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DISTRIBUTIONS OF ARITHMETICAL FUNCTIONS. SOME RESULTS AND PROBLEMS

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Dedicated to my friends, Professors Ferenc Schipp and Péter Simon

1. Introduction

1.1. Notations

$\mathbb{N}$ = set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the sets of rational, real, complex numbers, respectively. $\mathcal{P}$ = set of primes; $\omega(n)$ = number of distinct prime factors of n; $\mathcal{P}_k = \{n \mid \omega(n) = k\}$; $\Omega(n)$ = number of prime power divisors of n; $\mathcal{N}_k := \{n \mid \Omega(n) = k\}$.

Let $\pi_k(x) = \#\{n \leq x \mid \omega(n) = k\}$, $N_k(x) = \#\{n \leq x \mid \Omega(n) = k\}$, $\pi(x) = \pi_1(x)$.

1.2. Definitions

Let $q \geq 2$ be an integer, $A_q = \{0, 1, \ldots, q-1\}$ (= set of $q-ary$ digits), and let $\varepsilon_j(n)$ be the digits in the $q-ary$ expansion of n:

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in A_q \quad (j = 0, 1, 2, \ldots). \tag{1.1}$$

It is clear that the right hand side of (1.1) is a finite sum.

$\mathcal{A}_q$ = *set of q-additive functions.* We say, that $f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$, and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n) q^j)$$

holds for every $n \in \mathbb{N}$.

Special $q-$additive functions: $\alpha(n)=\sum_{j=0}^{\infty}\varepsilon_j(n)$; $\quad\beta_l(n):=\#\{j\mid\varepsilon_j(n)=$ $=l\}$ defined for $l=1,\dots,q-1$; $\quad f(n)=cn$.

This notion was introduced by A.O. Gelfond [24].

$\mathcal{M}_q$ = *set of q-multiplicative functions.* We say, that $g:\mathbb{N}_0\to\mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0)=1$, and $g(n)=\prod_{j=0}^{\infty}g(\varepsilon_j(n)q^j)\quad(n\in\mathbb{N})$. Furthermore let $\overline{\mathcal{M}}_q=$ $=\{g\in\mathcal{M}_q,\quad|g(n)|=1\quad(n\in\mathbb{N})\}$. One can see easily, that $g(n)=z^{f(n)}\in$ $\in\mathcal{M}_q$, if $f\in\mathcal{A}_q,\quad z\in\mathbb{C}$.

$\mathcal{A}$ = *set of additive arithmetical functions.*

$f:\mathbb{N}\to\mathbb{R}$ belongs to $\mathcal{A}$, if $f(1)=0$ and

$$f(mn)=f(m)+f(n) \tag{1.2}$$

holds for every coprime pairs of integers m,n.

$\mathcal{A}^*$ = *set of completely additive arithmetical functions.*
We say that $f\in\mathcal{A}^*$, if $f\in\mathcal{A}$, and (1.2) holds for every $m,n\in\mathbb{N}$.

We say that f is strongly additive if $f\in\mathcal{A}$ and $f(p^k)=f(p)$ for $k\in$ $\in\mathbb{N},\quad p\in\mathcal{P}$.

$\mathcal{M}$ = *set of multiplicative arithmetical functions.* $g:\mathbb{N}\to\mathbb{C}$ belongs to $\mathcal{M}$ if $g(1)=1$ and

$$g(mn)=g(m)\cdot g(n)\qquad\text{for every coprime pairs of } m \text{ and } n. \tag{1.3}$$

$\mathcal{M}^*$ = set of completely multiplicative arithmetical functions. $g\in\mathcal{M}^*$, if $g\in\mathcal{M}$ and (1.3) holds for every $m,n\in\mathbb{N}$.

Furthermore, let $\overline{\mathcal{M}}=\{g\mid g\in\mathcal{M},\quad|g(n)|=1\quad(n\in\mathbb{N})\}$.

Let $\{e_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that it has a limit distribution, if

$$\lim_{x\to\infty}\frac{1}{x}\#\{n\le x\mid e_n<y\}=F(y) \tag{1.4}$$

exists for almost all y, and F is a distribution function.

1.3. Classical theorems in probabilistic number theory

(E-W): Erdős-Wintner theorem [1]: *If $f\in\mathcal{A}$, then it has a limit distribution, i.e.*

$$\lim x^{-1}\#\{n\le x\mid f(n)<y\}=F(y)$$

exists for almost all y, where F is a distribution function, if and only if the following three series

$$\sum_{|f(p)|<1} \frac{f(p)}{p}, \quad \sum_{|f(p)|<1} \frac{f^2(p)}{p}, \quad \sum_{|f(p)|\geq 1} \frac{1}{p} \tag{1.5}$$

converge. If this condition is satisfied, then $\varphi(\tau)$, the characteristic function of F can be written as

$$\varphi(\tau) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{m=1}^{\infty} \frac{1}{p^m} \exp(i\tau f(p^m))\right). \tag{1.6}$$

In (1.5), (1.6) p run over $\mathcal{P}$.

