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DISTRIBUTION OF ADDITIVE FUNCTIONS FOR SOME SUBSETS OF SHIFTED PRIMES

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Dedicated to Professor Ferenc Schipp on his seventieth, and Professor Péter Simon on his sixtieth anniversary

Abstract. Distribution of additive functions on some subsets of integers is investigated.

1. Introduction, notations

Let $\mathcal{M}$ be the set of complex valued multiplicative functions, $\mathcal{A}$ be the set of real valued additive functions, and let

$$\mathcal{M}_1 = \{f \in \mathcal{M},\ |f(n)| = 1 \quad (n \in \mathbb{N})\}.$$

$\omega(n)$ denotes the number of prime factors of n whilst $\Omega(n)$ counts the number of prime power divisors of n. p with or without indices always denotes primes. $\mathcal{P}$ stands for the whole set of the primes. Let furthermore $\{x\}$ be the fractional part of x and $\|x\|$ be the distance of x to the nearest integer. As usual $e(\alpha) :=$ $:= e^{2\pi i\alpha}$ for all real values of α, and $\lambda(E)$ denotes the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}$.

Let $\mathcal{P}_k = \{n \mid \omega(n) = k\}$ and $\mathcal{N}_k = \{n \mid \Omega(n) = k\}$. Further $\pi_k(x) =$ $= \#\{n \le x,\ \omega(n) = k\}$, $N_k(x) = \#\{n \le x,\ \Omega(n) = k\}$.

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A theorem of Erdős-Wintner [1] asserts that for an additive arithmetical function f the limit

$$\lim_{x\to\infty}\frac{1}{x}\#\{n\le x\mid f(n)<y\}=F(y) \tag{1.1}$$

exists for almost all y, and F is a distribution function, if and only if

$$\sum_{|f(p)|>1}\frac{1}{p}<\infty,\qquad \sum_{|f(p)|\le 1}\frac{f^2(p)}{p}<\infty. \tag{1.2)-(1.3}$$

and

$$\sum_{|f(p)|\le 1}\frac{f(p)}{p}\quad\text{converges.} \tag{1.4}$$

Let

$$a(x):=\sum_{\substack{p\le x\\ |f(p)|\le 1}}\frac{f(p)}{p}. \tag{1.5}$$

They proved also that

$$\frac{1}{x}\#\{n\le x\mid f(n)-a(x)<y\}$$

has a limit as $x\to\infty$ for almost all y, if and only if (1.2), (1.3) hold true.

H. Delange [2] proved that for $g\in\mathcal{M}_1$ the limit

$$\frac{1}{x}\sum_{n\le x}g(n)\to M(g) \tag{1.6}$$

exists and it is nonzero, if in the notation

$$h(p^\alpha)=g(p^\alpha)-g(p^{\alpha-1})\quad(\alpha=1,2,\ldots),\quad p\in\mathcal{P},$$
$$\sum_{\alpha=0}^{\infty}\frac{h(p^\alpha)}{p^\alpha}\ne 0\quad(p\in\mathcal{P}),$$

and

$$\sum_{p\in\mathcal{P}}\frac{1-g(p)}{p}\quad\text{is convergent.} \tag{1.7}$$

If this condition is satisfied then

$$M(g) = \prod_p \left(\sum_{\alpha=0}^{\infty} \frac{h(p^\alpha)}{p^\alpha} \right).$$

I. Kátai proved in [3] that the convergence of the series (1.2), (1.3), (1.4) imply the existence of the limit distribution of $f \in \mathcal{A}$ on the set of shifted primes, i.e. that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x | f(p+1) < y\} = F^*(y) \quad a.a.\ y$$

if (1.2), (1.3), (1.4) are convergent. He proved also that the convergence of (1.2), (1.3) imply the existence of

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x \mid f(p+1) - a(x) < y\}$$

for almost all y.

