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CONSTRUCTION OF WALSH-LIKE SYSTEMS

B. Király (Pécs, Hungary)

Dedicated to
Professor Ferenc Schipp *on the occasion on his* 70*th birthday*
and to
Professor Péter Simon *on the occasion on his* 60*th birthday*

Abstract. We give a general way to construct a Walsh-like biorthogonal system. With this system we can give an efficient interpolation algorithm. The Walsh-Fourier coefficients with respect to this product system can be computed by a generalized Fast Fourier Transform.

1. Introduction

The Rademacher functions r_n $(n \in \mathbb{N})$ can be derived from the basic function r by dilation:

$$r_n(x) := r(2^n x) \quad (x \in [0,1), n \in \mathbb{N}),$$

$$r(x) := \begin{cases} 1, & x \in [k,\ k+\frac{1}{2}),\ k \in \mathbb{Z}, \\ -1, & x \in [k+\frac{1}{2},\ k+1),\ k \in \mathbb{Z}. \end{cases} \tag{1.1}$$

The Walsh system is the product system of the Rademacher system, i.e.

$$w_m = \prod_{k=0}^{\infty} r_k^{m_k}, \quad m = \sum_{k=0}^{\infty} m_k 2^k, \quad m_k \in \{0,\ 1\}. \tag{1.2}$$

The w_m $(m \in \mathbb{N})$ Walsh system is a complete orthonormal system with respect to the scalar product

$$\langle f, g\rangle = \int_0^1 f(t) \cdot \overline{g}(t) dt. \tag{1.3}$$

The subsystem w_m $(m < 2^N)$ is orthonormal with respect to the discrete scalar product

$$[f, g]_{X_N} = \frac{1}{2^N} \sum_{x \in X_N} f(x) \cdot \overline{g}(x), \tag{1.4}$$

where X_N is the set of discretization, defined by

$$X_N = \left\{ \frac{k}{2^N} : \; 0 \leq k < 2^N \right\}. \tag{1.5}$$

The discrete Walsh system is also complete with respect to the set of the discrete functions $f : X_N \to \mathbb{R}$.

Using the fast Walsh transform (FWT) the discrete Walsh-Fourier-coefficients (DWFC) $c_n(f) = [f, w_n]_{X_N}$ $(0 \leq n < 2^N)$ can be computed by $\mathcal{O}(N \cdot 2^N)$ operations. To reconstruct the discrete function $f : X_N \to \mathbb{R}$ from the DWFC we need the same number of operations, i.e. the number of operations is of the same order as in the case of DFT using FFT.

2. Generalized product system

In this paper our aim is the generalization of the construction of the Walsh system. This is important because in this way we can obtain fast algorithms for the computation of the coefficients and we can obtain FFT-like transforms.

To generalize the construction of the Walsh system first we give the generalization of the basic function of the Rademacher system. For this we will need the following definitions.

Denote $\mathcal{I}$ an interval in $\mathbb{R}$. The map $A : \mathcal{I} \to \mathcal{I}$ is called twofold, if for every $y \in \mathcal{I}$ there exists exactly two points $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ such that

$$A(x_1) = A(x_2) = y. \tag{2.1}$$

The function $\varphi^{(1)}$, $\left[\varphi^{(0)}\right]$ *is called odd* [*even*] *with respect to the map* A, *if* $A(x') = A(x'') = y$ *and* $x' \neq x''$ *imply*

$$\varphi^{(j)}(x') = (-1)^j \varphi^{(j)}(x'') \quad (j = 0,\ 1). \tag{2.2}$$

Let

$$A_0(x) := x, \quad A_{n+1}(x) := A(A_n(x)) \qquad (x \in \mathcal{I}, n \in \mathbb{N}). \tag{2.3}$$

It is easy to see that $A_n : \mathcal{I} \to \mathcal{I}$ is a 2^n-fold map.

The functions $\varphi_n^{(1)}(x) := \varphi^{(1)}(A_n(x))$ $(x \in \mathcal{I}, n \in \mathbb{N})$ are called *Rademacher-like functions.* The twofold map A is the analogue of the dilation in the original Rademacher-construction and the function $\varphi^{(1)}$ is the analogue of the basic function r.

