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RECENT EXTENSIONS OF MONOTONE SEQUENCES AND SOME APPLICATIONS

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Dedicated to Professor Ferenc Schipp, for his 70th birthday,
and to Professor Péter Simon, for his 60th birthday

Abstract. Recently several new kinds of sequences were introduced for extending a lot of classical results in Fourier analysis. To the best of our knowledge, the class of sequences of rest bounded variation was the first one defined to extend the notion of monotonicity into a new direction utilizing two specific properties of the decreasing null-sequences. The aim of the present writing is to give a short survey on the progress made newly in this field.

Novel definitions and their introducers with some applications

It is well known that there are several interesting classical theorems in Fourier analysis having assumptions determined by certain monotonicity of the coefficients; and many of them have been generalized in different ways.

In this paper we intend to recall some recent results pertaining to one type of the generalizations, which has been initiated, by one of my fortunate observation, at least in my view.

The crucial idea has appeared already in 1997 in the papers [8] and [9], where we defined a new subclass H_S^ω of the classical function class H^ω and studied its embedding relations in connection with a known function class $S_p(\underline{\lambda})$ defined by the partial sums of Fourier series.

In the class H_S^ω we considered sine series with coefficients satisfying the conditions: $c_n \to 0$ and

$$(D1) \qquad \sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c}) c_m$$

for all $m \in \mathbb{N}$. Such a sequence $\mathbf{c} := \{c_n\}$ we called a *rest bounded variation sequence* (in symbol, $\mathbf{c} \in RBVS$).

This notion has not appeared again up to 2001.

Then in [10], among others, we generalized the classical theorem of *Chaundy* and *Jolliffe* [5] (1916) replacing the *monotonic null-sequence* (MS) coefficients by a sequence $\mathbf{b} \in RBVS$.

The new theorem reads as follows:

Theorem 1. *If a sequence* $\mathbf{b} := \{b_n\}$ *belongs to the class* $RBVS$*, then the condition* $nb_n \to 0$ *as* $n \to \infty$ *is both necessary and sufficient for the uniform convergence of series*

$$(1) \qquad g(x) := \sum_{n=1}^{\infty} b_n \sin nx.$$

In [11] (2002), we answered a problem due to *S.A. Telyakovskiĭ* (personal communication): Is the class $RBVS$ *comparable* to the class of *classical quasi-monotone sequences*, ($CQMS$), that is, if

$$(D2) \qquad b_{n+1} \le b_n \left(1 + \frac{\alpha}{n}\right) \quad (\text{or equivalently if } n^{-\beta} b_n \downarrow 0,\ \beta > 0)$$

holds for some constant $\alpha > 0$ and all $n \ge n_0(\alpha)$?

This problem has its interest in the fact that *S.M. Shah* [21] (1962) verified that the result of *Chaundy* and *Jolliffe* remains valid if the monotonicity condition in it is replaced by condition (D2).

The Telyakovskiĭ's problem implicitly includes the question, which result is better, that of Shah or ours.

We proved that *these classes are not comparable*. To verify this assertion we gave the following sequences:

The sequence $\mathbf{d} := \{d_n\}$ with $\mu_m := 2^{2^m}$ and

$$d_n := \begin{cases} 1/m^2\mu_{m+1}, & \text{if} \quad n = \mu_m, \\ d_{\mu_m} \prod_{\ell=\mu_m}^{n-1} \left(1 + \frac{1}{\ell}\right), & \text{if} \quad \mu_m + 1 \le n \le m\mu_m, \\ d_{m\mu_m}, & \text{if} \quad m\mu_m < n < \mu_{m+1} \end{cases}$$

belongs to $CQMS$, but $\mathbf{d} \notin RBVS$.

On the other hand the sequence $\mathbf{c} := \{c_n\}$ with $0 < q < 1/2,\ c_1 = 1$ and

$$c_n := \begin{cases} q^m, & \text{if} \quad 2^{m-1} < n < 2^m, \\ 2q^m, & \text{if} \quad n = 2^m, \end{cases} \qquad m = 1, 2, \dots,$$

belongs to $RBVS$, but $\mathbf{c} \notin CQMS$.

