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SOME CONVERGENCE AND DIVERGENCE RESULTS WITH RESPECT TO SUMMATION OF FOURIER SERIES ON ONE AND TWO–DIMENSIONAL UNBOUNDED VILENKIN GROUPS

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This paper is dedicated
to Professor Ferenc Schipp on the occasion of his 70th birthday
and to Professor Péter Simon on the occasion of his 60th birthday

Abstract. It is a highly celebrated problem in dyadic harmonic analysis the pointwise convergence (or divergence) of the Fejér means of functions on unbounded Vilenkin groups. We give a résumé of the very recent developments concerning this matter both in the point of view of the one and two dimensional cases.

1. Introduction, some known results

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [39, 1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \ldots\}, \mathbb{P} := \mathbb{N} \setminus \{0\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, with the group operation mod m_k addition. Since the group is discrete, then

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every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m.

Furthermore, let $L^p(G_m)\,(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$ $(f \in L^1)$.

The definition of the maximal function of integrable function f and the definition of the maximal Hardy space on Vilenkin groups is as follows

$$f^* := \sup_{n \in \mathbb{N}} |E_n f| = M_n \int_{I_n(x)} f d\mu,$$

$$H^1 := \left\{ f \in L^1(G_m) : f^* \in L^1(G_m) \right\}, \quad \|f\|_{H^1} := \|f^*\|_1.$$

The atomic Hardy space is defined by functions called atom: An atom is a function $a : G_m \to L^\infty(G_m)$ either $a = 1$ or supp $a \subset I_n(x)$, $\int_{I_n(x)} a = 0$, $\|a\|_\infty \le 1/\mu(I_n(x))$ for some interval $I_n(x)$.

Then the function f belongs to the atomic Hardy space, that is $f \in H(G_m)$ if and only if $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ $\lambda_i \in \mathbb{C}$, a_i is an atom $(i \in \mathbb{N})$. Moreover, $H(G_m)$ is a Banach space with the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H(G_m)$. If the sequence m is bounded, then $H(G_m) = H^1(G_m)$, Moreover, $\|f\|_H \sim \sim \|f\|_{H^1}$. That is, the two norms are equivalent.

If the sequence m is not bounded, then the situation changes. That is, in this situation we have

$$H(G_m) \underset{\neq}{\subset} H^1(G_m).$$

This inconvenience comes from that there are ”too few” intervals. In order to overcome this difficulty Simon defined more intervals on unbounded Vilenkin groups [34]. If the sequence m is not bounded, then define the set of intervals in a different way. That is, we have “more” intervals.

$I \subset G_m$ is called an interval if $I = \bigcup_{k \in U} I_n(x,k)$ where U is obtained from:

$$U^0_{n,0} = \left\{0, ..., m_n - 1\right\},$$

$$U^1_{n,0} = \left\{0, ..., \left[\frac{m_n}{2}\right] - 1\right\}, \quad U^1_{n,1} = \left\{\left[\frac{m_n}{2}\right], ..., m_n - 1\right\},$$

$$U^2_{n,0} = \left\{0, ..., \left[\frac{[m_n/2] - 1}{2}\right] - 1\right\},$$

$$U^2_{n,1} = \left\{\left[\frac{[m_n/2] - 1}{2}\right], ..., \left[\frac{m_n}{2}\right] - 1\right\}, ...$$

and so on, where $I_n(x,k) := \{y \in G_m : y_j = x_j \ (j < n), y_n = k\}$. Simon [34]: the two Hardy spaces coincide.

Let a be a nonnegative real. We say that the function $f \in L^1(G_m)$ belongs to the logarithm space $L(\log^+ L)^a(G_m)$ if the integral

$$\|f\|_{L(\log^+ L)^a} := \int_{G_m} |f(x)| \left(\log^+(|f(x)|)\right)^a d\mu(x)$$

is finite. The positive logarithm $\log^+$ is defined as

$$\log^+(x) := \begin{cases} \log x & \text{if } x > \exp(1), \\ 1 & \text{otherwise.} \end{cases}$$

Let X and Y be either $L(\log^+ L)^a(G_m)$ or $L^p(G_m)$ for some $1 \le p \le \infty$, and $a \ge 0$ with norms $\|.\|_X$ and $\|.\|_Y$. We say that operator T is of type (X, Y) if there exists an absolute constant $C > 0$ for which $\|Tf\|_Y \le C\|f\|_X$ for all $f \in X$. If $X = Y = L^p(G_m)$ then we often say that T is of type (p, p) instead of type (L^p, L^p). T is of weak type (L^1, L^1) (or weak type $(1, 1)$) if there exists an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \le C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps a function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \le \infty$ (see e.g. [4]).