We can generalize the definition of the product system. Let us start from the following finite collection of the systems

$$\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}\}, \quad n = 0,\ 1, \ldots,\ N-1. \tag{2.4}$$

Then

$$\Phi_\ell := \prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot 2^k$$

is the generalized product system of the systems (2.4). It is really the generalization of idea of the product system, because in special case when $\varphi_m^{(0)} = 1$ and $\varphi_m^{(1)} = r_m$ we reobtain the Walsh system.

Let denote by Φ_ℓ $(0 \le \ell < 2^N - 1)$ the product system of

$$\phi_n \quad (0 \le n < N-1)$$

and by Ψ_ℓ $(0 \le \ell < 2^N - 1)$ the product system of

$$\Upsilon_n = \{\psi_n^{(0)},\ \psi_n^{(1)}\} \quad (0 \le n < N-1).$$

Then the mixed Dirichlet kernel of the product systems is defined by

$$D_{2^N}(s,t) := \sum_{\ell=0}^{2^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t). \tag{2.5}$$

From the definition of the product system we get the following explicit form for the mixed Dirichlet kernel:

$$D_{2^N}(s,t) = \sum_{\ell=0}^{2^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t) = \sum_{\ell=0}^{2^N-1} \prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}(s) \cdot \overline{\psi}_k^{(\ell_k)}(t) =$$

$$= \prod_{k=0}^{N-1} \left(\varphi_k^{(0)}(s) \cdot \overline{\psi}_k^{(0)}(t) + \varphi_k^{(1)}(s) \cdot \overline{\psi}_k^{(1)}(t) \right).$$

In this paper we investigate systems of the form

$$\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}\}, \quad \varphi_n^{(j)}(x) := \varphi^{(j)}(A_n(x)) \quad (j = 0, 1, x \in \mathcal{I}). \tag{2.6}$$

Restricting the functions to the set

$$Y_N := \{x \in \mathcal{I} : A_N(x) = x_0\} = \{x_k^N : k = 0, 1, \cdots, 2^N - 1\} \tag{2.7}$$

we can discretize the generalized product system.

Let us denote by Φ_ℓ $(0 \le \ell < 2^N)$ the generalized product system of the system $\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}\}$ $(0 \le n < N)$, defined by 2.6. Assume that $\varphi^{(0)}(x) \neq 0$ and $\varphi^{(1)}(x) \neq 0$, if $x \in Y_N$ and restart the construction from the finite set of functions

$$\Upsilon := \{\psi^{(0)},\ \psi^{(1)}\} = \left\{ \frac{1}{\overline{\varphi}^{(0)}},\ \frac{1}{\overline{\varphi}^{(1)}} \right\}. \tag{2.8}$$

Let us denote by Ψ_ℓ $(0 \le \ell < 2^N - 1)$ the generalized product system of system

$$\Upsilon_n = \{\psi_n^{(0)},\ \psi_n^{(1)}\} \quad (\psi_n^{(j)}(x) = \psi^{(j)}(A_n(x)), \quad x \in \mathcal{I}, \quad 0 \le n < N-1), \tag{2.9}$$

i.e.

$$\Psi_\ell := \prod_{k=0}^{N-1} \psi_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot 2^k. \tag{2.10}$$

Theorem 2.1. *The generalized product system* $\{\Phi_\ell,\ 0 \le \ell < 2^N - 1\}$ *defined by (2.6) and the generalized product system* $\{\Psi_\ell,\ 0 \le \ell < 2^N - 1\}$ *given by (2.10) are biorthogonal with respect to the discrete scalar product of the set* Y_N*, i.e.*

$$[\Phi_\ell, \Psi_m]_{Y_N} = \delta_{\ell m} \quad (0 \le \ell,\ m < 2^N).$$

Proof. The mixed Dirichlet kernel of the generalized product systems is

$$D_{2^N}(s,t) := \sum_{m=0}^{2^N-1} \Phi_m(s)\cdot\overline{\Psi}_m(t) \quad (s,t\in Y_N). \tag{2.11}$$