Then $nd_n \to 0$ and $nc_n \to 0$. This result almost *prescribes* to consider the theorems utilizing the classical quasi-monotone sequences ($CQMS$) as an assumption and to try to prove analogous theorems for $RBVS$equences.

Essentially, the "*carrier*" of the sequences $RBVS$ has launched by the result that *the classes $RBVS$ and $CQMS$ are not comparable.*

In [11] three theorems having this character are proved. We recall only one which gives a certain answer to Question 6.12 of *R.P. Boas* raised in [2].

Theorem 2. *If $1 < p < \infty$ and $\underline{\lambda} := \{\lambda_n\} \in RBVS$ then $x^{-\gamma}\varphi(x) \in \in L^p(0,\pi),\ (1/p) - 1 < \gamma < 1/p$, if and only if*

$$\sum_{n=1}^{\infty} n^{p\gamma+p-2}\lambda_n^p < \infty,$$

where φ stands for f or g and the λ_n are its associated Fourier coefficients, moreover

$$f(x) := \sum_{n=1}^{\infty} a_n \cos nx. \tag{2}$$

In [12] (2004) we defined the *head bounded variation sequences*, by analogy with the rest bounded variation sequences, and showed that the classical monotonicity conditions can be moderated in four theorems of *P. Chandra* [3], [4]. These theorems deal with the order of approximation.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers is called of *Head Bounded Variation*, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$(D3) \qquad \sum_{n=0}^{m-1} |c_n - c_{n+1}| \le K(\mathbf{c})c_m$$

for all $m \in \mathbb{N}$.

In the same year we ([13]) generalized the definition (D1) with a view to broaden the class of $RBVS$. Our new definition reads as follows:

Let $\underline{\gamma} := \{\gamma_n\}$ be a fixed sequence of positive numbers. We say that a nonnegative null-sequence $\mathbf{c}$ belongs to the class $\gamma RBVS$ if

$$(D4) \qquad \sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c}, \underline{\gamma})\gamma_m$$

holds for all $m \in \mathbb{N}$, in symbol $\mathbf{c} \in \gamma RBVS$, and call $\mathbf{c}$ a $\underline{\gamma}$ *rest bounded variation sequence.*

It is clear that (D4) implies that $c_n \le K(\mathbf{c})\gamma_m$ $(n \ge m)$, but it does not exhibit that $\mathbf{c} \in AMS$, where AMS (or QMS) denotes the set of *almost* (*quasi*) *monotonic sequences*, that is, if

$$(D5) \qquad (0 \le) c_n \le K(\mathbf{c})c_m, \quad \text{for} \quad n \ge m.$$

Namely we also showed that *the classes AMS and $\gamma RBVS$ are not comparable.* The following example shows this.

Let

$$c_n := 2^{-m} + (-1)^n 2^{-m-1}, \quad \text{if} \quad 2^m \le n < 2^{m+1}.$$

Then (D5) holds with $K(\mathbf{c}) = 8$, thus $\mathbf{c} \in AMS$, but

$$\sum_{n=2^m}^{2^{m+1}} |c_n - c_{n+1}| \ge \frac{1}{2} \quad \text{for any} \quad m,$$

therefore (D4) does not hold.

We mention another new property of the sequence $\mathbf{c} \in \gamma RBVS$. If one $c_k = 0$, then (D4) does not imply that all of the terms with index $n > k$ are zero, too, while (D1) and (D5) claim this.

By all means (D4) gives much greater freedom for the sequence $\mathbf{c}$ than (D1) does, consequently, in general, $\gamma RBVS$ is a larger class than $RBVS$; they are identical only if $\gamma_n = Kc_n$.

By launching the notion $\gamma RBVS$ we have intended to generalize the theorems having conditions with sequences belonging to the classes *MS, AMS, CQMS* or *RBVS*. But using the classes $\gamma RBVS$ with suitable sequences $\underline{\gamma}$, it turned out that the plenty of rope of the sequences $\mathbf{c} \in \gamma RBVS$, it is not possible. Consequently we could prove generalization of this type only for the sufficient part of the known theorems.

This *shortcoming* gave the next task to find such a class of sequences whose terms yield *necessary and sufficient conditions*, too.

But before recalling this notion we cite some definitions given by *Le, Yu, Zhou, Tikhonov* and me motivated by the class $RBVS$.