(E-K): Erdős-Kac theorem [2]: *Let $f(n)$ be a strongly additive function, such that $|f(p)| \leq 1 \quad (p \in \mathcal{P})$. Let*

$$A(x) = \sum_{p\leq x} f(p)\cdot p^{-1}, \quad B^2(x) = \sum_{p\leq x} f^2(p)\cdot p^{-1}, \quad (0 \leq) B(x) \to \infty \quad (x \to \infty).$$

Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x \;\middle|\; \frac{f(n) - A(x)}{B(x)} < y \right\} = \Phi(y), \tag{1.7}$$

$\Phi =$ standard normal law.

(T-K): Turn-Kubilius inequality: *Let $f \in \mathcal{A}$,*

$$E(x) = \sum_{p^k\leq x} \frac{f(p^k)}{p^k}\left(1 - \frac{1}{p}\right), \quad D^2(x) = \sum_{p^k\leq x} \frac{f^2(p^k)}{p^k}.$$

Then

$$\sum_{n\leq x} |f(n) - E(x)|^2 \leq cxD^2(x).$$

Here c is an absolute constant, $c = 32$ is appropriate (see [4], Ch. IV, p. 147).

(D): Delanges's theorem [3]:

1. *Let $g \in \overline{\mathcal{M}}_q, \quad m_j = \frac{1}{q}\sum_{b\in A_q} g(cq^j)$. Then*

$$\lim_{x\to\infty} x^{-1} \left| \sum_{n\leq x} g(n) \right| = a$$

always exists, it is nonzero, if and only if $m_j \neq 0 \quad (j = 0, 1, 2, \dots)$ *and* $(0 \leq) \sum \mathrm{Re}(1 - m_j) < \infty$. *Furthermore,*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} g(n) = M(g) \tag{1.8}$$

exists, and $M(g) \neq 0$, *if* $m_j \neq 0 \quad (j = 0, 1, 2, \dots)$, *and* $\sum(1 - m_j)$ *is convergent. If these conditions hold, then*

$$M(g) = \prod_{j=0}^{\infty} m_j. \tag{1.9}$$

2. Let $f \in \mathcal{A}_q$. f *has a limit distribution, i.e.*

$$\lim_{x\to\infty} x^{-1} \#\{n \leq x \mid f(n) < y\} = F(y) \qquad a.a.$$

if and only if both of the series

$$\sum_{j=0}^{\infty} \sum_{b\in A_q} f(bq^j), \quad \sum_{j=0}^{\infty} \sum_{b\in A_q} f^2(bq^j) \tag{1.10}$$

are convergent. If X_j *are independent random variables,*

$$P(X_j = f(bq^j)) = 1/q \quad \text{if} \quad b \in A_q, \tag{1.11}$$

then

$$F(y) = P\left(\sum_{j=0}^{\infty} X_j < y\right).$$

The following assertion is an almost immediate consequence of known theorem for the sum of independent random variables:

Theorem 1. *Let* $|f(bq^j)| \leq 1 \quad (b \in A_q, j \in \mathbb{N}_0)$, $\quad \mu_j = \frac{1}{q} \sum_{b\in A_q} f(bq^j)$,

$$\sigma_j^2 = \frac{1}{q} \sum_{b\in A_q} (f(bq^j) - \mu_j)^2, \quad N = N(x) = \left[\frac{\log x}{\log q}\right], \tag{1.12}$$

$$E(x) := \sum_{j=0}^{N} \mu_j; \quad D^2(x) = \sum_{j=0}^{N} \sigma_j^2. \tag{1.13}$$

Assume that $D(x) \to \infty \quad (x \to \infty)$. *Then*

$$\lim x^{-1} \# \left\{ n \le x \mid \frac{f(n) - E(x)}{D(x)} < y \right\} = \Phi(y).$$

2. Distribution of q-additive functions on some subsets of integers

The analogous of Theorem D and 1 are considered for the following subsets:

1. the set $\mathcal{P}$,
2. the set $P(\mathbb{N})$, where P is a polynomial over $\mathbb{Z}$,
3. the set $P(\mathcal{P})$,
4. on the sets $\mathcal{P}_k,\ \mathcal{N}_k$.

Let $\mathcal{B}$ be an infinite sequence of non-negative integers, $A_{\mathcal{B}}(x) := \#\{n < x \mid n \in \mathcal{B}\}$. For some collection of $(0 \le) l_1 < \ldots < l_r (\le N), \quad N = N(x) = \left[\frac{\log x}{\log q}\right], \quad b_1, \ldots, b_r \in A_q$, let

$$A_{\mathcal{B}}\left(x \mid \tfrac{\underline{l}}{\underline{b}}\right) = \#\{n < x \mid n \in \mathcal{B}, \quad \varepsilon_{l_j}(n) = b_j, \quad j = 1, \ldots, r\}.$$

It is clear that

$$\sum_{\underline{b}} A_{\mathbb{N}_0}\left(q^N \mid \tfrac{\underline{l}}{\underline{b}}\right) = A_{\mathbb{N}_0}(q^N),$$

and a similar relation with some error holds by x instead of q^N, uniformly in $\underline{l}, \underline{b}$. One can hope that such kind of relation holds for $A_{\mathcal{B}}\left(x | \tfrac{\underline{l}}{\underline{b}}\right)$ for a large class of sequences $\mathcal{B}$.

2.1. Distribution of $f \in \mathcal{A}_q$ on the set $P(\mathbb{N})$ and on $P(\mathcal{P})$

In a joint paper of N.L. Bassily and myself [5] the following assertion has been proved.