Let $M_0 := 1, M_{n+1} := m_n M_n$ $(n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\}, \quad i \in \mathbb{N}),$$

where only a finite number of n_i's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that

$$\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & \text{if } x_n \ne 0, \\ m_n & \text{if } x_n = 0 \end{cases} \quad (x \in G_m, \ n \in \mathbb{N}).$$

The n-th Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\text{mod } m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k\oplus n} = \psi_k\psi_n, \quad \psi_n(x+y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1 \quad (k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int\limits_{G_m} f\bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f,$$

$$K_n(y,x) = K_n(y-x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(y-x),$$

$$\left(n \in \mathbb{P}, y, x \in G_m, \quad \hat{f}(0) := \int\limits_{G_m} f, \quad S_0 f = D_0 = K_0 = 0,\ f \in L^1(G_m)\right).$$

It is well-known that

$$S_n f(y) = \int\limits_{G_m} f(x) D_n(y-x) d\mu(x),$$

$$\sigma_n f(y) = \int\limits_{G_m} f(x) K_n(y-x) d\mu(x)$$

$$(n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0), \\ 0 & \text{if } x \notin I_n(0), \end{cases}$$

$$S_{M_n} f(x) = M_n \int\limits_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Next, we introduce some notations with respect to the theory of two-dimensional Vilenkin systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequences $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequences (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstanding.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\hat{f}(n_1, n_2) := \int\limits_{G_m \times G_{\tilde{m}}} f(x^1, x^2) \bar{\psi}_{n_1}(x^1) \bar{\psi}_{n_2}(x^2) d\mu(x^1, x^2),$$

$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(y^1) \psi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y, x) = D_{n_1}(y^1 - x^1) D_{n_2}(y^2 - x^2) :=$$

$$:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \psi_{k_1}(y^1) \psi_{k_2}(y^2) \bar{\psi}_{k_1}(x^1) \bar{\psi}_{k_2}(x^2),$$

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f,$$

$$K_{n_1,n_2}(y, x) = K_{n_1,n_2}(y - x) := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} D_{k_1,k_2}(y - x),$$

$$(y = (y^1, y^2), \quad x = (x^1, x^2) \in G_m \times G_{\tilde{m}}).$$

It is also well-known that

$$\sigma_{n_1,n_2} f(y) = \int\limits_{G_m \times G_{\tilde{m}}} f(x) K_{n_1,n_2}(y - x) d\mu(x),$$

$$S_{M_{n_1}, \tilde{M}_{n_2}} f(x) = M_{n_1} \tilde{M}_{n_2} \int\limits_{I_{n_1}(x^1) \times I_{n_2}(x^2)} f(y) d\mu(y) = (E^1_{n_1} \otimes E^2_{n_2}) f(x).$$

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on one and two-dimensional unbounded Vilenkin groups.

Fine [5] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C,α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [24]. Schipp [30] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. He proved that $\sigma^* := \sup |\sigma_n|$ is of weak type (L^1, L^1).

For the proof that σ^* is bounded from H^1 to L^1 see Schipp and Simon [32] and also Fujii [7].

The theorem of Schipp is generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [38], and later to bounded Vilenkin systems by Pál and Simon [27].

Now, what about the Vilenkin groups with unbounded generating sequences? The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [28]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [28] that for an arbitrary sequence m $(\sup_n m_n = \infty)$ and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [28] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable.

Moreover, the result of Price also implies that for each unbounded Vilenkin group G_m one can give an integrable function $f \in L^1(G_m)$ such that even the special subsequence of the Fejér means $\sigma_{M_n} f$ does not converge to the function in the Lebesgue norm L^1.

On the other hand, norm convergence of the full partial sums for L^p, $p > 1$, is known for the unbounded case. This result is proven by Schipp [31], Simon [33] and Young [47]. This trivially implies the norm convergence $\sigma_n f \to f$ for all $f \in L^p$, where $1 < p < \infty$. But what positive can be said with respect to the L^1 case?

The concept of Nörlund logarithmic means is as follows

$$t_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \quad \text{where} \quad l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

For further information with respect to Nörlund logarithmic means on Walsh-Paley systems see some papers of Gát and Goginava and Tkebuchava [19, 18, 20]. In their paper Gát and Goginava [19] proved (for Walsh-Paley system), that there exists an $f \in L^1$ such that

$$\|t_n f - f\|_1 \not\to 0.$$

On the other hand, Blahota and Gát [3] proved that the Nörlund logarithmic means have better approximation properties on some unbounded Vilenkin groups, than the Fejér means. Namely:

Theorem 1. *If* $f \in L^1$ *and*

$$\limsup_{n\in\mathbb{N}} \frac{\sum\limits_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty,$$

then

$$\|t_{M_n} f - f\|_1 \to 0.$$

In the case $f \in C$ the convergence holds in the supremum norm. This means that in the case of some unbounded Vilenkin groups the behavior of the Nörlund means t_{M_n} is better than the behavior of the Fejér means σ_{M_n}.