The next new class wider than $RBVS$ was defined by *Le* and *Zhou* [7] (2005). The class of sequences satisfying their condition is an artful generalization of our class $RBVS$. Their definition reads as follows:

If there exists a natural number N such that

$$(D6) \qquad \sum_{n=m}^{2m} |c_n - c_{n+1}| \le K(\mathbf{c}) \max_{m \le n < m+N} c_n$$

holds for all m, then we say that the sequence $\mathbf{c}$ belongs to class $GBVS$, that is, $\mathbf{c}$ is a *sequence of Group Bounded Variation*.

Le and *Zhou* also showed that our Theorem 1 can be improved, replacing the class $RBVS$ by the wider class $GBVS$. This is a very nice result, and its proof is very skillful. They also verified that *the class $GBVS$ is wider than the class $CQMS$*, thus they proved that $RBVS \cup CQMS \subset GBVS$ also holds. We have seen $RBVS$ is not wider than $CQMS$.

After appearing the crucial paper of Le and Zhou we also analyzed the embedding relations of this new class $GBVS$ and some others. In [15] (2007, but received July 2005), we analyzed the relationships of three well-known and four recently defined classes of numerical sequences.

Now we recall two definitions have not been given earlier.

We say that a sequence $\mathbf{c}$ is *locally quasi monotone* ($LQMS$) if (D5) holds for $m \le n \le 2m,\ m = 1, 2, \dots$.

A sequence $\mathbf{c}$ is called *quasi β-power monotone* ($Q\beta MS$) if for some positive β and $n \ge m$

$$(D5^*) \qquad n^\beta c_n \le K(\mathbf{c}) m^\beta c_m$$

holds.

First we raised the question: What additional condition on the sequence $\mathbf{c}$ implies that for such sequences the classes $RBVS$ and $GBVS$ are equivalent? The answer is *the quasi β-power-monotonicity*, namely we proved that

$$GBVS \cap Q\beta MS \subset RBVS \ (\subset GBVS).$$

We also verified that the class $GBVS$ is *not comparable* to the classes $QMS, LQMS$ and $Q\beta MS$, that is, using the notation $\not\sim$,

$$QMS \not\sim GBVS, \quad LQMS \not\sim GBVS \quad \text{and} \quad Q\beta MS \not\sim GBVS.$$

Furthermore we also verified the following class-relations:

$$MS \not\sim Q\beta MS, \quad RBVS \not\sim Q\beta MS, \quad QMS \not\sim CQMS, \quad CQMS \not\sim Q\beta MS.$$

The following embedding relations follow easily from the definitions.

$$MS \subset RBVS \subset QMS \subset LQMS, \quad MS \subset CQMS, \quad MS \subset GBVS,$$

$$CQMS \subset LQMS, \quad Q\beta MS \subset QMS, \quad Q\beta MS \subset LQMS,$$

$$RBVS \subset GBVS.$$

In [17] (2006) unifying the advantages of the definitions (D4) and (D6) we defined a further new class of sequences, denoted by $\gamma GBVS$, which is wider than any of the classes $GBVS$ and $\gamma RBVS$.

A null-sequence $\mathbf{c}$ *belongs to* $\gamma GBVS$ *if*

$$(D7) \qquad \sum_{n=m}^{2m} |\Delta c_n| \le K(\mathbf{c}, \underline{\gamma})\gamma_m, \quad (\Delta c_n = c_n - c_{n+1})$$

holds, where $\underline{\gamma}$ is a given sequence of nonnegative numbers. We underline that in (D7) the sequence $\underline{\gamma}$ may have *infinitely many zero terms*, too; but not in (D4).

For academic fairness we write that in [23] (2007, available on line 18 April 2006) *S. Tikhonov* introduced his main definition as follows: a complex $\mathbf{c} := \{c_n\}$ is said to be *general monotone*, or $\mathbf{c} \in GM$, if

$$(D8) \qquad \sum_{n=m}^{2m-1} |\Delta c_n| \le K(\mathbf{c})|c_m|.$$

If $c_n \ge 0$ then (D8) is almost (D6) with $N = 1$.