Theorem 2. *Let* $P(x) \in \mathbb{Z}[x]$, *with positive leading term,* $r = \deg P$. *Let* $f \in \mathcal{A}_q$, *such that* $\sup_{j \in \mathbb{N}_0} \max_{b \in \mathcal{A}_q} |f(bq^j)|$ *is finite. Assume furthermore, that* $D(x) \cdot (\log x)^{-1/3} \to \infty \quad (x \to \infty)$. *Then*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n < x \;\middle|\; \frac{f(P(n)) - E(x^r)}{D(x^r)} < y \right\} = \Phi(y), \tag{2.1}$$

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\#\left\{p<x \;\Big|\; \frac{f(P(n))-E(x^r)}{D(x^r)}<y\right\}=\Phi(y). \tag{2.2}$$

Here $E(x)$ and $D(x)$ is defined in (1.12), (1.13).

Remark. The proof is based upon theorems of I.M. Vinogradov and L. K. Hua for exponential sums. By using appropriate estimates we deduced that for every fixed h,

$$A_{P(\mathbb{N})}\left(x \mid {}^{l_1,\ldots,l_h}_{b_1,\ldots,b_h}\right)=\frac{x}{q^h}+\mathcal{O}(x\cdot(\log x)^{-\lambda}) \tag{2.3}$$

and

$$A_{P(\mathcal{P})}\left(x \mid {}^{l_1,\ldots,l_h}_{b_1,\ldots,b_h}\right)=\frac{\pi(x)}{q^h}+\mathcal{O}(x\cdot(\log x)^{-\lambda}) \tag{2.4}$$

whenever $N=\left[\frac{\log x}{\log q}\right]$,

$$N^{1/3}\le l_1<l_2<\ldots<l_h\le rN-N^{1/3} \tag{2.5}$$

and $b_1,\ldots,b_h\in\mathcal{A}_q$. Here λ is an arbitrary constant. By using these estimates one can deduce that the moments $a_k(x)$, $b_k(x)$ are close to $c_k(x)$, where

$$a_k(x)=x^{-1}\sum_{n\le x}\left(\frac{f(P(n))-E(x^r)}{D(x^r)}\right)^k;$$

$$b_k(x)=\frac{1}{\pi(x)}\sum_{p\le x}\left(\frac{f(P(p))-E(x^r)}{D(x^r)}\right)^k;$$

$$c_k(x):=\frac{1}{x^r}\sum_{n\le x^r}\left(\frac{f(n)-E(x^r)}{D(x^r)}\right)^k.$$

Since $\lim\limits_{x\to\infty}c_k(x)=\lim\limits_{x\to\infty}a_k(x)=\lim\limits_{x\to\infty}b_k(x)$, and $\lim c_k(x)=\mu_k$, $\mu_k=k$-th moment of Φ, by using the Frechet-Shohat theorem the proof will be finished.

Theorem 3. *Let $f\in\mathcal{A}_q$, assume that both series in (1.10) are convergent. Let $P\in\mathbb{Z}[x]$, with positive leading term. Then the sequences $f(P(n))$ $(n\in\mathbb{N})$, $f(P(p))$ $(p\in\mathcal{P})$ have limit distribution.*

Remarks. In [6] it is proved that for $f \in \mathcal{A}_q$

$$\frac{1}{x}\sum_{n\le x}(f(P(n)) - E(q^{kN+d}))^2 \le c_1 D^2(q^{kN+d}) \tag{2.6}$$

and

$$\frac{1}{\pi(x)}\sum_{p\le x}(f(P(p)) - E(q^{kN+d})) \le c_2 D^2(q^{kN+d}) \tag{2.7}$$

hold, if $x \ge x_0$. Here c_1, c_2 are absolute constants, $N = \left[\frac{\log x}{\log q}\right]$, $d \ge 0$ is a constant defined so that $\max_{y\le x} P(y) \le q^{kN+d}$, if $x \ge x_0$.

Furthermore, the quantity of integers $n < q^N$ for which $P(n) \equiv a \pmod{q^M}$, and similarly, the number of primes $p < q^N$ for which $P(p) \equiv a \pmod{q^M}$ can be estimated quite well if $M = [\log N]$, say. Using this, and the inequalities (2.6), (2.7), Theorem 3 can be deduced easily.

In [7] we proved that the convergence of the series in (1.10) are necessary for the existence of the limit distribution in the case $f(p)$.

The necessity of the convergence of the series' in (1.10) is not known in general for $P(n)$, $P(p)$ if $\deg P \ge 2$.

2.2. Distribution of $f \in \mathcal{A}_q$ on the set of primes

As we mentioned in § 2.1, the convergence of the series (1.10) are necessary for the existence of the limit distribution of $f(p)$ $(p \in \mathcal{P})$. We proved in [8]

Theorem 4. *Let $f \in \mathcal{A}_q$. Assume that $\sum_{j=0}^{\infty}\sum_{b\in A_q} f^2(bq^j) < \infty$. Let*

$$E_N := \frac{1}{q}\sum_{j=0}^{N}\sum_{b\in A_q} f(bq^j).$$

Then

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\#\{p < x \mid f(p) - E_{N(x)} < y\} = F(y)$$

exists for a.a. y, F is a distribution function. Here $N(x) = \left[\frac{\log x}{\log q}\right]$.