Then the biorthogonality of the systems is equivalent to the following property of the Dirichlet kernel:

$$D_{2^N}(s,t) = 2^N\delta_{s,t} \quad (s,t\in Y_N). \tag{2.12}$$

By the definition of the generalized product system and by (2.8) we have

$$\begin{aligned} D_{2^N}(s,t) &= \sum_{m=0}^{2^N-1} \Phi_m(s)\cdot\overline{\Psi}_m(t) = \\ &= \prod_{k=0}^{N-1}\left(\varphi_k^{(0)}(s)\cdot\overline{\psi}_k^{(0)}(t) + \varphi_k^{(1)}(s)\cdot\overline{\psi}_k^{(1)}(t)\right) = \\ &= \prod_{k=0}^{N-1}\left(\varphi_k^{(0)}(s)\cdot\frac{1}{\varphi_k^{(0)}(t)} + \varphi_k^{(1)}(s)\cdot\frac{1}{\varphi_k^{(1)}(t)}\right). \end{aligned} \tag{2.13}$$

In the case when $s=t$, we have

$$\varphi_m^{(0)}(s)\cdot\frac{1}{\varphi_m^{(0)}(t)} = \varphi_m^{(1)}(s)\cdot\frac{1}{\varphi_m^{(1)}(t)} = 1 \tag{2.14}$$

for every m $(0\le m<2^N)$, from this it follows that $D_{2^N}(s,s)=2^N$.

If $s\neq t$, then there exists a first common preimage x and a number p $(0\le \le p<N)$ such that

$$x' = A_p(s) \neq A_p(t) = x'', \tag{2.15}$$

and in the same time $x = A_{p+1}(s) = A_{p+1}(t)$.

We know that $\varphi^{(0)}$ is even with respect to the map A, so $\varphi^{(0)}(x')=\varphi^{(0)}(x'')$ and $\varphi^{(1)}$ is odd with respect to the map A, so $\varphi^{(1)}(x')=-\varphi^{(1)}(x'')$ and from this it follows that

$$\begin{aligned} &\varphi_p^{(0)}(s)\cdot\frac{1}{\varphi_p^{(0)}(t)} + \varphi_p^{(1)}(s)\cdot\frac{1}{\varphi_p^{(1)}(t)} = \\ &= \varphi^{(0)}(A_p(s))\cdot\frac{1}{\varphi^{(0)}(A_p(t))} + \varphi^{(1)}(A_p(s))\cdot\frac{1}{\varphi^{(1)}(A_p(t))} = \\ &= \varphi^{(0)}(x')\cdot\frac{1}{\varphi^{(0)}(x'')} + \varphi^{(1)}(x')\cdot\frac{1}{\varphi^{(1)}(x'')} = 1-1=0. \end{aligned} \tag{2.16}$$

This means that there exists a zero factor in the product of (2.13), which implies that $D_{2^N}(s,t) = 0$ if $s \neq t$. ■

3. Interpolation formula and FFT

Let us define by

$$S_m f := \sum_{n=0}^{m-1} [f, \Psi_n]_{Y_N} \Phi_n \tag{3.1}$$

the m-th partial sum of the biorthogonal expansion of the function $f : \mathcal{I} \to \mathbb{C}$.

Theorem 3.1. *Then $S_{2^N} f$ partial sums interpolate the function f in the points of Y_N, i.e.*

$$(S_{2^N} f)(x) = f(x), \quad x \in \ Y_N. \tag{3.2}$$

Proof. Using the definition of the discrete scalar product and the property of the mixed Dirichlet kernel proved in the theorem we obtain that for every $x \in Y_N$:

$$S_N f(x) = \sum_{m=0}^{2^N-1} [f, \Psi_m]_{Y_N} \phi_m(x) = \sum_{m=0}^{2^N-1} \left(\frac{1}{2^N} \sum_{y \in Y_N} f(y) \overline{\Psi}_m(y) \right) \phi_m(x) =$$

$$= \frac{1}{2^N} \sum_{y \in Y_N} f(y) \sum_{m=0}^{2^N-1} \phi_m(x) \overline{\Psi}_m(y) = \frac{1}{2^N} \sum_{y \in Y_N} f(y) D_{2^N}(x,y) = f(x).$$