In his paper Tikhonov presented several interesting properties of the GM-sequences, and studied pointwise and L_1-convergence, among others. Finally, in the same paper, he defined a generalization of GM as follows:

Let $\underline{\beta} := \{\beta_n\}$ be a positive sequence. The sequence of complex numbers $\mathbf{c} := \{c_n\}$ is said to be β-*general monotone*, or $\mathbf{c} \in GM(\underline{\beta})$, if

$$(D9) \qquad |c_m| + \sum_{n=m}^{2m-1} |\Delta c_n| \le K(\mathbf{c}, \underline{\beta})\beta_n.$$

I dare state that recently it has become a popular topic how to generalize monotonicity, and generalize classical theorems in different themes for these new sequences.

If in (D4) $\gamma_n := \sum_{k=n}^{\infty} |\Delta c_k|$ then we get the *class* BVS_0 introduced in [14] (2006), that is, $\mathbf{c} \in BVS_0$ if

$$(D10) \qquad c_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |\Delta c_n| < \infty.$$

It is clear that a sequence $\mathbf{c} \in BVS_0$ may have zero and negative terms as well. This freedom is the reason that we could generalize only two theorems of *A.A. Konyushkov* [6] for sequences $\mathbf{c} \in BVS_0$, but with $\mathbf{c} \in RBVS$ we could establish five ones.

The next new class of sequences was defined first in [16] (2008) (received June 30, 2006), but the paper [18] using these sequences appeared sooner in 2007.

The reason of the new definition is the fact that there are many interesting results involving no restriction on the speed of the decrease of the coefficients of trigonometric series. In such cases we cannot use conditions of $\gamma RBVS$ or $\gamma GBVS$ type with a given sequence $\underline{\gamma}$, or we can prove only sufficient type results for these classes. Therefore we introduced the class of *mean rest bounded variation sequences*, where $\underline{\gamma}$ is defined by a certain arithmetical mean of the coefficients, e.g.,

$$(D11) \qquad \tilde{\gamma}_m := \frac{1}{m} \sum_{n=m}^{2m-1} c_n \quad \text{or} \quad \gamma_m^* := \frac{1}{m} \sum_{n \ge m/2}^{m} c_n.$$

It is easy to see that the class $\underline{\gamma}^* RBVS$ includes the class $RBVS$, consequently the almost monotone and monotone sequences, too; but $\tilde{\gamma} RBVS$ does not, in general (see e.g., $c_n = 2^{-n}$).

In spite of this we have used the class $\tilde{\gamma}RBVS$, because this is more convenient in applications, and used the notation $MRBVS$ for $\tilde{\gamma}RBVS$. We emphasize that the sequences of the class $MRBVS$ may have many zero terms, too.

Soon after defining the class $MRBVS$ we ([19] (2007)) defined the class $MGBVS$ as a subclass of $\gamma GBVS$, choosing

$$\gamma_m := \frac{1}{m} \sum_{n=m}^{2m-1} |c_n|.$$

The aim of these definitions were to produce necessary and sufficient conditions for the uniform convergence and boundedness if the coefficients of the sine series (1) belong to $MGBVS$.

In [19]) we proved, e.g., that *if* $b_n \geq 0$ *and* $\mathbf{b} := \{b_n\} \in MGBVS$*, then the sine series* (1) *is uniformly convergent if and only if*

$$B_m := \sum_{n=m}^{2m-1} b_n \to 0.$$

We dare say that this result is a radical generalization of the classical *Chaundy* and *Jolliffe* theorem, and many others proved recently.

If $B_n = O(1)$, we get *a necessary and sufficient condition for the uniform boundedness of* (1).

A generalization of the class $MGBVS$ was given by *D.S. Yu, P. Zhou* and *S.P. Zhou* in [24] (2007) as follows:

A nonnegative sequence $\mathbf{c} := \{c_n\}$ is said to be a *mean value bounded variation sequence* (in symbol $\mathbf{c} \in MVBVS$) if there is a $\lambda \geq 2$ such that

$$(D12) \qquad \sum_{n=m}^{2m} |\Delta c_n| \leq \frac{K(\mathbf{c})}{m} \sum_{n=[\lambda^{-1}m]}^{[\lambda m]} c_n.$$

Their main result is a generalization of our Theorem 2, namely they replace our assumption $\underline{\lambda} \in RBVS$ by the weaker condition $\underline{\lambda} \in MVBVS$.