Other hand, assume that there exists a function $a(x)$ for which

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\#\{p < x \mid f(p) - a(x) < y\} = G(y)$$

exists for a.a y, where G is a distribution function, then $\sum_j \sum_{b\in A_q} f^2(bq^j) < \infty$, *furthermore* $a(x) - E_{N(x)} \to c \quad (x\to\infty)$.

2.3. Distribution of q-additive functions on the set of integers having k prime factors

In our paper [9] with L. Germán we proved the following theorems.

Theorem 5. *Let* $f \in \mathcal{A}_q$, $\sup_{j\in\mathbb{N}_0, b\in A_q} |f(bq^j)| \le 1$. *Assume that* $D(x)(\log x)^{-1/3} \to \infty$. *Let* $1 \le B \le x^{1/3}$, $\quad (B,q)=1$. *Then*

$$\sup_{\substack{B<x^{1/3}\\(B,q)=1}} \sup_{y\in(-\infty,\infty)} \left| \frac{1}{\pi\left(\frac{x}{B}\right)} \# \left\{ p < \frac{x}{B} \;\middle|\; \frac{f(Bp)-E(x)}{D(x)} < y \right\} - \Phi(y) \right| \le \tau(x),$$

where $\tau(x) \to 0$ *as* $x \to \infty$.

Theorem 6. *Let* $f \in \mathcal{A}_q$ *and that both series in (1.10) are convergent. Let* $\xi_0, \xi_1, \dots$ *be independent random variables,* $P(\xi_j = f(bq^j)) = 1/q$ *if* $j \ge\ \ge 1$, $\quad b \in A_q$, *and* $P(\xi_0 = f(b)) = \frac{1}{\varphi(q)}$ *if* $(b,q)=1$, *and* 0 *if* $(b,q)>1$.

Let $\theta = \sum_{l=0}^{\infty} \xi_l$. (The right hand side is clearly convergent according to the 3 series theorem of Kolmogorov.)

Let $F(y) := P(\theta < y)$. *Let* $B \in \mathbb{N}$, $\quad (B,q)=1$, *and*

$$F_{x,B}(y) := \frac{1}{\pi\left(\frac{x}{B}\right)} \# \left\{ p < \frac{x}{B} : f(pB) < y \right\}.$$

Then

$$\max_{\substack{1\le B\le x^{1/3}\\(B,q)=1}} \sup_y |F_{x,B}(y) - F(y)| \le \delta_x,$$

$\delta_x \to 0 \quad (x \to \infty)$.

Let $J_x := [1, \delta(x)\log\log x]$, where $\delta(x) \to 0 \quad (x\to\infty)$ arbitrarily slowly.

Theorem 7. *Assume that* f *satisfies the conditions stated in Theorem 5. Then*

$$\sup_{k\in J_x} \sup_{y\in\mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \le x, n \in \mathcal{P}_k \;\middle|\; \frac{f(n)-E(x)}{D(x)} < y \right\} - \Phi(y) \right| \to 0$$

as $x \to \infty$, *and*

$$\sup_{k\in J_x}\sup_{y\in\mathbb{R}}\left|\frac{1}{N_k(x)}\#\left\{n\le x, n\in\mathcal{N}_k \;\Big|\; \frac{f(n)-E(x)}{D(x)}<y\right\}-\Phi(y)\right|\to 0,$$

as $x \to \infty$.

Theorem 8. *Let* $f\in\mathcal{A}_q$, F *be defined as in Theorem 6. Let*

$$H_x^{(k)}(y) := \frac{1}{\pi_k(x)}\#\{n<x, n\in\mathcal{P}_k, f(n)<y\},$$

$$G_x^{(k)}(y) := \frac{1}{N_k(x)}\#\{n<x, n\in\mathcal{N}_k, f(n)<y\}.$$

Then

$$\lim_{x\to\infty}\sup_{k\in J_x}|H_x^{(k)}(y)-F(y)|=0,$$

$$\lim_{x\to\infty}\sup_{k\in J_x}|G_x^{(k)}(y)-F(y)|=0,$$

at every continuity point y *of* F.

Remark. As it is known, F is continuous if $\sum P(\xi_j\ne 0)=\infty$, i.e., if there is a sequence $j_1<j_2<\dots$ and $b_1,b_2,\dots\in A_q$, such that $f(b_\nu q^{j_\nu})\ne 0$.

3. Linear combinations of q-additive functions

Let $(1\le)a_1<a_2<\dots<a_k(<q)$ be mutually coprime integers, each of which is coprime to q as well. Let $f_1,\dots,f_k\in\mathcal{A}_q$; $g_1,\dots,g_k\in\mathcal{M}_q$, and

(3.1) $$l(n) := f_1(a_1n)+\dots+f_k(a_kn),$$

(3.2) $$t(n) := g_1(a_1n)\cdots g_k(a_kn).$$

We say that a sequence e_n $(n=1,2,\dots)$ of real numbers is „*tight*”, if there is a sequence A_N such that $\limsup_{N\to\infty} q^{-N}\#\{n<q^N \mid |l(n)-A_N|>K\} := := C(K)\to 0$ as $K\to\infty$. We say that e_n is „*bounded in mean*”, if $C(K)\to 0$ at the choice $A_N=0$ $(N\in\mathbb{N})$.