Furthermore he proved that if (1.7) holds for some $g \in \mathcal{M}_1$, then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} g(p+1) = M^*(g),$$

where

$$M^*(g) = \prod_p \left(1 - \frac{1}{p-1} + \sum_{\alpha\ge 1} \frac{g(p^\alpha)}{p^\alpha} \right).$$

The necessity of the convergence of the series (1.2), (1.3), (1.4) to the existence of the limit distribution on the set of shifted primes has been proved by Hildebrand about 20 years later [4]. L. Germán [5] proved both the necessity and the sufficiency of the convergence of the three series ((1.2), (1.3), (1.4)) to the existence of the distribution of

$$\lim \frac{1}{\pi_k(x)} \#\{n \le x \mid n \in \mathcal{P}_k, \quad f(n+1) < y\},$$

where k may depend on x, up to $k \le \epsilon(x)\sqrt{\log\log x}$ with $\epsilon(x) \to 0$ as $x \to \infty$. We note that the assertions remain valid, if we change $n \in \mathcal{P}_k$ into $n \in \mathcal{N}_k$, and we write $N_k(x)$ instead of $\pi_k(x)$.

We are interested in the following: what can we say if p runs over a subset of the primes, or in general, if π_k runs over a subset of $\mathcal{P}_k$?

2. Definition of the special subset of primes

Let $I_1,\dots,I_t$ be sets in $[0,1)$, each of them is a union of finitely many intervals. Let $\beta_1,\dots,\beta_t$ be real numbers, such that $1,\beta_1,\dots,\beta_t$ are linearly independent over Q. Let

$$\mathbb{N}^* = \{n\in\mathbb{N} \mid \{\beta_j n\}\in I_j,\ j=1,\dots,t\},$$

$$\mathcal{P}^* = \mathcal{P}\cap\mathbb{N}^*,\quad \mathcal{P}_k^* = \mathcal{P}_k\cap\mathbb{N}^*,$$

$$\mathcal{N}_k^* = \mathcal{N}_k\cap\mathbb{N}^*.$$

Let $f_j(x)$ be a function defined on $[0,1)$ so that

$$f_j(x)=\begin{cases}1 & \text{if}\quad x\in I_j,\\ \\ 0 & \text{if}\quad x\in[0,1)\setminus I_j.\end{cases}$$

Let us extend the function over $\mathbb{R}$ to be periodic mod 1, i.e. let $f_j(x+k)= = f_j(x)\quad(k\in\mathbb{Z})$. Let the Fourier series of $f_j(x)$ be $\sum_{n=-\infty}^{\infty} a_n^{(j)}e(nx)$. It is clear that $|a_n^{(j)}|\le\frac{c_j}{|n|}$, where c_j may depend on I_j. Obviously holds also $|a_n^{(j)}|\le 1$.

Let $\Delta>0$ be a small positive constant,

$$f_j^{(\Delta)}(x)=\frac{1}{(2\Delta)^2}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} f_j(x+u_1+u_2)du_1du_2.$$

Let

$$\kappa(n)=\frac{\sin 2\pi n\Delta}{4\pi\Delta n}.$$

Then

$$f_j^{(\Delta)}(x)=\sum_{n=-\infty}^{\infty} b_n^{(j)}e(nx)$$

$$b_n^{(j)}=\kappa(n)^2a_n^{(j)},$$

$$|b_n^{(j)}|\le\min{}^2\left(1,\frac{1}{\Delta|n|}\right).$$

Let

$$I_j^{(-\Delta)}=\{x\mid (x-2\Delta,x+2\Delta)\subseteq I_j\},$$

$$I_j^{(\Delta)} = \{x \mid (x-2\Delta, x+2\Delta) \cap I_j \neq \emptyset\}.$$

Observe that

$$\lambda(I_j^{(\Delta)} \setminus I_j^{(-\Delta)}) \le c_j \Delta,$$

where c_j is the number of the endpoints of the intervals occurring in I_j.

Let the discrepancy of the sequence $y_1, \dots, y_N$ mod 1 be defined as usual by

$$\mathcal{D}_N(y_1, \dots, y_N) = \sup_{[\alpha,\beta)\subseteq[0,1)} \left| \frac{1}{N} \#\{j = 1, \dots, N \ \{y_j\} \in [\alpha, \beta)\} - (\beta - \alpha) \right|.$$

