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MAXIMAL OPERATORS OF FEJÉR MEANS OF WALSH–FOURIER SERIES

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Dedicated to Prof. Ferenc Schipp on his 70th birthday and to Prof. Péter Simon on his 60th birthday

Abstract. The main aim of this paper is to prove that there exists a martingale $f \in H_{1/2}$ such that the maximal Fejér operator and the conjugate Fejér operator does not belong to the space $L_{1/2}$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [6] showed that the maximal operator $\sigma^* f$ is of weak type (1, 1), from which the a.e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [7]). Fujii's theorem was extended by Weisz [9]. Namely, he proved that the maximal operator $\sigma^* f$ and the conjugate maximal operator $\widetilde{\sigma}_*^{(t)} f$ is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [8] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [11] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$. In [4] (see also [3]) the author proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

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In this paper we shall prove a stronger result than the unboundedness of the maximal operator from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$ in particular, we prove that there exists a martingale $f \in H_{1/2}(G)$ such that

$$\|\sigma^* f\|_{1/2} = +\infty$$

and

$$\|\widetilde{\sigma}_*^{(t)} f\|_{1/2} = +\infty.$$

2. Definitions and notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0,1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$
$$(x \in G,\ n \in \mathbf{N}).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$. Let $\overline{I}_n := G \backslash I_n$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N})$$

the k-th Rademacher function. If $n \in \mathbf{N}$, then $n = \sum\limits_{i=0}^{\infty} n_i 2^i$, where $n_i \in$ $\in \{0,1\}$ $(i \in \mathbf{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G,\ n \in \mathbf{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \overline{I}_n. \end{cases} \tag{1}$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M f(x) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_G f(x) w_i(x) d_\mu(x)$$

is said to be the i-th Walsh-Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left(\int_G |f(x)|^p d\mu(x) \right)^{1/p} \qquad (0 < p < +\infty).$$

The space weak-$L_p(G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the I_k dyadic interval of measure 2^{-k} will be denoted by $F_k \quad (k \in \mathbf{N})$.

Denote by $f = (f^{(n)}, n \in \mathbf{N})$ the martingale with respect to $(F_n,\ n \in \mathbf{N})$ (for details see, e.g. [10]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbf{N}} |f^{(n)}|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n\in\mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int\limits_{I_n(x)} f(u) d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$ then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \ldots)$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k\to\infty} \int\limits_G f^{(k)}(x) w_i(x) d\mu(x).$$

The Walsh-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from f.

For $n = 1, 2, \ldots$ and a martingale f the Fejér means of the Walsh-Fourier series of the function f is given by

$$\sigma_n f(x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j(f; x).$$

For a martingale

$$f \sim \sum_{n=0}^{\infty} (f_n - f_{n-1})$$

the conjugate transforms are defined by the martingale

$$\widetilde{f}^{(t)} \sim \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in [0,1)$ is fixed. Note that $\widetilde{f}^{(0)} = f$. As is well known, if f is an integrable function then the conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

Let

$$\rho_0 := r_0, \quad \rho_k := r_n, \quad \text{if } 2^{(n-1)} \leq k < 2^n.$$

Then the n-th partial sum of the conjugate transforms is given by

$$\widetilde{S}_n^{(t)} f(x) := \sum_{k=0}^{n-1} \rho_k(t) \widehat{f}(k) w_k(x).$$

The conjugate Fejér means of a martingale f are introduced by

$$\widetilde{\sigma}_n^{(t)} f(x) = \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{S}_j^{(t)} f(x) \quad (t \in [0,1);\ n \in \mathbf{P}).$$

For the martingale f we consider maximal operators

$$\sigma^* f = \sup_{n \in \mathbf{P}} |\sigma_n f(x)|, \quad \widetilde{\sigma}_*^{(t)} f = \sup_{n \in \mathbf{P}} |\widetilde{\sigma}_n^{(t)} f(x)|.$$

The n-th Fejér kernel of the Walsh-Fourier series is defined by

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

(a) $\int_I a \, d\mu = 0$;

(b) $\|a\|_\infty \leq \mu(I)^{-1/p}$;

(c) supp $a \subset I$.

The basic result of atomic decomposition is the following one.

Theorem A (Weisz [10]). *A martingale $f = (f^{(n)} : n \in \mathbf{N})$ is in H_p $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k,\ k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k,\ k \in \mathbf{N})$ of real numbers such that for every $n \in \mathbf{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \tag{2}$$

$$\sum_{k=0}^{\infty} |\mu_k\|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (2).