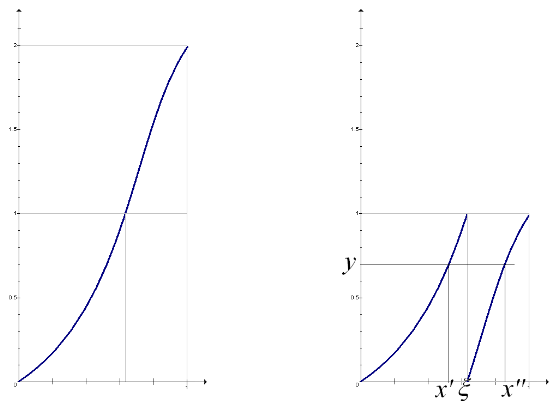
■

The Fast Walsh Transform can be used similarly to the case of UDMD systems (published in [6]). In this way we need $\mathcal{O}(N \cdot 2^N)$ operation to compute the coefficient of the biorthogonal expansion and to reconstruct the function. We remark that $\mathcal{O}(2^{2N})$ operations are required computing directly by definition.

4. Examples

The construction of the biorthogonal systems presented before is very general. Choosing special ϕ we can reobtain well known orthonormal and biorthogonal systems and we can also construct new biorthogonal systems with adaptive properties to the examined problem.

1. Let $\gamma : [0,\ 1) \to [0,\ 2)$ strictly monotone continuous bijective function and $\gamma_1(x) := \{\gamma(x)\}$, where $\{a\}$ denotes the fractional part of a. Sometimes we denote the function γ_1 with $\gamma_1(x) = \gamma(x)$ **mod** 1. The function $\gamma_1 : [0,\ 1) \to [0,\ 1)$ is a twofold map on the interval $\mathcal{I} = [0,\ 1)$.



function γ function γ_1

To develop the general construction in this special case we need an even function $\varphi^{(0)}$ and an odd function $\varphi^{(1)}$ with respect the twofold map γ_1. Let $\xi \in [0,\ 1)$ such that $\gamma(\xi) = 1$ and denote $\mathcal{I}_1$ the interval $[0,\ \xi)$ and $\mathcal{I}_2$ the interval $[\xi,\ 1)$. Then knowing the values of functions $\varphi^{(0)}$ and $\varphi^{(1)}$ on the interval $\mathcal{I}_1$ determines the values of these functions on the interval $\mathcal{I}_2$.

Specially using the function $\varphi^{(0)}(x) = 1$ and the function

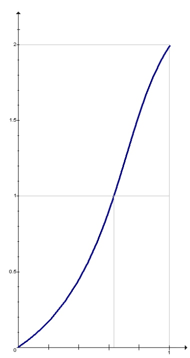
$$\varphi^{(1)}(x) := \begin{cases} 1 & 0 \le x < \xi, \\ -1 & \xi \le x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

we get a biorthogonal system.

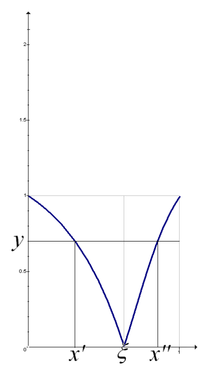
In the special case of $\gamma(x) = 2x$ the twofold map γ_1 is a dilation and in this special case we reobtain the Rademacher system and as its product system, the Walsh system.

- With this construction it is easy to give an adaptive interpolation method. Assume that we are interested in the properties of the function in the second part of the interval $\mathcal{I}$. Then we need a discrete set Y_N with higher density of points in the second part of $\mathcal{I}$ and lower density of points in the first part. If the twofold map γ_1 is derived from function γ as we have presented before and $\xi >> \frac{1}{2}$, then the density of points in $\mathcal{I}_2$ is much higher, because Y_N has the same number of points in $\mathcal{I}_1$ and $\mathcal{I}_2$.