According to a well-kown theorem of Erdős and Turán [7], we have

$$\mathcal{D}_N(y_1, \dots, y_N) \le C \left(\sum_{k=1}^{T} \frac{|\psi_k|}{k} + \frac{1}{T} \right), \tag{2.1}$$

where C is an absolute constant, T is an arbitrary integer and

$$\psi_k = \sum_{j=1}^{N} e(ky_j).$$

Let

$$s(n) := f_1(\beta_1 n) \cdots f_t(\beta_t n),$$

$$s_\Delta(n) := f_1^{(\Delta)}(\beta_1 n) \cdots f_t^{(\Delta)}(\beta_t n).$$

It is clear that $s(n) = 1$ if $n \in \mathbb{N}^*$ and $s(n) = 0$ if $n \notin \mathbb{N}^*$. Furthermore, $s(n) = s_\Delta(n)$ if $\{\beta_j n\} \in I_j^{(-\Delta)}$ for $j = 1, \dots, t$ or if $\{\beta_j n\} \in [0,1) \setminus I_j^{(\Delta)}$ for some $1 \le j \le t$ and $0 \le s_\Delta(n) \le 1$ holds always. Let $K \ge \left(\frac{1}{\Delta}\right)^4$. Then

$$\sum_{|n|\ge K} |b_n^{(j)}| \le 2 \sum_{n>K} \frac{1}{\Delta^2 n^2} \le 2\Delta^2.$$

Let

$$f_j^{(\Delta,K)}(x) = \sum_{|n|<K} b_n^{(j)} e(nx).$$

Then

$$|f_1^{(\Delta)}(\beta_1 n) \cdots f_t^{(\Delta)}(\beta_t n) - f_1^{(\Delta,K)}(\beta_1 n) \cdots f_t^{(\Delta,K)}(\beta_t n)| \le 2t\Delta^2.$$

Let

$$s_{\Delta,K}(n) := \sum_{|n_1|<K} \dots \sum_{|n_t|<K} b_{n_1}^{(1)} \dots b_{n_t}^{(t)} e((n_1\beta_1 + \dots + n_t\beta_t)n).$$

We obtain that

$$|s(n) - s_{\Delta,K}(n)| \le 2t\Delta^2$$

if $\{\beta_j n\} \in I_j^{(-\Delta)}$ $(j = 1, \dots, t)$, or $\{\beta_j n\} \in [0,1) \setminus I_j^{(\Delta)}$ for some $1 \le j \le t$ and

$$|s_{\Delta,K}(n)| \le 1 + 2t\Delta^2$$

otherwise.

3. Formulation of the results

Let

$$\pi^*(x) = \#\{p \le x, p \in \mathcal{P}^*\}, \qquad \pi_k^*(x) = \#\{n \le x, n \in \mathbb{N}^*, n \in \mathcal{N}_k\},$$

and

$$A := \lambda(I_1) \cdots \lambda(I_t).$$

Theorem 1. *Let $g \in \mathcal{M}_1$, and assume that*

$$\sum_{p \in \mathcal{P}} \frac{\mathrm{Re}(1 - g(p))}{p} \tag{3.1}$$

is convergent. Then

$$\frac{1}{A\pi(x)} \sum_{\substack{p \le x \\ p \in \mathcal{P}^*}} g(p+1) - \frac{1}{\pi(x)} \sum_{p \le x} g(p+1) \to 0 \quad (x \to \infty). \tag{3.2}$$

Especially, if

$$\sum_{p \in \mathcal{P}} \frac{1 - g(p)}{p} \qquad \text{is convergent,} \tag{3.3}$$

then

$$\frac{1}{A\pi(x)} \sum_{\substack{p\le x\\ p\in\mathcal{P}^*}} g(p+1) \to M^*(g). \tag{3.4}$$

Remarks. (1) Choosing $g(n)=1 \quad (n\in\mathbb{N})$ we obtain that

$$\pi^*(x) = (1+o_x(1))A\pi(x) \quad (x\to\infty).$$

(2) From Theorem 1 we obtain directly

Theorem 2. *Let $f\in\mathcal{A}$, and assume that (1.2), (1.3) hold. Let*

$$a(x) := \sum_{\substack{p\le x\\ |f(p)|\le 1}} \frac{f(p)}{p}.$$

Let

$$F_x^*(y) = \frac{1}{\pi^*(x)}\#\{p\le x, p\in\mathcal{P}^*,\ f(p+1)-a(x)<y\},$$

$$F_x(y) = \frac{1}{\pi(x)}\#\{p\le x, p\in\mathcal{P},\ f(p+1)-a(x)<y\}.$$

Then $F_x^*(y) \Longrightarrow F(y)$, where $F(y) := \lim_{x\to\infty} F_x(y)$ for almost all y.