3. Formulation of main result

Theorem 1. *There exists a martingale $f \in H_{1/2}(G)$ such that*

$$\|\sigma^* f\|_{1/2} = +\infty$$

and

$$\|\widetilde{\sigma}_*^{(t)} f\|_{1/2} = +\infty$$

for all $t \in G$.

4. Auxiliary propositions

Lemma 1. ([5]) *Let $2 < A \in \mathbf{P}$ and $q_A := 2^{2A} + 2^{2A-2} + \ldots + 2^2 + 2^0$. Then*

$$q_{A-1} |K_{qA-1}(x)| \geq 2^{2m+2s-3}$$

for $x \in I_{2A}(0, \ldots, 0, x_{2m} = 1, 0, \ldots, 0, x_{2s} = 1, x_{2s+1}, \ldots, x_{2A-1})$, $m = 0, 1, \ldots, A-3$, $s = m+2, m+3, \ldots, A-1$.

5. Proof of the theorem

Proof of Theorem 1. Let $\{m_k : k \in \mathbf{P}\}$ be an increasing sequence of positive integers such that

(3) $$\sum_{k=1}^{\infty}\frac{1}{m_k^{1/2}}<\infty,$$

(4) $$\sum_{l=0}^{k-1}\frac{2^{4m_l}}{m_l}<\frac{2^{4m_k}}{m_k},$$

(5) $$\frac{k2^{4m_{k-1}}}{m_{k-1}}\leq\frac{2^{2m_k}}{m_k}.$$

Let
$$f^{(A)}(x):=\sum_{\{k:\ 2m_k<A\}}\lambda_k a_k,$$
where
$$\lambda_k:=\frac{1}{m_k}$$
and
$$a_k(x):=2^{2m_k}(D_{2^{2m_k+1}}(x)-D_{2^{2m_k}}(x)).$$

It is to show that the martingale $f:=(f^{(0)},f^{(1)},\ldots,f^{(A)},\ldots)\in H_{1/2}(G)$. Indeed, since
$$f^{(A)}(x)=\sum_{k=0}^{\infty}\lambda_k S_{2^A}a_k(x)$$
from (3) and Theorem A we conclude that $f\in H_{1/2}(G)$.

We write

(6) $$\sigma_{q_{m_k}}f(x)=\frac{1}{q_{m_k}}\sum_{j=0}^{2^{2m_k}-1}S_jf(x)+\frac{1}{q_{m_k}}\sum_{j=2^{2m_k}}^{q_{m_k}-1}S_jf(x)=I+II.$$

Let $j\in\{2^{2m_k},\ldots,2^{2m_k+1}-1\}$ for some $k=1,2,\ldots$. Then it is evident that
$$\widehat{f}(j):=\lim_{A\to\infty}\widehat{f^{(A)}}(j)=\frac{2^{2m_k}}{m_k}$$
and $\widehat{f}(j)=0$, if $j\notin\{2^{2m_k},\ldots,2^{2m_k+1}-1\}$, $k=1,2,\ldots$.

Consequently, for $2^{2m_k} \le j < q_{m_k}$ we can write

$$
\begin{aligned}
S_j f(x) &= \sum_{v=0}^{2^{2m_{k-1}+1}-1} \widehat{f}(v) w_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}(v) w_v(x) = \\
&= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}(v) w_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}(v) w_v(x) = \\
&= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l}}{m_l} w_v(x) + \frac{2^{2m_k}}{m_k} \sum_{v=2^{2m_k}}^{j-1} w_v(x) = \\
&= \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} (D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)) + \frac{2^{2m_k}}{m_k} (D_j(x) - D_{2^{2m_k}}(x)).
\end{aligned} \tag{7}
$$

Applying (7) in II, we have

$$
\begin{aligned}
II &= \frac{(q_{m_k} - 2^{2m_k})}{q_{m_k}} \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} (D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)) + \\
&+ \frac{2^{2m_k}}{q_{m_k} m_k} \sum_{j=2^{2m_k}}^{q_{m_k}-1} (D_j(x) - D_{2^{2m_k}}(x)) = II_1 + II_2.
\end{aligned} \tag{8}
$$