D.S. Yu and *S.P. Zhou* [25] (2007) introduced a further *new class of sequences, called as* $NBVS$, to generalize $GBVS$. Essentially they extended the monotonicity from "one sided" to "two sided".

If $\mathbf{c} := \{c_n\}$ is a nonnegative sequence tending to zero and

$$(D13) \qquad \sum_{n=m}^{2m} |\Delta c_n| \leq K(\mathbf{c})(c_m + c_{2m}),$$

then we say $\mathbf{c}$ belongs to *the class* $NBVS$ ($TSGBVS$).

It is clear that if $\mathbf{c} \in GBVS$ (with $N = 1$) then $\mathbf{c} \in NBVS$, but the converse is not true. They proved that there exists a null-sequence $\mathbf{c}$ such that $\mathbf{c} \in NBVS$, but $\mathbf{c} \notin GBVS$ if $N = 1$. In their paper there are five theorems using the class $NBVS$ instead of $GBVS$.

We have cited 16 definitions, but as their applications, we have recalled only some results from the more than 100 theorems have been proved with sequences of the classes defined lately.

Finally we present two theorems having special feature. One of them is connected with the important *Sidon-Telyakovskiĭ class*, the other has an embedding character and it is a very sharp result.

A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class $\mathbf{S}$, in symbol : $\mathbf{a} \in \mathbf{S}$, if there exists a monotonically decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad |\Delta a_n| \leq A_n, \quad n \in \mathbb{N}.$$

In [20, 13] we proved:

Theorem 3. *(i) If* $\mathbf{a} := \{a_n\} \in MGBVS$ *and*

$$\mathbf{A} := \sum_{n=1}^{\infty} a_n < \infty,$$

then $\mathbf{a} \in \mathbf{S}$.

(ii) If $\mathbf{A} = \infty$*, then* $\mathbf{a} \in MRBVS \subset MGBVS$ *does not imply that* $\mathbf{a} \in \mathbf{S}$ *even under the additional condition*

$$\sum_{k=n}^{2n} a_k \to 0.$$

(iii) The assumptions $\mathbf{A} < \infty$ *and* $\mathbf{a} \in \mathbf{S}$ *do not imply* $\mathbf{a} \in MGBVS$.

Using the idea of *A.S. Belov* in a joint paper [1] we proved

Theorem 4. *Let* $0 < p < \infty$ *and* $\omega(\delta)$ *be a modulus of continuity. If* $\underline{\lambda} := \{\lambda_n\}$ *is an arbitrary sequence of nonnegative numbers, then the embedding relation*

$$H_{S,\underline{\gamma}} \subset S_p(\underline{\lambda})$$

holds if and only if

$$\Lambda_n^{1/p}\omega\left(\frac{1}{n}\right) \le K(\underline{\lambda}) \quad \textit{with} \quad \Lambda_n := \sum_{k=1}^{n} \lambda_k,$$

where

$$H_{S,\underline{\gamma}} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx \quad \textit{and} \quad \{b_n\} \in \gamma RBVS \right\}$$

with $\underline{\gamma} := \{\gamma_n\},\ \gamma_n := n^{-1}\omega(n^{-1})$ *and*

$$S_p(\underline{\lambda}) := \left\{ f : \left\| \sum_{n=1}^{\infty} \lambda_n |f - S_n|^p \right\|_C < \infty \right\}.$$

Very recently I have perceived a further new class of sequences introduced by *B. Szal* [22]. His definition reads as follows:

A null-sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers is called the *rest bounded* **second** *variation sequence*, or briefly $\mathbf{c} \in RBSVS$, if it has the property

$$(D14) \qquad \sum_{n=m}^{\infty} |c_n - c_{n+2}| \le K(\mathbf{c})c_m, \quad \text{for} \quad n \in \mathbb{N}.$$

He proved, among others, that there exists a sequence, which belongs to the class $RBSVS$, but it does not belong to $RBVS$ and QMS, furthermore $RBVS \subset RBSVS$.

This new class useful to prove theorems of sufficient type, namely, the condition

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} |a_k - a_{k+2}| \right)^p < \infty$$

implies that series (2) is L^p-integrable ($f \in L^p,\ 1 < p < \infty$) (see the cited book of Boas).

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