A}}I_1\right\|_L \leq \frac{1}{l_{p_A}}\sum_{j=1}^{2^{2A}-1}\frac{j\|\mathcal{K}_j^\kappa\|_L}{(p_A-j)(p_A-j-1)} \leq \frac{c}{l_{p_A}}\sum_{j=1}^{2^{2A}-1}\frac{1}{p_A-j} \leq c,$$

$$\left\|\frac{1}{l_{p_A}}I_2\right\|_L = \frac{1}{l_{p_A}}\frac{2^{2A}}{p_{A-1}}\|\mathcal{K}_{2^{2A}}^\kappa\|_L \leq c.$$

By Lemma 4 we get that

$$\left\|\frac{1}{l_{p_A}}II_4\right\|_L \geq c\|\mathcal{F}_{p_A-2^{2A}}^w \circ (\tau_{2A}\times\tau_{2A})\|_L \geq c\|\mathcal{F}_{p_{A-1}}^w\|_L \geq c\log^2 p_{A-1}.$$

That is,

$$\|\mathcal{F}_{p_A}^\kappa\|_L \geq \left\|\frac{1}{l_{p_A}}II_4\right\|_L - c\log p_{A-1} - c \geq C\log^2 p_A.$$

for A large enough.

To prove Theorem 6, we modify the function of Gát and Goginava given in the paper [8].

First, we construct the modified function $f \in C(I^2)$. We choose a monotonically increasing sequence of positive integers $\{n_k : k \geq 1\}$ such that the conditions (3) (that is, $n_k^2 \leq n_{k+1}$) and

$$\sum_{l=1}^{k-1}\frac{2^{2n_l}}{n_l^2} \leq \frac{2^{2n_k}}{n_k^2} \tag{13}$$

are satisfied.

Set

$$f_{n_k}(x,y) := \varphi_{n_k}(x)\varphi_{n_k}(y)\operatorname{sgn}\mathcal{F}_{p_{n_k}}^\kappa(x,y),$$

where p_{n_k} is defined in Lemma 5 and φ_{n_k} is defined in the proof of Theorem 4. Set

$$f(x,y) := \sum_{k=1}^{\infty}\frac{f_{n_k}(x,y)}{n_k^2}.$$

By the method of article [8] it is easy to see that

$$\omega(\delta,f)_C \leq O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and

$$\omega_1\left(\frac{1}{2^{2n_k}}, f_{n_i}\right)_C = \omega_2\left(\frac{1}{2^{2n_k}}, f_{n_i}\right)_C = O\left(\frac{2^{2n_i}}{2^{2n_k}}\right), \quad i = 1, 2, ..., k-1. \tag{14}$$

Proof of Theorem 6. The only fact we have to prove is that this function f is a counterexample function for Walsh-Kaczmarz system. To do this we show that $T^{\kappa}_{p_{n_k}}(f, 0, 0)$ diverges. Now, we write the expression $|T^{\kappa}_{p_{n_k}}(f, 0, 0) - f(0, 0)|$ into the analogous form of the inequality (6).

By Lemma 5 we get

$$I = \frac{c}{n_k^2}\int_0^1\int_0^1 \varphi_{n_k}(t)\varphi_{n_k}(s)|\mathcal{F}^{\kappa}_{p_{n_k}}(t, s)|dtds \geq \frac{c}{n_k^2}\|\mathcal{F}^{\kappa}_{p_{n_k}}\|_L \geq c > 0$$

for k big enough (for more details see the one-dimensional case, the inequality (7)).

By Lemma 3 of Goginava and the method of the paper [8] we have that

$$\|T^{\kappa}_n(f) - f\|_C \leq$$

$$\leq c\log^2 n\left\{\omega_1\left(\frac{1}{n}, f\right)_C + \omega_2\left(\frac{1}{n}, f\right)_C + \sqrt{\omega_1\left(\frac{1}{n}, f\right)_C \omega_2\left(\frac{1}{n}, f\right)_C}\right\}.$$

This and (14) give that

$$|T^{\kappa}_{p_{n_k}}(f_{n_i}, 0, 0)| \leq \|T^{\kappa}_{p_{n_k}}(f_{n_i}) - f_{n_i}\|_C \leq c\frac{n_k^2 2^{2n_i}}{2^{2n_k}} \quad i = 1, 2, ..., k-1.$$