2. Let $\gamma : [0,\ 1) \rightarrow [0,\ 2)$ strictly monotone continuous bijective function and $\gamma_1(x) := |\gamma(x) - 1|$.



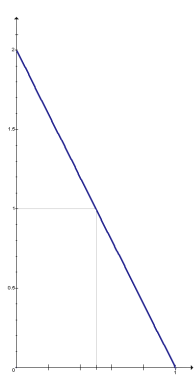
function γ



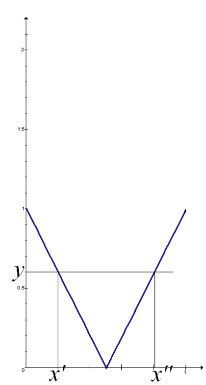
function γ_1

We can choose function ϕ similarly to the previous case.

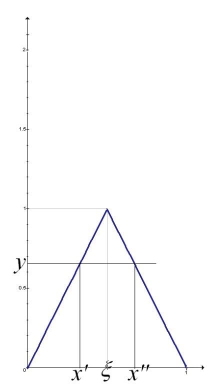
• In the special case of $\gamma(x) = 1 - 2x$, $\gamma_1(x) := |\gamma(x) - 1|$ is a twofold map on $(0,\ 1]$, but starting from $\gamma_2 := 1 - \gamma_1$ we get the Basic-spline B_1.



function $\gamma(x) = 1 - 2x$



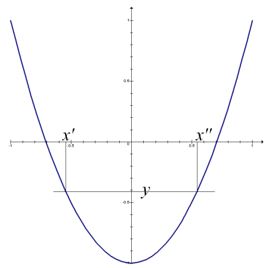
function $\gamma_1(x)$



function $\gamma_2(x) = B_1(2x)$

Using this function for the construction we get a system of spline functions.

3. Let $\mathcal{I} = (-1,\ 1)$, then $A(x) = 2x^2 - 1$ is a twofold map on $\mathcal{I}$ and by using $\varphi^0(x) = 1$ and $\varphi^{(1)}(x) = \varphi(x) = x$ we get the Walsh-Chebishev system, which was introduced in [2].



$T_2(x) = 2x^2 - 1$

In the mentioned article we got the function

$$\varphi_k(x) = \varphi(A_k(x)) = T_{2^k}(x) = \cos(2^k \arccos x).$$

Using the construction of the theorem we can get the same biorthogonal system obtained in [2] using a new way.

In [2] we proved that the product system Ψ_m of the system

$$\psi_k = \frac{2\varphi_k}{\varphi_{k+1} + 1}$$

is biorthogonal to product system Φ_m of system φ_k with respect to discrete scalar product of set X_N of Chebishev abscissas.

Let $\psi^{(0)} = 1$ and $\psi^{(1)} = \frac{1}{\overline{\varphi}^{(1)}}$ as in the theorem. Since

$$\begin{aligned} 2\cos^2(2^k \arccos x) &= \cos^2(2^k \arccos x) - \sin^2(2^k \arccos x) + \\ &\qquad + \cos^2(2^k \arccos x) + \sin^2(2^k \arccos x) = \\ &= \cos(2 \cdot (2^k \arccos x)) + 1, \end{aligned}$$

then

$$\begin{aligned} \psi_k^{(1)}(x) &= \frac{1}{\overline{\varphi}_k(x)} = \frac{1}{\varphi_k(x)} = \frac{1}{\cos(2^k \arccos x)} = \frac{2\cos(2^k \arccos x)}{2\cos^2(2^k \arccos x)} = \\ &= \frac{2\cos(2^k \arccos x)}{\cos(2 \cdot (2^k \arccos x)) + 1} = \frac{2\varphi_k(x)}{\varphi_{k+1}(x) + 1}. \end{aligned}$$

We remark that Example 3 could be derived from example 2 by transformation of $\mathcal{I}$.

