

THE θ -SUMMATION ON LOCAL FIELDS

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Dedicated to

*Prof. Ferenc Schipp on the occasion of his 70th birthday
and to*

Prof. Péter Simon on the occasion of his 60th birthday

Abstract. We will introduce the definition of the θ -summation on local fields. It will be given a convolution form of the partial sums of the θ -summation. It will be shown examples for the θ -summation in the dyadic and in the arithmetic case.

1. Preliminaries

Let \mathbb{Z} denote the set of integers, \mathbb{R}_+ represent the set of nonnegative real numbers and denote $(\mathbb{B}, \overset{\circ}{+}, \circ)$ the dyadic (logical) field, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ the arithmetic field and $(\mathbb{R}, +, \cdot)$ the usual real field.

Recall that \mathbb{B} is the set of binary infinite sequence $\underline{x} = (x_n, n \in \mathbb{Z})$, $x_n \in \{0, 1\}$ satisfying $\lim_{n \rightarrow -\infty} x_n = 0$, $\mathbb{B}^* := \mathbb{B} \setminus \{\underline{\Theta}\}$ ($\underline{\Theta} = (\Theta_n = 0, n \in \mathbb{Z})$).

The *norm* on \mathbb{B} is defined by

$$(1) \quad \|\underline{x}\| := 2^{-\pi(\underline{x})} \quad (\underline{x} \in \mathbb{B}),$$

where $\pi(\underline{\Theta}) := +\infty$ and for $\underline{x} \in \mathbb{B}^*$, $\pi(\underline{x}) := n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$.

The dyadic sum and dyadic product of $\underline{x}, \underline{t} \in \mathbb{B}$ is defined as

$$(2) \quad (\underline{x} \overset{\circ}{+} \underline{t})_n := x_n + t_n \pmod{2}, \quad (\underline{x} \overset{\circ}{\cdot} \underline{t})_n = \sum_{j=-\infty}^{\infty} x_j t_{n-j} \pmod{2} \quad (n \in \mathbb{Z}).$$

The arithmetic sum $\underline{x} \overset{\bullet}{+} \underline{t}$ of $\underline{x}, \underline{t} \in \mathbb{B}$ is defined as

$$(3) \quad (\underline{x} \overset{\bullet}{+} \underline{t})_n := r_n \quad (n \in \mathbb{Z}),$$

where the bits $q_n, r_n \in \{0, 1\}$ ($n \in \mathbb{Z}$) are defined recursively using the additive digits of $\underline{x}, \underline{t}$ as follows

$$q_n = r_n = 0 \quad \text{for } n < m := \min\{\pi(\underline{x}), \pi(\underline{t})\},$$

and

$$x_n + t_n + q_{n-1} = 2q_n + r_n \quad \text{for } n \geq m.$$

The arithmetic product $\underline{x} \bullet \underline{t}$ of the elements $\underline{x}, \underline{t} \in \mathbb{B}^*$ is defined as

$$(4) \quad (\underline{x} \bullet \underline{t})_n := r_n \quad (n \in \mathbb{Z}),$$

where the bits $q_n, r_n \in \{0, 1\}$ ($n \in \mathbb{Z}$) are defined recursively using the additive digits of $\underline{x}, \underline{t}$ as follows

$$q_n = r_n = 0 \quad \text{for } n < m := \min\{\pi(\underline{x}), \pi(\underline{t})\},$$

and

$$\sum_{j=-\infty}^{\infty} x_j t_{n-j} + q_{n-1} = 2q_n + r_n \quad \text{for } n \geq m.$$

Also define $\underline{x} \bullet \underline{\Theta} = \underline{\Theta}$ for all $\underline{x} \in \mathbb{B}$.

The fundamental sequence in \mathbb{B} is the sequence $\underline{e}^n := (\delta_{nj}, j \in \mathbb{Z})$, defined for each $n \in \mathbb{Z}$, where δ_{nj} represents the Kronecker delta.