The above theorems can be generalized as follow.

Theorem 3. *Let $g\in\mathcal{M}_1$, and assume that (3.1) holds. Then*

$$\sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \left| \frac{1}{\pi_k^*(x)} \sum_{\substack{n\le x\\ n\in\mathcal{P}_k^*}} g(n+1) - \frac{1}{\pi_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} g(n+1) \right| \to 0$$

$$(x\to\infty), \tag{3.5}$$

where $\varepsilon(x)\to 0$ as $x\to\infty$. Especially, if (3.3) holds then

$$\frac{1}{A\pi_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{P}_k^*}} g(n+1) \to M^*(g).$$

This theorem in the special case $\mathcal{P}_k^* = \mathcal{P}_k$ was proved in [6]. Similarly we have

Theorem 4. *Let $f \in \mathcal{A}$, and assume that (1.2), (1.3) hold. Let*

$$a(x) := \sum_{\substack{p \le x \\ |f(p)| \le 1}} \frac{f(p)}{p}.$$

Let

$$F_{k,x}^*(y) = \frac{1}{\pi_k^*(x)} \#\{n \le x,\ n \in \mathcal{P}_k^*,\ f(n+1) - a(x) < y\},$$

$$F_{k,x}(y) = \frac{1}{\pi_k(x)} \#\{n \le x,\ n \in \mathcal{P}_k,\ f(n+1) - a(x) < y\}.$$

Then uniformly for all $2 \le k \le \epsilon(x)\sqrt{\log\log x}$ we have

$$F_{k,x}^*(y) \Longrightarrow G(y) \qquad (x \to \infty),$$

where $G(y) := \lim_{x\to\infty} F_{k,x}(y)$ for almost all y.

Remark. Theorem 3 remains true if we substitute $\mathcal{P}_k$ by $\mathcal{N}_k$ and $\pi_k(x)$ by $N_k(x)$.

4. Proof of Theorem 1

Assume that (3.1) holds true.

Let $S(x) = \sum_{p \le x} g(p+1)$, $\quad S^*(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P}^*}} g(p+1)$. We have

$$S^*(x) = \sum_{p \le x} g(p+1)s(p),$$

where $s(\cdot)$ is defined in Section 2. Furthermore, we can change $s(p)$ by $s_{\Delta,K}(p)$, if $\{\beta_j p\} \notin I_j^{(\Delta)} \setminus I_j^{(-\Delta)}$. Let

$$E(n_1, \dots, n_t) = \sum_{p \le x} g(p+1) e((\beta_1 n_1 + \dots + \beta_t n_t)p). \tag{4.1}$$

Observe that $a_0^{(j)} = b_0^{(j)} = \lambda(I_j)$. Thus $a_0^{(1)} \dots a_t^{(1)} = A$. We have

$$(4.2)\qquad \begin{aligned} |S^*(x) - AS(x)| \le & \sum_{\substack{|n_1|<K,\dots,|n_t|<K \\ (n_1,\dots,n_t)\neq(0,\dots,0)}} \cdots \sum |b_{n_1}^{(1)}| \dots |b_{n_t}^{(t)}||E(n_1,\dots,n_t)| + \\ & + c\Delta\pi(x) + Error, \end{aligned}$$

where the *Error* comes from those primes p for which $\{\beta_j p\} \in I_j^{(\Delta)} \setminus I_j^{(-\Delta)}$ holds for at least one j.

We can estimate it by using the Erdős-Turán theorem. Let

$$(4.3)\qquad \psi_{k,j} = \sum_{p\le x} e(k\beta_j p).$$

Since $\lambda(I_j^{\Delta} \setminus I_j^{-\Delta}) \le c\Delta$, and $I_j^{\Delta} \setminus I_j^{-\Delta}$ is a union of finitely many intervals, therefore by choosing $T = \frac{1}{\Delta}$, we have

$$Error \le c\Delta + \sum_{j=1}^{T} \sum_{1\le k\le \frac{1}{\Delta}} \frac{1}{k}|\psi_{k,j}|.$$

According to a well-known theorem due to I.M. Vinogradov, αp is uniformly distributed for every irrational α, then for $\alpha = \beta_1, \dots \beta_t$ as well, consequently $\psi_{k,j} = o_x(1)\pi(x)$ as $x \to \infty$ for every k and j.