This implies that

$$II = O\left(\frac{n_k^2}{2^{2n_k}}\sum_{i=1}^{k-1}\frac{2^{2n_i}}{n_i^2}\right) = O\left(\frac{n_k^2}{2^{2n_k}}\frac{2^{2n_{k-1}}}{n_{k-1}^2}\right) = o(1) \quad \text{as } k \to \infty.$$

Since

$$\|\mathcal{F}^{\kappa}_n\|_L = O\left(\frac{1}{l_n}\sum_{l=1}^{n-1}\frac{\|D^{\kappa}_l\|_L^2}{n-l}\right) = O(\log^2 n)$$

and (3) hold, we write

$$III = O\left(\sum_{i=k+1}^{\infty}\frac{\|\mathcal{F}^{\kappa}_{p_{n_k}}\|_L}{n_i^2}\right) = O\left(\frac{n_k^2}{n_{k+1}^2}\right) = o(1) \quad \text{as } k \to \infty.$$

This completes the proof of this theorem.

Now, we use the function $g \in L(I^2)$ constructed by Gát and Goginava [8].

We choose a monotonically increasing sequence of positive integers $\{m_k : k \geq 1\}$ such that the condition (8) (that is, $2m_{k-1} \leq m_k$) and

$$\sum_{l=1}^{k-1} \frac{2^{2m_l}}{m_l^2} < \frac{2^{2m_k}}{m_k^2} \tag{15}$$

are satisfied. Set

$$g(x,y) := \sum_{j=1}^{\infty} g_j(x,y), \qquad \text{where} \qquad g_j(x,y) := \frac{D_{2^{2m_j+1}}(x) D_{2^{2m_j+1}}(y)}{m_j^2}.$$

In the article [8] it is proved that

$$\omega(\delta, g)_L = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Thus, the only thing we have to prove is that the function g is really a counterexample function for the Walsh-Kaczmarz system, too.

Proof of Theorem 7. To prove this theorem we use the analogue of the inequality (12). First, we investigate I. We have that

$$T_{p_{m_k}}^{\kappa}(g_j) = \frac{1}{m_j^2} S_{2^{2m_j+1},2^{2m_j+1}}(\mathcal{F}_{p_{m_k}}^{\kappa}) = \frac{\mathcal{F}_{p_{m_k}}^{\kappa}}{m_j^2} \quad \text{for } j \geq k.$$

Lemma 5 and condition (8) yield

$$I = \left\| \sum_{j=k}^{\infty} \frac{1}{m_j^2} S_{2^{2m_j+1},2^{2m_j+1}}(\mathcal{F}_{p_{m_k}}^{\kappa}) \right\|_L = \sum_{j=k}^{\infty} \frac{\|\mathcal{F}_{p_{m_k}}^{\kappa}\|_L}{m_j^2} \geq \frac{c}{m_k^2} \|\mathcal{F}_{p_{m_k}}^{\kappa}\|_L \geq c > 0$$

for k big enough.

Second, we discuss II. Using (8) we write

$$II \leq \sum_{j=k}^{\infty} \frac{1}{m_j^2} = O\left(\frac{1}{m_k^2}\right) = o(1) \quad \text{as } k \to \infty.$$

At last, we discuss III. From (11) it is easy to get that

$$\omega_1(\delta, g_l)_L = \omega_2(\delta, g_l)_L = O(2^{2m_l}\delta/m_l^2) \text{ for } l = 1, 2, ..., k-1,\ \ \delta > 0.$$

By Lemma 3 and

$$\omega_{1,2}\left(\frac{1}{i}, \frac{1}{i}, g_l\right)_L \leq 2\omega_1\left(\frac{1}{i}, g_l\right)_L$$

we write that

$$\|S_{i,i}^{\kappa}(g_l) - g_l\|_L \leq O\left(\frac{2^{2m_l}\log^2 i}{i m_l^2}\right)$$

and

$$\begin{aligned}\|T_{p_{m_k}}(g_l) - g_l\|_L &\leq \frac{c}{m_k}\sum_{i=1}^{p_{m_k}-1}\frac{\|S_{i,i}^{\kappa}(g_l) - g_l\|_L}{p_{m_k} - i} = \\ &= O\left(\frac{2^{2m_l}\log^2 p_{m_k}}{m_k m_l^2}\sum_{i=1}^{p_{m_k}-1}\frac{1}{i(p_{m_k} - i)}\right) = \\ &= O\left(\frac{2^{2m_l}\log^2 p_{m_k}}{2^{2m_k} m_l^2}\right).\end{aligned}$$