Since

$$D_{j+2^{2m_k}}(x) = D_{2^{2m_k}}(x) + w_{2^{2m_k}}(x) D_j(x)$$

for II_2, we write

$$
\begin{aligned}
|II_2| &= \frac{2^{2m_k}}{q_{m_k} m_k} \left| \sum_{j=0}^{q_{m_{k-1}}-1} (D_{j+2^{2m_k}}(x) - D_{2^{2m_k}}(x)) \right| = \\
&= \frac{2^{2m_k}}{q_{m_k} m_k} \left| w_{2^{2m_k}}(x) \sum_{j=0}^{q_{m_{k-1}}-1} D_j(x) \right| = \\
&= \frac{2^{2m_k}}{m_k} \frac{q_{m_{k-1}}}{q_{m_k}} \left| K_{q_{m_{k-1}}}(x) \right|.
\end{aligned} \tag{9}
$$

Since

$$D_{2^n}(x) \le 2^n,$$

from (4) we can write

$$|II_1| \leq c \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{10}$$

Combining (8)-(10) we get

$$|II| \geq \frac{c}{m_k} q_{m_k-1} \left| K_{q_{m_k-1}}(x) \right| - \frac{c2^{4m_{k-1}}}{m_{k-1}}. \tag{11}$$

Let $j < 2^{2m_k}$. Then from (4) we can write

$$|S_j f(x)| \leq \sum_{v=0}^{2^{2m_{k-1}+1}-1} |\widehat{f}(v)| \leq c \frac{2^{4m_{k-1}}}{m_{k-1}},$$

$$I \leq c \frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_{k-1}}} |S_j f(x)| \leq c \frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{12}$$

Combining (6), (11) and (12) we get

$$|\sigma_{q_{m_k}} f(x)| \geq \frac{c}{m_k} q_{m_k-1} \left| K_{q_{m_k-1}}(x) \right| - c \frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{13}$$

Let $x \in I_{2m_k}(0, \ldots, 0, x_{2l} = 1, 0, \ldots, 0, x_{2s} = 1, x_{2s+1}, \ldots, x_{2m_k-1})$, for some $l = [m_k/2], [m_k/2]+1, \ldots, m_k - 3, \quad s = l+2, l+3, \ldots, m_k - 1$, then from Lemma 1 and (5) we have

$$|\sigma_{q_{m_k}} f(x)| \geq \frac{c}{m_k} 2^{2l+2s} - c \frac{2^{4m_{k-1}}}{m_{k-1}} \geq \frac{c}{m_k} 2^{2l+2s-1}.$$

Hence we can write

$$\int_G |\sigma^* f(x)|^{1/2} d\mu(x) \geq \int_G |\sigma_{q_{m_k}} f(x)|^{1/2} d\mu(x) \geq$$

$$\geq \sum_{l=[m_k/2]}^{m_k-1} \sum_{s=l}^{m_k-1} \int_{I_{2m_k}(0,\ldots,0,x_{2l}=1,0,\ldots,0,x_{2s}=1,x_{2s+1},\ldots,x_{2m_k-1})} |\sigma_{q_{m_k}} f(x)|^{1/2} d\mu(x) \geq$$

$$\geq \frac{c}{m_k^{1/2}} \sum_{l=[m_k/2]}^{m_k-3} \sum_{s=l}^{m_k-1} \frac{2^{2m_k-2s}}{2^{2m_k}} 2^{l+s} \geq$$

$$\geq \frac{c}{m_k^{1/2}} \sum_{l=[m_k/2]}^{m_k-3} \sum_{s=l}^{m_k-1} \frac{2^l}{2^s} \geq c m_k^{1/2} \to \infty \quad \text{as} \quad k \to \infty,$$

$$\|\sigma^* f\|_{1/2} = +\infty.$$

From the simple calculation we obtain that

$$\widetilde{S}_j^(t) f(x) = \sum_{l=0}^{k-1} r_{2m_l}(t) \frac{2^{2m_l}}{m_l} (D_{2^{2m_l+1}} - D_{2^{2m_l}}(x)) +$$

$$+ r_{2m_k}(t) \frac{2^{2m_k}}{m_k} (D_j(x) - D_{2^{2m_k}}(x)) \quad \text{for} \quad 2m_k \leq j < q_{m_k}$$

and

$$\left| \widetilde{S}_j^{(t)} f(x) \right| \leq \sum_{v=0}^{2^{2m_{k-1}+1}-1} |\widehat{f}(v)| \leq c \frac{2^{4m_{k-1}}}{m_{k-1}} \quad \text{for} \quad j < 2m_k.$$

Then the estimation of $\left| \widetilde{\sigma}_*^{(t)} f(x) \right|$ is analogous to the estimation of $|\sigma^* f(x)|$ and we have

$$\|\widetilde{\sigma}_*^{(t)} f\|_{1/2} = +\infty.$$

Theorem 1 is proved.

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