Thus

$$Error \le 2c\Delta \qquad \text{if} \quad x > x_2.$$

Now we estimate (4.1).

Let $Y_x \to \infty$ arbitrarily slowly. Let

$$g_1(n) := \prod_{\substack{p^\alpha \| n \\ p\le Y_x}} g(p^\alpha), \quad g_2(n) := \prod_{\substack{p^\alpha \| n \\ p> Y_x}} g(p^\alpha).$$

We know that there is complex number $\tau(x), |\tau(x)| = 1$, such that

$$(4.4)\qquad \sum_{p\le x} |g_2(p+1) - \tau(x)| \le \varepsilon(Y_x)\pi(x),$$

where $\varepsilon(Y_x) \to 0$, if $Y_x \to \infty$. (For the proof of (4.4) see [3].) Let

$$E_Y(n) := \prod_{\substack{p^\alpha \| n \\ p< Y_x}} p^\alpha.$$

One can prove that

$$\frac{1}{\pi(x)}\#\{p\le x \mid E_{Y_x}(p+1)>Y_x^{A_x}\}\to 0 \quad (x\to\infty) \tag{4.5}$$

if $A_x\to\infty$ as $x\to\infty$.

Consequently

$$|E(n_1,\dots,n_t)|\le \varepsilon(Y_x)\pi(x)+\left|\sum_{p\le x} g_1(p+1)e((n_1\beta_1+\ldots+n_t\beta_t)p)\right|.$$

Let $\eta=n_1\beta_1+\ldots+n_t\beta_t$. Let $u(m)$ be defined by $\sum_{d|m}u(d)=g_1(m)$, i.e. let $u(p^\alpha)=g(p^\alpha)-g(p^{\alpha-1})$, if $p\le Y_x$, and 0 otherwise. Observing (4.5), we have

$$\sum g_1(p+1)e(\eta p)=\sum u(d)\sum_{\substack{p+1\equiv 0 \pmod d\\ p\le x}} e(\eta p)+o(\pi(x)),$$

where d runs over those integers for which $P(d)\le Y_x$, and $d\le Y_x^{A_x}$.

Now we estimate

$$\sum_{\substack{p+1\equiv 0(d)\\ p\le x}} e(\eta p)=\frac{1}{d}\sum_{a=0}^{d-1} e\left(-\frac{a}{d}\right)\sum_{p\le x} e\left(\left(\eta+\frac{a}{d}\right)p\right). \tag{4.6}$$

Since $\eta+\frac{a}{d}$ is irrational, therefore according to a classical result of I.M. Vinogradov (4.6) is $o\left(\frac{\pi(x)}{d}\right)$ for every fixed d. For a fixed $n_1,\dots,n_t$ we can choose such an Y_x, A_x tending to infinity for which

$$\max_{\substack{P(d)\le Y_x\\ d\le Y_x^{A_x}}}\left|\frac{1}{\pi(x,d,-1)}\sum_{\substack{p\le x\\ p+1\equiv 0(d)}} e(\eta p)\right|\to 0 \qquad \text{as} \quad x\to\infty.$$

Since we have only finitely many choices for $n_1,\dots,n_t$ we can choose the minimum of Y_x, and A_x. Hence we obtain that

$$\sum_{\substack{|n_1|<K,\dots,|n_t|<K\\ (n_1,\dots,n_t)\ne(0,\dots,0)}}\cdots\sum |b_{n_t}^{(1)}|\dots|b_{n_t}^{(t)}||E(n_1,\dots,n_t)|=o(\pi(x)) \quad (x\to\infty).$$

Here we used also the fact that

$$\sum |b_{n_j}^{(j)}| < \infty.$$

Since Δ is arbitrarily small, the proof is complete.

5. Proof of Theorem 3

Let

$$S_k(x) = \sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} g(n+1), \quad S_k^*(x) = \sum_{\substack{n\le x\\ n\in\mathcal{P}_k^*}} g(n+1),$$

where $2 \le k \le \varepsilon(x)\sqrt{\log\log x}$, $\varepsilon(x) \to 0$ as $x \to \infty$.