We investigated the existence of the distribution of $l(n)$ and the mean value of $t(n)$ in our papers [10], [11], [12]. In [11] we proved

Theorem 9. *The sequence $l(n)$ defined in (3.1) is tight if and only if there exist suitable real numbers $\gamma_1, \dots, \gamma_k$ such that for the functions $\psi_l(n) := := f_l(n) - \gamma_l(n)$, the relations*

$$D_l^2 := \sum_{j=0}^{\infty} \sum_{b \in A_q} \psi_l^2(bq^j) < \infty \tag{3.3}$$

are satisfied.

Let

$$A_N^{(l)} = \frac{1}{q} \sum_{j=0}^{N-1} \sum_{b \in A_q} \psi_l(bq^j), \tag{3.4}$$

$$E_N = \sum_{l=1}^{k} A_N^{(l)}. \tag{3.5}$$

Theorem 10. *Assume that the conditions of Theorem 9 are satisfied. Then*
(1)

$$\lim q^{-N} \#\{n < q^N \mid l(n) - E_N < y\} \quad (= F(y))$$

exists for a.a. y. E_N is defined by (3.5);
(2) the sequence $l(n)$ has a limit distribution if additionally E_N has a finite limit as $N \to \infty$.

Let

$$\mu_l(u) = \frac{1}{q} \sum_{c \in A_q} f_l(cq^u), \quad p(u) = \sum_{l=1}^{k} \mu_l(u),$$

$$\pi_u(c_1, \dots, c_k) = \frac{1}{q^{u+1}} \#\{n < q^{u+1} \mid \varepsilon_u(a_j n) = c_j, \quad j = 1, \dots, k\},$$

$$\tau_u = \sum_{c_1, \dots, c_k \in A_q} (f_1(c_1 q^u) + \dots + f_k(c_k q^u) - p(u))^2 \pi_u(c_1, \dots, c_k),$$

$$\lambda_u := \sum_{j=1}^{k} \sum_{b \in A_q} f_j^2(bq^u).$$

We proved in [11] that for $u \geq 2 \quad d_1\lambda_u \leq \tau_u \leq d_2\lambda_u$ holds with positive constants d_1, d_2, which may depend at most on q.

Theorem 11. *Let $a_1, \ldots, a_k$ and $l(n)$ be as earlier. Let $\sigma_N^2 = \sum_{u=0}^{N-1} \tau_u$. Assume furthermore that*

$$\max_{l=1,\ldots,k} \max_{c\in A} \frac{|f_l(cq^M)|}{\sigma_M} \to 0 \qquad as \quad M \to \infty.$$

Then

$$\lim \frac{1}{x} \# \left\{ n < x \;\Big|\; \frac{l(n) - F(N_x)}{\sigma_{N_x}} < y \right\} = \Phi(y),$$

where $(N =)N_x = \left[\frac{\log x}{\log q}\right], \quad F(N) = \sum_{u=0}^{N-1} p(u).$

4. Linear combinations of q-additive functions over $\mathcal{P}$

We keep the notations introduced in §3. In [13] the following assertion is proved.

Theorem 12. *We have*

(1) over the set $\mathcal{P}$ is tight if and only if it is tight over $\mathbb{N}$;

(2) has limit distribution over $\mathcal{P}$, if it has a limit distribution over $\mathbb{N}$.

We can prove furthermore the following

Theorem 13. *Assume that the condition stated in Theorem 11 holds true. Let $K_N = [\log N]$, $N = N_x$. Assume that $\sigma_{K_N}^2/\sigma_N^2 \to 0$, $(\sigma_N^2 - -\sigma_{N-K_N}^2)/\sigma_N^2 \to 0$ as $N \to \infty$. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p < x \;\Big|\; \frac{l(p) - F(N)}{\sigma_N} < y \right\} = \Phi(y),$$

5. Mean-value of products of q-multiplicative functions having absolute value 1

Let $t(n)$ be as in (3.2), assume that $g_j \in \overline{\mathcal{M}}_q \quad (j = 1, \ldots)$. Let $t_j(n) = t(nq^j), \quad M_j(x) = \sum_{n<x} t_j(n), \quad m_j(N) = q^{-N} M_j(q^N),$

$$\alpha_j = \liminf_{N\to\infty} |m_j(N)|, \quad \beta_j = \limsup_{N\to\infty} |m_j(N)|.$$

We proved the following assertion in [10].

Theorem 14.

(1) We have $\alpha_j = \beta_j \quad (j = 1, 2, \ldots)$. If $\beta_j > 0$ for some j, then $\beta_l \to 1$.

(2) The relation $\beta_l \to 1$ holds, if there exists an integer j, and $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ such that

$$q^j(a_1\gamma_1 + \ldots + a_k\gamma_k) = integer,$$

and

$$\sum_{s=0}^{\infty} \sum_{b\in A_q} Re(1 - g_l(bq^s)e(-\gamma_l bq^s)) < \infty$$

holds for $l = 1, \ldots, k$. Here $e(\alpha) := \exp(2\pi i\alpha)$.

6. Distribution of q-additive functions on the set of integers characterized by the number of given digits

In [14] we proved the following Theorem 15.