On the other hand, this can not be said in general, that is for the means t_n. That is, Blahota and Gát proved [3]:

If $\log m_n = O(n^\delta)$ for some $0 < \delta < 1/2$, then there exists an $f \in L^1$ such that

$$\|t_n f - f\|_1 \not\to 0.$$

It is surprising that the behavior of the Nörlund logarithmic means is worse than the behavior of the Fejér means in the Walsh-Paley or in the bounded Vilenkin case, but the situation changes on a class of unbounded Vilenkin groups. For the time being it is an open question that it is possible to give an unbounded generating sequence m such that we would have the norm convergence $\|t_n f - f\|_1 \to 0$ for all integrable functions f.

We already have written about the behavior of the Nörlund logarithmic means. Another weighted mean of the partial sums of the Fourier series is the logarithmic mean, which seems to be very similar to the Nörlund ones:

$$u_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k}, \quad \text{where} \quad l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is easy to see in the trigonometric, Walsh and bounded Vilenkin case, that for each integrable function f the logarithmic means $u_n f$ converge to f both in norm and a.e. This is a trivial consequence of the nice properties of the Fejér means and the Abel transformation. On the other hand, if we investigate these means on unbounded Vilenkin groups then the situation is different. Namely, for the time being there is no result with respect to convergence or divergence of these means of integrable functions.

With respect to the Walsh-Paley system Simon proved [35] that for each function belonging the Hardy space H we have the norm convergence

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{\|S_k f - f\|_1}{k} \to 0.$$

This result was generalized for unbounded Vilenkin systems by Gát [8] and for the two-parameter Walsh-Fourier series by Weisz [42]. The two dimensional Vilenkin case is due to Simon and Weisz [37]. They proved

$$\frac{1}{\log N \log M}\sum_{\substack{0\le k\le N, 0\le l\le M\\ 1/\alpha\le k/l\le\alpha}}\frac{\|S_{k,l} f - f\|_1}{kl} \to 0,$$

where $\alpha > 1$ is some constant. More precisely, their result is much more general. It also concerns the Hardy spaces H^p for $0 < p < 1$.

Now, turn back to the Fejér means. Nurpeisov [26] gave a necessary and sufficient condition of the uniform convergence of the Fejér means $\sigma_{M_n} f$ of continuous functions on unbounded Vilenkin groups. Namely, define the uniform modulus of continuity as

$$\omega_n(f) := \sup_{h\in I_n(0), x\in G_m} |f(x+h) - f(x)|.$$

Let ω be a real sequence with property $\omega_n \searrow 0$. We say that f belongs to the Hölder class H^ω if $\omega_n(f) \le \omega_n$ for all $n \in \mathbb{N}$. Nurpeisov [26] proved: a necessary and sufficient condition that the means $\sigma_{M_n} f$ of the Fourier series of the continuous function f converge uniformly to f on an unbounded Vilenkin group for all f belonging to the Hölder class ω is that

$$\omega_{n-1}(f)\log(m_n) = o(1).$$

Since the uniform modulus of continuity can be any nonincreasing real sequence which converges to zero (for the proof see [29, 6]), then as a consequence of this it is possible to give a sequence m increasing enough fast, and a function even in the Lipschitz class Lip(1), such that the M_nth Fejér means do not converge to the function uniformly.

So, it seems that it is impossible to give a (Hölder) function class such that the uniform convergence of the Fejér means would hold for all functions in this class if there is no condition on sequence m at all.

It also seems that some difference could occur in the case of Nörlund logarithmic means. For the time being there is no result is known with respect to this issue.

Concerning the a.e. convergence and Fejér means on unbounded Vilenkin groups we can say a bit more. Namely, in 1999 the author [10] proved:

Theorem 2. *If $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere.*

This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. One might say that this result is an easy consequence of the result of Carleson, that is the a.e. convergence $S_n f \to f$ for functions $f \in L^p(G_m)$, where $p > 1$. The "only problem" is that to prove this a.e. convergence result of the partial sums is the one of the greatest open problems in the theory of Fourier analysis on Vilenkin groups.