Assume that (3.1) holds. We can argue as in the proof of Theorem 1.

In [5] it was proved that

$$\sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \frac{1}{\pi_k(x)} \#\{n \le x, n \in \mathcal{P}_k, P(n) < x^{1-\delta}\} \to 0 \quad (x\to\infty) \tag{5.1}$$

for all fixed $0 < \delta \le 1$. First we overestimate the number of those $n \in \mathcal{P}_k, n \le x$ for which $\{\beta_j n\} \in I_j^{\Delta} \setminus I_j^{-\Delta}$ holds for at least one j. We shall prove that

$$\sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \frac{1}{\pi_k(x)} \left| \sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} e(l\beta_j n)\right| \to 0, \tag{5.2}$$

for every nonzero integer l and $j = 1, \ldots, t$.

Hence, by the theorem of Erdős and Turán we obtain that

$$\limsup_{x\to\infty} \sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \frac{1}{\pi_k(x)} \#\{n \le x, n \in \mathcal{P}_k, \{\beta_j n\} \in I_j^{\Delta} \setminus I_j^{-\Delta}\} \le c\Delta. \tag{5.3}$$

Let us define $g_1(n), g_2(n), Y_x, A_x$ as in the proof of Theorem 1. We can argue as earlier. We have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} g_1(n+1)e(\eta n) &= \sum_{\substack{P(d)\le Y_x\\ d\le Y_x^{A_x}}} u(d) \sum_{\substack{n+1\equiv 0 \pmod d\\ n\le x\\ n\in\mathcal{P}_k}} e(\eta n) = \\ &= \sum_{\substack{P(d)\le Y_x\\ d\le Y_x^{A_x}}} \frac{u(d)}{d} \sum_{a=0}^{d-1} e\left(-\frac{a}{d}\right) \sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} e\left(\left(\eta + \frac{a}{d}\right) n\right).\end{aligned} \tag{5.4}$$

We shall prove that for every irrational κ,

$$\sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \frac{1}{\pi_k(x)}\left|\sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} e(\kappa n)\right| \to 0. \tag{5.5}$$

Hence we obtain (5.2), and that (5.4) divided by $\pi_k(x)$ tends to zero uniformly as $2\le k\le\varepsilon(x)\sqrt{\log\log x}$.

Finally we prove (5.5). We shall write $n\in\mathcal{N}_k$ as $n=pm, p=P(n)$. Taking into account (5.1), we obtain that

$$\sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} e(\kappa n) = o_x(1)\pi_k(x) + \sum_{\substack{m\le x^\delta\\ m\in\mathcal{P}_{k-1}}}\sum_{p\le\frac{x}{m}} e(\kappa mp). \tag{5.6}$$

Let $\tau=\frac{x}{(\log x)^{30}}$. In order to estimate

$$\Sigma_m := \sum_{p\le\frac{x}{m}} e(\kappa mp), \tag{5.7}$$

we shall approximate κm by a rational number $\frac{a_m}{q_m}$ satisfying

$$\left|\kappa m-\frac{a_m}{q_m}\right| < \frac{1}{q_m\tau}, \quad q_m<\tau. \tag{5.8}$$

We shall use the following lemma due to I.M. Vinogradov. (A proof can be found in [11], Corollary 16.3, page 142.)

Lemma 1. *Let y be a large number and assume that $R\le q\le\frac{y}{R}$, $1\le$ $\le R\le y^{\frac14}$, $(a,q)=1$, $\left|\beta-\frac{a}{q}\right|\le\frac{1}{q^2}$. Then*

$$\sum_{p\le y} e(\beta p) \ll \frac{y}{\sqrt{R}}\cdot(\log y)^{16}.$$

If $q_m>(\log x)^{40}$, then we can apply Lemma 1, and get that

$$|\Sigma_m| \ll \frac{\frac{x}{m}}{\log^2\frac{x}{m}}.$$

Let us assume that $q_m \le (\log x)^{40}$. By using Lemma 3.1 in Vaughan [12], after partial summation, we obtain that