Theorem 15. *Let $f \in \mathcal{A}_q$, the series in (1.10) be convergent. Let $\underline{r}^{(N)} = (r_1^{(N)}, \ldots, r_{q-1}^{(N)})$ be such a sequence for which*

$$\left|\frac{qr_j^{(N)}}{N} - 1\right| < \delta_N \quad (j = 1, \ldots, N-1),$$

where $\delta_N \to 0 \quad (N \to \infty)$. Let

$$S_N(\underline{r}^{(N)}) = \{n < q^N \mid \beta_l(n) = r_l^{(N)}, \quad l = 1, \ldots, q-1\},$$

$$M_N(\underline{r}^{(N)}) = \# S_N(\underline{r}^{(N)}).$$

Then

$$\lim_{N\to\infty} \frac{1}{M_N(r^{(N)})} \#\{n \in S_N(\underline{r}^{(N)}) \mid f(n) < y\} = F(y)$$

for almost all y, *where* $F(y) := P(\sum X_j < y)$, X_j *are defined in (1.11).*

Theorem 16. *Let* $f \in \mathcal{A}_2$, *such that* $\sum f(2^j)$, $\sum f^2(2^j)$ *are convergent. Let* $\xi_0, \xi_1, \xi_2, \dots$ *be random variables,* $P(\xi_\nu = 0) = 1 - \eta$, $P(\xi_\nu = f(2^\nu)) = = \eta$, $\theta_\eta = \sum \xi_\nu$, $F_\eta(y) = P(\theta_\eta < y)$. *Then*

$$\lim_{N\to\infty} \max_{\frac{k}{N}\in[\delta,1-\delta]} \left| \frac{1}{\binom{N}{k}} \#\{n < 2^N, \alpha(n) = k, f(n) < y\} - F_{\frac{k}{N}}(y) \right| = 0$$

for almost all y. *It holds for all* $y \in \mathbb{R}$ *if* $f(2^\nu) \neq 0$ *holds for infinitely many* ν.

Theorem 17. *Let* $f \in \mathcal{A}_2$, $|f(2^j)|$ *be bounded. Let* $h_N \in \mathcal{A}_2$ *be defined by* $h_N(2^j) := f(2^j) - \frac{A_N}{N}$, *where* $A_N = \sum_{j=0}^{N-1} f(2^j)$. *Let*

$$\sigma_N^2(\eta) := (1-\eta)\eta \sum_{j=0}^{N-1} h_N^2(2^j).$$

Assume that $\sigma_N^2(1/2) \to \infty$ $(N \to \infty)$. *Let* $\delta > 0$ *be an arbitrary constant,* $\delta < 1/2$. *Then*

$$\lim_{N\to\infty} \sup_{\frac{k}{N}\in[\delta,1-\delta]} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N, \alpha(n) = k \;\middle|\; \frac{f(n) - \frac{k}{N}A_N}{\sigma_N\left(\frac{k}{N}\right)} < y \right\} - \right.$$
$$\left. -\Phi(y) \right| = 0.$$

In [15] we proved

Theorem 18. *Let* $f \in \mathcal{A}_2$. *Assume that there exists some* $\xi \in (0,1)$ *and a sequence* k_N *of integers such that* $k_N/N \to \xi$ $(N \to \infty)$ *and*

$$\frac{1}{\binom{N}{k_N}} \#\{n < 2^N \mid \alpha(n) = k_N, f(n) < y\} \to G(y) \quad (N \to \infty)$$

for a.a y, *where* G *is a distribution function. Then* $G(y) = F_\xi(y)$ *(defined in Theorem 16) and both series in (1.10) are convergent.*

7. On q-multiplicative functions taking a fixed value on the set of primes

In [17] we proved

Theorem 19. *Let $q \geq 2$ be fixed. Then there exists a constant $c = c(q)$ such that for every $q \in \mathcal{M}_q$ for which $g(p)$ =constant if p runs over $\mathcal{P}$, then there exists $k \in [1, c]$ such that $g^k(nq) = 1$ holds for every $n \in \mathbb{N}$.*

In [18] the following assertion has been proved.

Theorem 20. *Let $q \geq 2$, $\quad B(\geq 1)$ be an arbitrary constant. Then there exists a constant $c_1 := c_1(q, B)$ with the following property.*

Let $g \in \overline{\mathcal{M}}_q$ for which there exists a sequence $N_1 < N_2 < \dots$ of integers and a sequence $k_\nu \in [1, B \log n_\nu]$, $\quad k_\nu \in \mathbb{N}$ and α_ν such that

$$\begin{aligned} g(\pi) = \alpha_\nu \quad if \quad & \pi \in [q^{N_\nu}, 4q^{N_\nu}] \\ and \quad & \pi \in \mathcal{P}_{k_\nu}, \end{aligned} \tag{7.1}$$

where $\mathcal{P}_{k_\nu}$ is the whole set of integers with exactly k prime factors.

Then there is an integer $r \in [1, c_2]$, such that $g^r(nq) = 1 \quad (n \in \mathbb{N})$.