The zero element of \mathbb{B} is the sequence $\underline{\Theta}$ in both cases. The dyadic additive inverse of $\underline{x} \in \mathbb{B}$ is itself, the arithmetic additive inverse of $\underline{x} \in \mathbb{B}$ is its reflection, \underline{x}^- ($\underline{x}_j^- = 0$ for $j \leq \pi(x)$ and $\underline{x}_j^- = 1 - \underline{x}_j$ for $j > \pi(x)$). The identity of the dyadic and arithmetic multiplication is $\underline{e} := \underline{e}^0$ (see Schipp-Wade [1]). The multiplicative inverse of $\underline{x} \in \mathbb{B}^* := \mathbb{B} \setminus \{\underline{\Theta}\}$ will be denoted by x° and x^\bullet , respectively.

The norm (1) is non-Archimedean and multiplicative in both cases, i.e.

$$\begin{aligned}\|\underline{x} \overset{\circ}{+} \underline{t}\| &\leq \max\{\|\underline{x}\|, \|\underline{t}\|\}, & \|\underline{x} \circ \underline{t}\| &= \|\underline{x}\| \cdot \|\underline{t}\| \quad (\underline{x}, \underline{t} \in \mathbb{B}), \\ \|\underline{x} \overset{\bullet}{+} \underline{t}\| &\leq \max\{\|\underline{x}\|, \|\underline{t}\|\}, & \|\underline{x} \bullet \underline{t}\| &= \|\underline{x}\| \cdot \|\underline{t}\| \quad (\underline{x}, \underline{t} \in \mathbb{B}).\end{aligned}$$

The functions $\rho^\circ(\underline{x}, \underline{t}) := \|\underline{x} \overset{\circ}{+} \underline{t}\|$ ($\underline{x}, \underline{t} \in \mathbb{B}$), $\rho^\bullet(\underline{x}, \underline{t}) := \|\underline{x} \overset{\bullet}{+} \underline{t}\|$ ($\underline{x}, \underline{t} \in \mathbb{B}$) are metrics on \mathbb{B} . The dyadic addition and dyadic multiplication are continuous with respect to the metric ρ° , arithmetic addition and arithmetic multiplication are continuous with respect to the metric ρ^\bullet .

Thus $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ are locally compact topological fields (see Schipp-Wade [1]).

The interval with rank $n \in \mathbb{Z}$, and center $\underline{x} \in \mathbb{B}$ is defined as

$$I_n(\underline{x}) := \{\underline{t} \in \mathbb{B} : x_j = t_j \ (j < n)\}.$$

It is known (see Schipp-Wade [1]) that the image of every interval under an addition and under a multiplication is again an interval in both cases. In fact, if $J = I_n(\underline{b})$ ($n \in \mathbb{Z}$, $\underline{a}, \underline{b} \in \mathbb{B}$), then

$$(5) \quad \begin{aligned}\underline{a} \overset{\circ}{+} J &:= \{\underline{a} \overset{\circ}{+} \underline{x} : \underline{x} \in J\} = I_n(\underline{a} \overset{\circ}{+} \underline{b}), \\ \underline{a} \circ J &:= \{\underline{a} \circ \underline{x} : \underline{x} \in J\} = I_{n+\pi(\underline{a})}(\underline{a} \circ \underline{b}), \\ \underline{a} \overset{\bullet}{+} J &:= \{\underline{a} \overset{\bullet}{+} \underline{x} : \underline{x} \in J\} = I_n(\underline{a} \overset{\bullet}{+} \underline{b}), \\ \underline{a} \bullet J &:= \{\underline{a} \bullet \underline{x} : \underline{x} \in J\} = I_{n+\pi(\underline{a})}(\underline{a} \bullet \underline{b}).\end{aligned}$$

Thus the measure μ defined on the set of intervals by

$$\mu(I_n(\underline{x})) := 2^{-n} \quad (n \in \mathbb{Z}, \underline{x} \in \mathbb{B})$$

is translation invariant and the extension of μ to the Borel-sets of \mathbb{B} coincides with the Haar-measure of $(\mathbb{B}, \overset{\circ}{+})$ and $(\mathbb{B}, \overset{\bullet}{+})$ normalized by $\mu(I_0(\Theta)) = 1$.