$$|\Sigma_m| \ll \frac{x}{q_m m \log \frac{x}{m}}.$$

Thus

$$\frac{1}{\pi_k(x)} \left| \sum_{\substack{n\le x\\ n\in\mathcal{P}_k}} e(\kappa n) \right| \le o_x(1) + c\frac{\pi(x)}{\pi_k(x)} \frac{1}{L} \sum_{\substack{q_m\ge L\\ m\in\mathcal{P}_{k-1}\\ m\le x}} \frac{1}{m} + \frac{1}{\pi_k(x)} \sum_{\substack{q_m<L\\ m\in\mathcal{P}_{k-1}\\ m<x^\delta}} |\Sigma_m|. \tag{5.9}$$

The second term on the right hand side of (5.9) is $\le \frac{c}{L}$. It remains to estimate the last term.

Let $l \le L$, and consider those $m \in \mathcal{P}_{k-1}$, for which $q_m = l$. Assume that these numbers are $m_1 < \ldots < m_T (< x^\delta)$. Then $\left| m_j\kappa - \frac{a_{m_j}}{l} \right| < \frac{1}{l\tau}$, and so $\left| l\kappa - \frac{a_{m_j}}{m_j} \right| < \frac{1}{m_j\tau}$ for every $j = 1, \ldots, T$. Assume that $T \ge 2$. Then

$$\left| \frac{a_{m_u}}{m_u} - \frac{a_{m_v}}{m_v} \right| \le \frac{1}{\tau}\left(\frac{1}{m_u} + \frac{1}{m_v}\right) \tag{5.10}$$

and this may hold, only if $\frac{a_{m_u}}{m_u} = \frac{a_{m_v}}{m_v}$. Let

$$\frac{R}{S} = \frac{a_{m_j}}{m_j} \quad (j = 1, \ldots, T) \quad (R, S) = 1.$$

Thus $Rm_j \equiv 0 \pmod S$, and so $m_j \equiv 0 \pmod S$ $(j = 1, \ldots, T)$. $S = S_l$ cannot be bounded as $x \to \infty$. Doing this for $l = 1, \ldots, L$, the last sum on the right hand side of (5.9) is less than

$$\ll \frac{\pi(x)}{\pi_k(x)} \sum_{l=1}^{L} \frac{1}{l} \sum_{\substack{\nu S_l\in\mathcal{P}_{k-1}\\ \nu S_l<x^\varepsilon}} \frac{1}{\nu S_l} \ll \frac{k}{\log\log x} \left\{ \frac{1}{1S_1} + \ldots + \frac{1}{LS_L} \right\}. \tag{5.11}$$

In [5] it was proved that

$$\sup_{2\le k\le\varepsilon(x)\sqrt{\log\log x}} \frac{1}{\pi_k(x)} \#\{n \le x, n \in \mathcal{P}_k, p(n) < \exp\exp(\sqrt{\log\log x})\} \to 0$$

$$(x \to \infty),$$

thus the right hand side of (5.11) tends to zero as $x \to \infty$, we obtain that (5.9) is $o_x(1)$, uniformly as $2 \le k \le \varepsilon(x)\sqrt{\log\log x}$.

The proof of Theorem 3 is complete.

6. Multiplicative functions on the set $\mathbb{N}_k$

Let $\xi_{k,x} = \frac{k}{\log\log x}$, δ be a positive constant, $\delta < \frac{1}{2}$. In [8] it was proved that for every irrational α,

$$\sup_{\delta \le \xi_{k,x} \le 2-\delta} \sup_{g \in \mathcal{M}_1} \frac{1}{\pi_k(x)} \left| \sum_{\substack{n \le x \\ n \in \mathcal{P}_k}} g(n)e(n\alpha) \right| \to 0 \qquad \text{as} \quad x \to \infty. \tag{6.1}$$