This and (15) immediately give that

$$\begin{aligned}III &\leq \sum_{l=1}^{k-1}\|T_{p_{m_k}}^{\kappa}(g_l) - g_l\|_L = O\left(\frac{\log^2 p_{m_k}}{2^{2m_k}}\sum_{l=1}^{k-1}\frac{2^{2m_l}}{m_l^2}\right) = \\ &= O\left(\frac{m_k^2}{2^{2m_k}}\frac{2^{2m_{k-1}}}{m_{k-1}^2}\right) = o(1) \quad \text{as } k \to \infty.\end{aligned}$$

Summarizing our results on I, II, III we conclude that

$$\overline{\lim_{k\to\infty}}\|T_{p_{m_k}}^{\kappa}(g) - g\|_L > 0.$$

This completes the proof of this theorem.

5. On the two-dimensoional Nörlund logarithmic means

We define the two-dimensional logarithmic means and kernels of rectangular partial sums by

$$\mathbf{t}^{\alpha}_{n,m}(f) := \frac{1}{l_n l_m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S^{\alpha}_{k,l}(f)}{(n-k)(n-l)},$$

$$\mathbf{F}^{\alpha}_{n,m} := \frac{1}{l_n l_m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{D^{\alpha}_{k,l}}{(n-k)(n-l)} = F^{\alpha,1}_{n} F^{\alpha,2}_{m}$$

($\alpha = w$ or κ). For the Walsh system this mean was discussed by Gát and Goginava in the article [7]. Now, we investigate the behavior of two-dimensional logarithmic means of rectangular partial sums with respect to the double Walsh-Kaczmarz system.

The two-dimensional logarithmic method can be given by the help of a positive rectangular matrix which satisfies regularity conditions (for more details see [7]).

Moreover,

$$\|\mathbf{t}^{\kappa}_{n,m}(f) - f\|_X \le \frac{1}{l_n l_m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{\|S^{\kappa}_{k,l}(f) - f\|_X}{(n-k)(n-l)},$$

where $X = C(I^2)$ or $L(I^2)$. These and Lemma 3 immediately give the following

Theorem 8. *Let either $X := C(I^2)$ or $X := L(I^2)$. Let $f \in X$ and*

$$\omega(\delta, f)_X = o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right),$$

then $\|\mathbf{t}^{\kappa}_{n,m}(f) - f\|_X \to 0$ as $n, m \to \infty$.

In this section we investigate the sharpness of this result. Namely, we state the following theorems:

Theorem 9. *There exists a function $f \in C(I^2)$ such that*

$$\omega(\delta, f)_C = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $\mathbf{t}^{\kappa}_{n,n}(f,0,0)$ *diverges.*

To prove Theorem 9 we use Lemma 2 and modify the counterexample function given in the article [7]. This is a mixed function of the functions in the previous sections.

We choose a monotonically increasing sequence of positive integers $\{n_k : k \geq 1\}$ such that the conditions (3) and (13) are satisfied. Set

$$f(x,y) := \sum_{k=1}^{\infty} \frac{f_{n_k}(x) f_{n_k}(y)}{n_k^2},$$

where f_{n_k} is defined in Theorem 3.

This function satisfies the conditions of our theorem. The proof of that fact, that f is a counterexample function for Walsh-Kaczmarz system, goes analogously to the proofs of Theorems 3 and 6 (for more details see [7]). Therefore, it is left to the reader.

Theorem 10. *There exists a function* $g \in L(I^2)$ *such that*

$$\omega(\delta, g)_L = O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $\mathbf{t}^{\kappa}_{n,n}(g)$ *does not converge to* g *in* L*-norm.*

The function g constructed in the proof of Theorem 7 with a monotonically increasing sequence of positive integers $\{m_k : k \geq 1\}$ which satisfies the conditions (8) and (15) will be good. The proof goes analogously to the proofs of Theorems 4 and 7 (for more details see the article [7]). Therefore, it is left to the reader. At last, we note that the proof of this theorem is based on Lemma 2.

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