The set \mathbb{B} can be identified with the set of nonnegative real numbers \mathbb{R}_+ and \mathbb{B}^* with $(0, \infty)$. This identification gives a useful interpretation of the concepts defined above. Define the map $\alpha : \underline{x} \mapsto x$ ($\underline{x} \in \mathbb{B}, x \in \mathbb{R}_+$) by

$$\alpha(\underline{x}) := x := \sum_{n=-\infty}^{\infty} x_n 2^{-n-1}.$$

From this point of view, we call x_n ($n \in \mathbb{Z}$) the binary coefficients of x . The map α takes $I_n(\underline{x})$ to a dyadic interval of the form $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$ ($n \in \mathbb{Z}$),

where $k = \sum_{j=-\infty}^{n-1} x_j 2^{-j-1}$. The map α is 1-1 except for a denumerable set of \mathbb{B} . All definitions above (dyadic addition, multiplication, etc.) correspond to a similar one defined on \mathbb{R}_+ . That is why we can and will work on \mathbb{R}_+ except for \mathbb{B} in the following.

In the sequel w_x ($x \in \mathbb{R}_+$) stands for the generalized Walsh function defined on \mathbb{R}_+ . We recall (see Schipp-Wade-Simon [2], Chapter 9 and [1], Chapter 3) that the generalized Walsh function w_x is defined by

$$(6) \quad w_x(t) := (-1)^{(x \circ t)^{-1}}, \quad (x \circ t)^{-1} = \sum_{n=-\infty}^{\infty} x_n t_{-1-n} \pmod{2},$$

where $x_n, t_n \in \{0, 1\}$ are the binary coefficients in the dyadic representation. If $x = n \in \mathbb{N} := \{0, 1, 2, \dots\}$ is a natural number, then w_n coincides with the 1-periodic extension of the corresponding Walsh-Paley function. The functions w_y ($y \in \mathbb{R}_+$) behave like characters with respect to dyadic addition, i.e. if $x, t \in \mathbb{R}_+$ and $x \overset{\circ}{+} t$ is dyadic irrational then

$$(7) \quad w_y(x \overset{\circ}{+} t) = w_y(x)w_y(t).$$

The function

$$(8) \quad w(t) := e^{2\pi i \frac{t-1}{2}} = (-1)^{t-1} \quad (t \in \mathbb{R}_+)$$

is called the basic character of the additive group of dyadic field. The Walsh-Paley functions can be expressed by the basic character in the form

$$(9) \quad w_x(t) := w(x \circ t) \quad (x, t \in \mathbb{R}_+)$$

and consequently

$$w_y(x) = w_x(y), \quad w_{t \circ x}(y) = w_x(t \circ y), \quad w_x(y) = w_{[x]}(y)w_{[y]}(x) \quad (x, y, t \in \mathbb{R}_+),$$

where $[x]$ is the integer part of $x \in \mathbb{R}_+$.