8. Mean value of $g \in \overline{\mathcal{M}}_q$ over $\mathcal{P}$

Then main problem we are interested in is to investigate the sum

$$P(x) := \sum_{p \leq x} g(p). \tag{8.1}$$

Let

$$S(x \mid \alpha) := \sum_{\substack{l < x \\ (l,q)=1}} g(l) e(l\alpha) \quad (e(\beta) = \exp(2\pi i \beta)). \tag{8.2}$$

We would like to give necessary and sufficient conditions for g to satisfy

$$\frac{P(x)}{\pi(x)} \to 0 \quad (x \to \infty). \tag{8.3}$$

One can see easily that (8.3) implies that

$$x^{-1}S(x \mid r) \to 0 \quad (x \to \infty) \tag{8.4}$$

holds for every $r \in \mathbb{Q}$.

In [19] we formulated our

Conjecture 1. *(8.2) holds if and only if (8.4) holds for every* $r \in \mathbb{Q}$.

Conjecture 2. *If* $g \in \mathcal{M}_q, \quad |g(n)| \le 1$, *and*

$$\lim \frac{1}{\pi(x)} \sum_{p \le x} g(p) \quad (=: M_q)$$

exists, and $M_q \ne 0$, *then*

$$\sum_j \sum_{a \in A_q} (1 - g(aq^j)) \quad \textit{is convergent}, \tag{8.5}$$

$$M_q = \left\{ \frac{1}{\varphi(q)} \sum_{(a,q)=1} g(a) \right\} \prod_{j=1}^{\infty} \left\{ \frac{1}{q} \sum_{a \in A_q} g(aq^j) \right\}. \tag{8.6}$$

Presently we can prove only the following weaker assertion.

Let $Y(x)$ be a monotonically increasing function such that $Y(x) \to \infty$ and $\frac{\log Y(x)}{\log x} \to 0$ as $x \to \infty$. Let $\mathcal{N}_x := \{n \le x, p(n) > Y(x)\}$, where $p(n)$ is the smallest prime factor of n. Let $N(x) = \#(\mathcal{N}_x)$.

Let L be the strongly multiplicative function such that

$$L(p^a) = L(p) = \begin{cases} \frac{1}{p-2} & \text{if} \quad p > 2 \quad \text{and} \quad p \nmid q, \\ 0 & \text{otherwise.} \end{cases}$$

In [20] the following assertion has been proved.

Let

$$U(x) = \sum_{n \in \mathcal{N}_x} g(n), \tag{8.7}$$

where $g \in \overline{\mathcal{M}}_q$. Then

$$\left|\frac{U(x)}{N(x)}\right|^2 \le \sum_{d<D} \frac{L(d)}{d} \sum_{a=0}^{d-1} \left|q^{-M} S\left(q^M \left| \frac{a}{d}\right.\right)\right|^2 + \tag{8.8}$$
$$+ \frac{c_1}{D} + o_x(1),$$

where M is an arbitrary integer, for which $q^M \in [q^{-1}x^{1/4}, qx^{1/4}]$, $c_1 = c_1(q)$ is a positive constant, the constant implicitly standing in $o_x(1)$ depends only on the choice of $Y(x)$ (and does not depend on g). $D > 0$ is an arbitrary number.

Remark. (1) (8.8) shows that the fulfilment of (8.4) (for every $r \in \mathbb{Q}$) implies that

$$\frac{U(x)}{N(x)} \to 0 \quad (x \to \infty).$$

(2) We are able to prove that Conjecture 1 follows from Conjecture 3.
Let

$$T_{l_1,l_2}^{(M)} = T_{l_1,l_2} =$$
$$= \#\{p_1, p_2 \in \mathcal{P}, \quad p_2 - p_1 = l_2 - l_1, \quad p_1 \equiv l_1 \pmod{q^M}, p_1 \le x\}.$$

Conjecture 3. *There exists a constant* $0 < \delta < \frac{1}{2}$ *such that for* $M = [\delta N]$, $N = \left[\frac{\log x}{\log q}\right]$ *we have*

$$\sum_{\substack{l_1,l_2<q^M \\ (l_i,q)=1 \\ l_1\ne l_2}} \left|T_{l_1,l_2}^{(M)} - \frac{x}{\varphi(q^M)(\log x)^2} H(l_2 - l_1)\right| < \frac{\varepsilon(x) x q^M}{(\log x)^2} \tag{8.9}$$

with a suitable function $\varepsilon(x) \to 0$ $(x \to \infty)$, *where*

$$H(d) = \prod_{\substack{p\mid d \\ p\nmid q}} \left(1 + \frac{1}{p-2}\right).$$

9. Distribution of additive functions on the set of shifted integers the number of prime factors of which is fixed

Kátai proved in [21] that $f \in \mathcal{A}$ has a limit distribution on the set $\{p+1 \mid p \in \mathcal{P}\}$, i.e.

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p < x \mid f(p+1) < y\} = \tilde{F}(y), \tag{9.1}$$

exists for a.a. y, and $\tilde{F}$ is a distribution function, if the series in (1.5) are convergent.

A. Hildebrand proved twenty years later in [22] that if (9.1) holds, then the series in (1.5) converge. L. Germán proved the following assertion.