However, it is possible to step further in the direction of space $L^1(G_m)$. In 2001 Simon [36] proved the following theorem with respect to the Fejér means of L^1 functions. A sequence m is said to be strong quasi-bounded if

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} < C \log m_n.$$

Then every bounded m is quasi-bounded, and there are also some unbounded ones. Let m be strong quasi-bounded. Then for all $f \in L^1(G_m)$

$$\sigma_{M_n} f(x) - f(x) = o(\max(\log m_0, \ldots, \log m_{n-1})).$$

Later, in 2003, the author of this paper [12] improved this result, and gave a partial answer for the L^1 case. He discussed this partial sequence of the sequence of the Fejér means. Namely,

Theorem 3. *if $f \in L^1(G_m)$, then ([12]) $\sigma_{M_n} f \to f$ almost everywhere, where m is any sequence.*

In my opinion, it is highly likely that the methods of the papers [10, 12] can be applied and improved in order to prove the a.e. relation $\sigma_n f \to f$ for all $f \in L \log^+ L$ and m. Anyway, it is not an easy task...

With respect to another class of unbounded Vilenkin groups Gát proved the original Lebesgue theorem. This class is called "rarely unbounded". What does it mean?

If there exists a constant C and $L \in \mathbb{P}$ such that for all $i, j \in \mathbb{P}$ we have

$$\frac{\min(m_i, m_{i+j})}{(m_{i+1} \cdot \ldots \cdot m_{i+j-1})^L} \leq C \tag{1}$$

(the empty product is defined to be 1, and the constant C may depend on the sequence m - of course), then we call the Vilenkin group G_m a rarely unbounded Vilenkin group. Every bounded Vilenkin group is a rarely unbounded Vilenkin group. Unfortunately, not all unbounded ones are rarely unbounded, since for instance the rarely unboundedness implies the inequality $\min(m_i, m_{i+1}) \leq C$. So, e.g. if (m_n) tends to plus infinity, then G_m is not rarely unbounded. On the other hand, there are many unbounded Vilenkin groups, which are rarely unbounded ones.

In paper [17] one can find

Theorem 4. *Let G_m be a rarely unbounded Vilenkin group. Then the operator σ^* is of weak type $(1,1)$.*

A straightforward consequence of Theorem 4 is the proof of the Fejér-Lebesgue theorem on rarely unbounded Vilenkin groups. That is,

Theorem 5. *Let G_m be a rarely unbounded Vilenkin group, and $f \in L^1(G_m)$. Then we have the a.e. relation $\sigma_n f \to f$.*

It is also interesting to add that the concept of rarely unbounded Vilenkin groups is natural in the point of view of the Carleson's theorem. Since it can be proved that if the theorem of Carleson holds on every rarely unbounded Vilenkin group, then it also holds on every Vilenkin groups.

At last, we mention a (H, L) and a $(L(\log^+ L)^{a+1},\ L(\log^+ L)^a)$ type inequality with respect to the one-dimensional Fejér means of integrable functions on unbounded Vilenkin groups.

Define the maximal operator $\sigma^\dagger f := \sup\limits_{n\in\mathbb{N}} |\sigma_{M_n} f|$, where f is an integrable function. Simon [34] proved that the maximal operator $\sup |\sigma_n f|$ is not a bounded one from the atomic Hardy space H to the Lebesgue space L^1, but for the "smaller" operator $\sigma^\dagger$ we have it [12] and we also have the following inequality [16]:

Theorem 6. *Let $f \in L(\log^+ L)^{a+1}$, and $a \geq 0$. Then we have*

$$\|\sigma^\dagger f\|_{L(\log^+ L)^a} \leq C_a(\|f\|_{L(\log^+ L)^{a+1}} + 1).$$

In the proof of this theorem and Theorems 3 and 4 the following operator plays a fundamental role: For an integrable function f we define

$$H_1 f(y) := \sup_{A\in\mathbb{N}} \left| M_{A-1} \int\limits_{I_{A-1}(y)\setminus I_A(y)} f(x) \frac{1}{1-r_{A-1}(y-x)} d\mu(x) \right|.$$

Much depends on the fact that operator H_1 is of weak type (L^1, L^1). In order to step further to discuss σ^*, for instance to prove that it is also of weak type (L^1, L^1) it would be necessary to discuss the operator $\sup_n H_1(f\bar{\psi}_n)$.