In the sequel v_x ($x \in \mathbb{R}_+$) defined on \mathbb{R}_+ stands for the characters of the additive group of arithmetic field. We recall (see Schipp-Wade [1], Chapter 2) that the basic character in this case is

$$(10) \quad v(t) := e^{2\pi i \left(\frac{t-1}{2} + \frac{t-2}{2^2} + \dots \right)} \quad (t \in \mathbb{R}_+)$$

and $v_x (x \in \mathbb{R}_+)$ can be expressed in the form

$$(11) \quad v_x(t) := v(x \bullet t) = e^{2\pi i \left(\frac{(x \bullet t) - 1}{2} + \frac{(x \bullet t) - 2}{2^2} + \dots \right)} \quad (t \in \mathbb{R}_+),$$

where $(x \bullet t)_k$ ($k \in \mathbb{Z}$) are the binary coefficients of $x \bullet t$ in the dyadic representation. If $x = n \in \mathbb{N} := \{0, 1, 2, \dots\}$ is a natural number, then v_n coincides with the 1-periodic extension of the characters of the subgroup $([0, 1), \overset{\bullet}{+})$. The functions v_x has also the character property, if $x, t \in \mathbb{R}_+$ and $x + t$ is dyadic irrational, then

$$(12) \quad v_y(x + t) = v_y(x)v_y(t).$$

Moreover

$$v_y(x) = v_x(y), \quad v_{t \bullet x}(y) = v_x(t \bullet y), \quad v_x(y) = v_{[x]}(y)v_{[y]}(x') \quad (x, y, t \in \mathbb{R}_+),$$

where $[x]$ is the integer part of $x \in \mathbb{R}_+$, $x' = x - [x]$.

In the sequel $\epsilon_x (x \in \mathbb{R})$ defined on \mathbb{R} stands for the characters of the real field $(\mathbb{R}, +)$. We recall (see Schipp-Wade [1], Chapter 3) that in this case the basic character is

$$(13) \quad \epsilon(t) := e^{2\pi i t} \quad (t \in \mathbb{R})$$

and $\epsilon_x (x \in \mathbb{R})$ is defined by

$$(14) \quad \epsilon_x(t) := \epsilon(x \cdot t) = e^{2\pi i x t} \quad (t \in \mathbb{R}).$$

If $x = n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ is an integer, then ϵ_n coincides with the 1-periodic extension of the characters of the subgroup $([0, 1), + \bmod 1)$. The functions ϵ_x have also the character property, if $x, t \in \mathbb{R}$, then

$$(15) \quad \epsilon_y(x + t) = \epsilon_y(x)\epsilon_y(t).$$

Let $L^p = L^p(\mathbb{R}_+)$ ($1 \leq p \leq \infty$) denote the usual Banach spaces. We have introduced three fields (the real field, the logical field and the arithmetic field) and thus have three distinct Fourier transforms. Let $(\mathbb{F}, +, \cdot)$ represent one of the fields. Let \mathbb{F}^\sharp denote \mathbb{Z} , when $\mathbb{F} = \mathbb{R}$, \mathbb{N} , when $\mathbb{F} = \mathbb{R}$.

Given $f \in L^1 := L^1(\mathbb{F})$ the Fourier transform of f is the function on \mathbb{F} defined by

$$(16) \quad (\mathcal{F}f)(x) := \int_{\mathbb{F}} f(t)\bar{u}(x \cdot t)dt \quad (x \in \mathbb{F}),$$

where u is the basic character of $(\mathbb{F}, +)$, and \bar{u} is the complex conjugation of u . Specially the trigonometric Fourier transform is denoted by

$$(17) \quad \hat{f}(x) := \int_{-\infty}^{\infty} f(t)\bar{\epsilon}(x \cdot t)dt \quad (x \in \mathbb{R}),$$

the Walsh-Fourier transform is denoted by

$$(18) \quad f^\circ(x) := \int_0^{\infty} f(t)w(x \circ t)dt \quad (x \in \mathbb{R}_+),$$

and the Fourier transform in the arithmetic case is denoted by

$$(19) \quad f^\bullet(x) := \int_0^{\infty} f(t)\bar{v}(x \bullet t)dt \quad (x \in \mathbb{R}_+).$$

It is known that, if $f, \mathcal{F}f \in L^1$, then the inversion formula is true. Namely,

$$(20) \quad f(x) = \int_{\mathbb{F}} (\mathcal{F}f)(t)u(x \cdot t)dt \quad \text{for a.a. } x \in \mathbb{F}$$

(see Schipp-Wade-Simon [2]).