The inequality

$$\sup_{\delta \le \xi_{k,x} \le 2-\delta} \sup_{g \in \mathcal{M}_1} \frac{1}{N_k(x)} \left| \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} g(n)e(n\alpha) \right| \to 0 \qquad \text{as} \quad x \to \infty \tag{6.2}$$

holds as well. Arguing as in the proof of Theorem 1, we obtain

Theorem 5. *Let S be as in §2, $\delta > 0$. Then*

$$\sup_{\delta \le \xi_{k,x} \le 2-\delta} \sup_{g \in \mathcal{M}_1} \frac{1}{N_k(x)} \left| A \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} g(n)e(n\alpha) - \sum_{\substack{n \le x \\ n \in N_k^*}} g(n)e(n\alpha) \right| \to 0 \tag{6.3}$$

$$\text{as} \quad x \to \infty,$$

furthermore

$$\sup_{\delta \le \xi_{k,x} \le 2-\delta} \sup_{g \in \mathcal{M}_1} \frac{1}{\pi_k(x)} \left| A \sum_{\substack{n \le x \\ n \in \mathcal{P}_k}} g(n)e(n\alpha) - \sum_{\substack{n \le x \\ n \in \mathcal{P}_k^*}} g(n)e(n\alpha) \right| \to 0 \tag{6.4}$$

$$\text{as} \quad x \to \infty.$$

Hence, and from the results proved in [9, 10] we obtain immediately the following assertions.

Assume that $g \in \mathcal{M}_1$, and (1.7) is convergent. Let $0 < \delta < \frac{1}{2}$, $\delta \leq \leq \eta \leq 2 - \delta$,

$$e_p(\eta) = \left(1 - \frac{\eta}{p}\right)\left(1 + \frac{g(p)\eta}{p} + \frac{g(p^2)\eta^2}{p^2} + \dots\right),$$

$M_\eta(g) := \prod_p e_p(\eta)$. The product is convergent.

Theorem 6. *If (1.7) is convergent, then*

$$\lim_{x\to\infty} \sup_{\delta \leq \xi_{k,x} \leq 2-\delta} \left| \frac{1}{AN_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k^*}} g(n) - M_{\xi_{k,x}}(g) \right| = 0$$

and

$$\lim_{x\to\infty} \sup_{\delta \leq \xi_{k,x} \leq 2-\delta} \left| \frac{1}{A\pi_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{P}_k^*}} g(n) - M_{\xi_{k,x}}(g) \right| = 0.$$

Theorem 7. *Let $f \in \mathcal{A}$, $f(p^\alpha)$ is bounded as p^α runs over the prime powers. Let*

$$A_x := \sum_{p \leq x} \frac{f(p)}{p}, \qquad f^*(p^\alpha) := f(p^\alpha) - \xi_{\alpha,x_2} A_x, \qquad B_x^2 = \sum_{p \leq x} \frac{1}{p}(f^*(p))^2.$$

Assume that f^* is extended to $\mathbb{N}$ so that $f^* \in \mathcal{A}$. Let $B_x \to \infty$.

Then

$$\lim_{x\to\infty} \max_{\delta \leq \xi_{k,x} \leq 2-\delta} \max_{y \in \mathbb{R}} \left| \frac{1}{AN_k(x)} \# \left\{ n \leq x, n \in \mathcal{N}_k^*, \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} \right\} - \Phi(y) \right| = 0.$$

Here

$$\Phi(y) = \frac{1}{2\pi} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du.$$

Theorem 8. *Let $f \in \mathcal{A}$. Assume that the 3 series in (1.2), (1.3), (1.4) are convergent. Let*

$$F_{k,x}^*(y) := \frac{1}{\#\{n \leq x, n \in \mathcal{N}_k^*\}} \#\{n \leq x, n \in \mathcal{N}_k^*, f(n) < y\}.$$

Furthermore, for some $\eta \in (0,2)$*, and* $p \in \mathcal{P}$ *let* $\xi_p = \xi_p(\eta)$ *be the random variable distributed by*

$$P(\xi_p = f(p^\alpha)) = \left(1 - \frac{\eta}{p}\right)\left(\frac{\eta}{p}\right)^\alpha \quad (\alpha = 0, 1, 2, \ldots).$$

Assume that $\xi_p (p \in \mathcal{P})$ *are completely independent. Let*

$$\Theta(\eta) = \sum_p \xi_p(\eta).$$

As we know from the theorem of Kolmogorov, the right hand side is convergent. Let $F_\eta(y) := P(\Theta_p(\eta) < y)$*. Then*

$$\lim_{x\to\infty} \max_{\delta \le \xi_{k,x} \le 2-\delta} \max_{y\in\mathbb{N}} |F^*_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0.$$

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