4. Let a from the complex unit disk and z from the unit circle, i.e.

$$a, z \in \mathbb{C} \quad |a| < 1, \quad |z| = 1.$$

Then the function

$$B_a(z) = \frac{z - a}{1 - \overline{a}z}$$

is called Blaschke function. It is proved in [5] that for $t \in \mathbb{R}$ B_a is of the form

$$B_a\left(e^{it}\right) = e^{i\beta_a(t)},$$

where

$$\beta_a(t) = \phi + \gamma_s(t - \phi),\ s := \log\left(\frac{1+r}{1-r}\right),\ (a = (re^{i\phi}) \in \mathbb{B},\ t \in \mathbb{R})$$

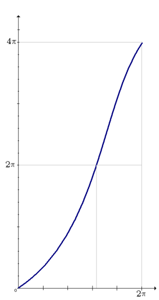
and $\gamma_s(t) := 2\arctan(e^s \tan(t/2))$. Because $\beta_a : [0,\ 2\pi] \to [0,\ 2\pi]$ is a continuous strictly monotone function, $B_a(z)$ is a bijective map on $\mathbb{T}$.

Let $B_{a_1}(z)$ and $B_{a_2}(z)$ two Blaschke functions. Then for $z = e^{it}$ we have

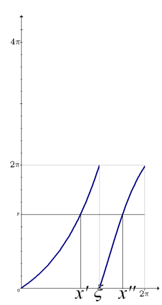
$$B_{a_1}(z) \cdot B_{a_2}(z) = e^{i(\beta_{a_1}(t) + \beta_{a_2})(t)} = e^{i\beta(t)},$$

where $\beta : [0,\ 2\pi] \to [0,\ 4\pi]$ is a strictly monotone continuous bijective map.

Let $\beta_1(t) := \beta(t) \bmod 2\pi$. It is easy to see that $\beta_1 : [0,\ 2\pi] \to [0,\ 2\pi]$ is a twofold map.



function $\beta(x)$



function $\beta_1(x)$

Using the twofold map

$$A = e^{i\beta_1(t)}$$

we can construct rational product systems (see [1], [3]).

5. Further opportunities for generalization

1. In the original construction we got 2^n-fold map A_n as iteration of the twofold map A. Instead of this we can use a series of twofold maps $a_1, a_2, \dots$ and $A_n = a_n \circ a_{n-1} \circ \dots \circ a_1$.

2. We can generalize the definition of the product system. Let $m > 1$, $m \in \mathbb{N}$ and start from the following finite system of the sets

(5.1) $$\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}, \dots,,\ \varphi_n^{(m-1)}\}, \quad n = 0,\ 1, \dots,\ N-1.$$

Then
$$\Phi_\ell := \prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot m^k$$
is the generalized product system of the system (5.1). In the special case when $m = 2$ and $\varphi_n^{(0)} = 1$ and $\varphi_n^{(1)} = \varphi_n$ we reobtain the original definition of product system.

Let $\{\Phi_\ell,\ 0 \le \ell < m^N - 1\}$ denote the generalized product system of $\{\phi_n,\ 0 \le n < N\}$ and $\{\Psi_\ell,\ 0 \le \ell < m^N - 1\}$ the generalized product system of $\{\Upsilon_n = \{\psi_n^{(0)},\ \psi_n^{(1)}, \dots, \psi_n^{(m-1)}\},\ 0 \le n < N\}$. Then the mixed Dirichlet kernel of the product systems is defined by

(5.2) $$D_{m^N}(s,t) := \sum_{\ell=0}^{m^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t).$$

Remark 5.1. *The mixed Dirichlet kernel of the product systems can be written in the product form*

$$D_{m^N}(s,t) = \sum_{\ell=0}^{m^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t) = \sum_{\ell=0}^{m^N-1} \prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}(s) \cdot \overline{\psi}_k^{(\ell_k)}(t)$$
$$= \prod_{k=0}^{N-1} \sum_{j=0}^{m-1} \varphi_k^{(j)}(s) \cdot \overline{\psi}_k^{(j)}(t)$$

This can be used to construct multidimensional biorthogonal product systems.

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B. Király
University of Pécs
kiralyb@gamma.ttk.pte.hu