What can be said in the case of two-dimensional functions? This is "another story". For double trigonometric Fourier series Marcinkiewicz and Zygmund [23] proved that $\sigma_{m,n}f \to f$ a.e. as $m,n\to\infty$ provided the integral lattice points (m,n) remain in some positive cone, that is provided $\beta^{-1} \le m/n \le \beta$ for some fixed parameter $\beta \ge 1$. It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded but this fails to hold for the one-dimensional Walsh-Fejér kernels. This growth difference is exacerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

In 1992 Móricz, Schipp and Wade [25] proved that $\sigma_{2^{n_1},2^{n_2}}f \to f$ a.e. for each two dimensional function $f\in L^1$, when $n_1,n_2\to\infty$, $|n_1-n_2|\le\alpha$ for some fixed α. Later, Gát [9] and Weisz [41] proved this for the whole sequence, that is, the theorem of Marcinkiewicz and Zygmund with respect to the Walsh-Paley system. For the bounded Vilenkin case see the paper of Weisz [44], and the paper of Blahota and the author [2]. In the paper [2] the authors generalize this theorem with respect to two-dimensional bounded Vilenkin-like systems.

If we do not provide a "cone restriction" for the indices in $\sigma_{n,k}f$ that is, we discuss the convergence of this two-dimensional Fejér means in the Pringsheim sense, then the situation changes. In 1992 Móricz, Schipp and Wade [25] proved with respect to the Walsh-Paley system that $\sigma_{n,k}f \to f$ a.e. for each two dimensional function $f \in L\log^+ L$, when $\min\{n,k\}\to\infty$. Later, in 2002 Weisz generalized [44] this with respect to two-dimensional bounded Vilenkin systems.

In 2000 Gát proved [11] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty}\delta(t) = 0$. Gát proved the existence of a two variable function $f\in L^1$ such that $f\in L\log^+ L\delta(L)$, and $\sigma_{n,k}f$ does not converge to f a.e. as $\min\{n,k\}\to\infty$. This theorem of Gát [11] is generalized on bounded two dimensional Vilenkin groups [15]. This divergence result has not been proved for unbounded two dimensional Vilenkin groups yet. It is interesting in the following point view. It is very usual that to prove some divergence results with respect to unbounded Vilenkin systems is easier or less complicated than in the case of bounded Vilenkin systems or in the Walsh-Paley setting. That is, - when we try to determine the maximal convergence space of the two dimensional Vilenkin-Fejér means in the Pringsheim setting, - we have a little bit unusual situation.

What "positive" can be said in the two-dimensional case with respect to unbounded Vilenkin systems? In 1997 Wade proved [40] the following. Let

$$\beta_{k,j} := \max\left\{m_0, \ldots, m_{k-1}, \tilde{m}_0, \ldots, \tilde{m}_{j-1}\right\}.$$

The sequence m is called δ-quasi bounded, $0 \le \delta < 1$, if the sums

$$\sum_{j=0}^{n-1} m_j/(m_{j+1} \ldots m_n)^\delta$$

are (uniformly) bounded. Let the generating sequences $m, \tilde{m}$ be δ-quasi bounded. Then for all $f \in L^1(G_m \times G_{\tilde{m}})$ we have

$$\sigma_{M_n, \tilde{M}_k} f(x) - f(x) = o(\beta_{n,k} \beta_{n+r,k+r}^{2r}),$$

as $n, k \to \infty$, provided that $|n-k| < \alpha$, where $\alpha, r \in \mathbb{N}$ are some constants for almost every $x \in G_m \times G_{\tilde{m}}$.

On the other hand, there was nothing concerning the pointwise convergence before the following result of the author. In [14] he proved

Theorem 7. *Let $f \in (L \log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1}, \tilde{M}_{n_2}} f \to f$ almost everywhere, where* $\min\{n_1, n_2\} \to \infty$ *provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$.*

Here it is necessary to emphasize that in this paper $m, \tilde{m}$ can be any sequences.

It seems also to be interesting to discuss the almost everywhere convergence of Marcinkiewicz means $\frac{1}{n} \sum_{j=0}^{n-1} S_{j,j} f$ of integrable functions on two-dimensional unbounded Vilenkin groups. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. For the trigonometric, Walsh-Paley and bounded Vilenkin case see the papers of Zhizhiasvili, Weisz and Gát [48, 43, 13]. With respect to the Walsh case see also the papers of Goginava [21, 22]. Some results can also be found in [45, 46]. It is highly likely that by the application of the method of the proof of the a.e. relation $\sigma_{M_n} f \to f$ (on unbounded Vilenkin groups), it would be possible to prove the a.e. relation

$$\frac{1}{M_n}\sum_{j=0}^{M_n-1} S_{j,j} f \to f$$

with respect to unbounded Vilenkin systems for every integrable f.

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