Theorem 21. *Let* $f \in \mathcal{A}$, $2 \le k \le \varepsilon(x)\sqrt{\log\log x}$, $\varepsilon(x) \to 0$ *as* $x \to \infty$*. Let*

$$F_{k,x}(z) := \frac{1}{\pi_k(x)} \#\{n \le x \mid n \in \mathcal{P}_k, f(n+1) < z\}.$$

Assume that there is a sequence $k = k_x$ *and a distribution function* F *such that* $F_{k_x,x} \Rightarrow F$*. Then the 3 series in (1.5) are convergent.*

Conversely, assume that the series in (1.5) are convergent. Then, with a distribution function $G(y)$,

$$\max_{2\le k\le \varepsilon(x)\sqrt{\log\log x}} |F_{k,x}(y) - G(y)| \to 0 \quad (x \to \infty)$$

if y *is a continuity point of* G*. Consequently* $F = G$*. The characteristic function* $\varphi(t)$ *of* $F(y)$ *is given by*

$$\varphi(t) = \prod_{p\in\mathcal{P}} \left(\left(1 - \frac{1}{p-1}\right) + \sum_{\alpha=1}^{\infty} \frac{e^{itf(p^\alpha)}}{p^\alpha} \right).$$

10. Distribution of additive functions over $\mathcal{P}_k, \mathcal{N}_k$

In [16] we proved the following theorems.

Theorem 22. *Let $f \in \mathcal{A}$, and assume that the series in (1.5) are convergent. For $\eta \in (0,2)$ let $\xi_p = \xi_p(\eta)$ be the random variable distributed by $P(\xi_p = f(p^\alpha)) = \left(1-\frac{\eta}{p}\right)\left(\frac{\eta}{p}\right)^\alpha$. Assume that $\xi_p \quad (p \in \mathcal{P})$ are completely independent, $\theta(\eta) := \sum_{p\in\mathcal{P}} \xi_p(\eta)$. Let $F_\eta(y) := P(\theta(\eta) < y)$. Let furthermore*

$$G_{k,x}(y) := \frac{1}{N_k(x)} \#\{n \le x, n \in \mathcal{N}_k, f(n) < y\}. \tag{10.1}$$

Assume furthermore that

$$\sum_{f(p)\neq 0} 1/p = \infty. \tag{10.2}$$

Let $\xi_{k,x} = \frac{k}{\log\log x}$, $\quad 0 < \delta < 1/2$. Then

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[\delta,2-\delta]} \max_{y\in\mathbb{R}} |G_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0. \tag{10.3}$$

If (10.2) does not hold, then

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[\delta,2-\delta]} |G_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0 \tag{10.4}$$

for every y which is a continuity point of $F_1(y)$.

Theorem 23. *Let f be as in Theorem 22. Assume that $f(2^\alpha) = 0 \quad (\alpha = 1,2,\ldots)$. Let $\delta > 2$, $\quad A > 2+\delta$ be constants. Then*

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[2+\delta,A]} |G_{k,x}(y) - F_2^*(y)| = 0$$

for every y, which is a continuity point of F_2^*. Here $F_2^*(y) = P\left(\sum_{p>2} \xi_p(2) < y\right)$.

Theorem 24. *Let $g \in \overline{\mathcal{M}}$, and assume that*

$$\sum_p \frac{1-g(p)}{p}$$

is convergent. Let $\delta > 0$ be fixed, $0 < \eta < 2$,

$$e_p(\eta) = \left(1-\frac{\eta}{p}\right)\left(1+\frac{g(p)\eta}{p}+\frac{g(p^2)\eta^2}{p^2}+\cdots\right),$$

$M_\eta(g) = \prod_p e_p(\eta)$.

We have

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[\delta,2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} g(n) - M_{\xi_{k,x}}(g) \right| = 0.$$

Here $\xi_{k,x} = \frac{k}{\log\log x}$.

Theorem 25. *Let g be as in Theorem 24. Assume furthermore that $g(2^\alpha) = 1 \quad (\alpha = 1,2,\ldots)$. Let $0<\delta, \quad A > 2+\delta$ be constants. Then*

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[2+\delta,A]} \left| \frac{1}{N_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} g(n) - M_2^*(g) \right| = 0,$$

where

$$M_2^*(g) = \prod_{p>2} e_p(2).$$

Theorem 26. *Let $f \in \mathcal{A}, \quad f(p^\alpha) = \mathcal{O}(1)$, if $p \in \mathcal{P}$. Let $A_x = \sum_{p\le x} \frac{f(p)}{p}, \quad f^* \in \mathcal{A}$ is defined on prime powers p^α by $f^*(p^\alpha) = f(p^\alpha) - \frac{\alpha A_x}{\log\log x}$. Let $B_x^2 = \sum_{p\le x} \frac{1}{p}\left(f^*(p)\right)^2$. Let $B_x \to \infty$. Then*

$$\max_{\xi_{k,x}\in[\delta,2-\delta]}{}_k \max_{y\in\mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x \;\middle|\; \frac{f^*(n)}{B_x\sqrt{\xi_{k,x}}} < y, n\in\mathcal{N}_k \right\} - \Phi(y) \right| \to 0$$

as $x\to\infty$. Here $\delta > 0$ is an arbitrary constant.

Theorem 27. *Let f, f^*, A_x, B_x be as in Theorem 26. Let $0<\delta, \quad A > 2+\delta$ be constants. Then*

$$\max_{2+\delta\le\xi_{k,x}\le A}{}_k \max_{y\in\mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x \;\middle|\; \frac{f^*(n)}{B_x\sqrt{2}} < y \right\} - \Phi(y) \right| \to 0 \quad (x\to\infty).$$

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