The **dilation operator** generated by $b \in \mathbb{F}^*$ is defined by

$$(\delta_b f)(x) := f(x \cdot b) \quad (x \in \mathbb{F}),$$

and the **translation operator** generated by $h \in \mathbb{F}$ is defined by

$$(\tau_h f)(x) := f(x + h) \quad (x \in \mathbb{F}).$$

It is known (see Schipp-Wade [1]) that

$$(21) \quad (\mathcal{F} \circ \delta_b)(f) = \frac{1}{\|b\|} (\delta_{b^{-1}} \circ \mathcal{F})(f) \quad (f \in L^1(\mathbb{F}), b \in \mathbb{F}^*).$$

2. The θ -summation

We begin with the definition of the θ -summation. We will do it with a function $\theta : [0, \infty) \rightarrow \mathbb{R}$. We consider only functions θ with the property:

$$(22) \quad \sum_{k=0}^{\infty} |\theta(k \circ 2^{-n})| = \sum_{k=0}^{\infty} |\theta(k \cdot 2^{-n})| < \infty \quad (n \in \mathbb{N}).$$

Definition 1. Let $z = (z_n, n \in \mathbb{N})$ be a bounded real sequence, and θ a function with the property (22). The series

$$(23) \quad \sum_{k=0}^{\infty} z_k$$

is called **θ -summable**, if the sequence

$$(24) \quad t_n := \sum_{k=0}^{\infty} \theta(k \cdot 2^{-n}) z_k \quad (n \in \mathbb{N})$$

is convergent. The limit of the sequence $(t_n, n \in \mathbb{N})$ is called the **θ -sum** of the series (23).

We examine, when will this summation be permanent. Assume that the series (23) is convergent, and use the following notations:

$$(25) \quad s_0 := 0, \quad s_k := \sum_{j=0}^{k-1} z_j \quad (k \in \mathbb{N}^*).$$

After applying the Abel-rearrangement for (24), we get

$$(26) \quad t_n = \sum_{k=1}^{\infty} (\theta((k-1) \cdot 2^{-n}) - \theta(k \cdot 2^{-n})) s_k \quad (n \in \mathbb{N}).$$

Applying the theorem of Toeplitz (see Zygmund [6]) we get, that the summation (24) is permanent if and only if for the double series

$$(27) \quad \alpha_{nk} := \theta((k-1) \cdot 2^{-n}) - \theta(k \cdot 2^{-n}) \quad (k \in \mathbb{N}^*, n \in \mathbb{N}),$$

the conditions

$$(28) \quad \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |\alpha_{nk}| < \infty, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \alpha_{nk} = 1, \quad \lim_{n \rightarrow \infty} \alpha_{nk} = 0 \quad (k \in \mathbb{N}^*),$$

hold. Since

$$(29) \quad \sum_{k=1}^{\infty} \alpha_{nk} = \theta(0),$$

it follows that if for the function θ condition (22) hold and

$$(30) \quad i) \theta \in BV(\mathbb{R}_+), \quad ii) \theta(0) = 1, \quad iii) \lim_{t \rightarrow 0^+} \theta(t) = \theta(0)$$

are satisfied then the summation, which is generated by θ is permanent. Here BV denotes the set of functions with bounded variation on \mathbb{R}^+ .

Assume now that the double sequence $\theta(k \cdot 2^{-n})$ has the following property:

$$(31) \quad \theta(2^k \cdot 2^{-n}) = \theta((2^k + \ell) \cdot 2^{-n}) \quad (0 \leq \ell < 2^k, k, n \in \mathbb{N}).$$

In such a case using again the Abel arrangement, we get

$$(32) \quad t_n = \theta(0)s_1 + \sum_{k=0}^{\infty} \theta(2^k \cdot 2^{-n}) \sum_{\ell=2^k}^{2^{k+1}-1} z_\ell = \theta(0)s_1 + \sum_{k=0}^{\infty} \theta(2^k \cdot 2^{-n})(s_{2^{k+1}} - s_{2^k})$$

for all $n \in \mathbb{N}$, and so

$$(33) \quad t_n = \theta(0)s_1 + \sum_{k=1}^{\infty} (\theta(2^{k-1} \cdot 2^{-n}) - \theta(2^k \cdot 2^{-n})) s_{2^k} - \theta(2^{-n})s_1.$$

It follows from the Toeplitz-theorem that, if the function θ fulfils the conditions (30) and (27), then

$$(34) \quad \lim_{n \rightarrow \infty} s_{2^n} = s \quad \Rightarrow \quad \lim_{n \rightarrow \infty} t_n = s \quad (s \in \mathbb{R}).$$

It is easy to see that, if $\theta(x) = \theta(\|x\|)$ ($x \in \mathbb{R}^+$), then θ has the property (31).

3. θ -summation of Fourier-series

Let $f : [0, 1) \rightarrow \mathbb{R}$ be an integrable function, and θ a function with the property (22). We will denote the Walsh-Fourier coefficients of f with $c_k^w(f)$ ($k \in \mathbb{N}$), the Fourier-coefficients of f with respect to the system $(v_k, k \in \mathbb{N})$ with $c_k^v(f)$, and the trigonometric Fourier coefficients of f with $c_k^\xi(f)$ ($k \in \mathbb{Z}$).

Let $(u_k, k \in \mathbb{F}^\sharp)$ represent one of the three systems and denote

$$c_k(f) = \int_0^1 f(t)\bar{u}_k(t)dt \quad (k \in \mathbb{F}^\sharp)$$

the Fourier coefficient of f with respect to this system. The linear combinations of the functions u_k ($k \in \mathbb{F}^\sharp$) are called the trigonometric polynomial of \mathbb{F} .

For $1 \leq p \leq \infty$ denote X_p the closure in L^p -norm of the trigonometric polynomial of \mathbb{F} . It is known that $X_p = L^p[0, 1)$ if $p < \infty$. In the case $\mathbb{F} = \mathbb{B}$ the space X_∞ is the set of W-continuous functions and if $\mathbb{F} = \mathbb{R}$ then X_∞ is the collection of one periodic continuous functions. The norm of X_p will be denoted by $\|\cdot\|_p$.

The Fourier series of f is called θ -**summable**, if the sequence

$$(35) \quad \sigma_n^\theta(f) := \sum_{k \in \mathbb{F}^\sharp} c_k(f)\theta(k \cdot 2^{-n})u_k \quad (n \in \mathbb{F}^\sharp)$$

converges. For the trigonometric system see [3], [4].

We will show that σ_n^θ can be expressed as a convolution operator:

Theorem 1. *Assume that the function θ fulfils the conditions (30) and (22). If $\theta, \hat{\theta} \in L^1_{\mathbb{F}}$ and $f \in X_p$ ($1 \leq p \leq \infty$), then*

$$(36) \quad (\sigma_n^\theta f)(x) = 2^n \int_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t)dt \quad (x \in [0, 1)),$$

and for all $f \in X_p$

$$(37) \quad \|\sigma_n^\theta f - f\|_p \rightarrow 0 \quad (n \rightarrow \infty).$$

Proof. We will use the following notation:

$$(V_n^\theta f)(x) = 2^n \int_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t)dt \quad (x \in \mathbb{F}).$$

The operators $V_n^\theta, \sigma_n^\theta : X_p \rightarrow X_p$ are bounded. Indeed, since $|c_n(f)| \leq \|f\|_p$ for the operator σ_n^θ we get

$$\|\sigma_n^\theta f\|_p \leq \sum_{k=0}^{\infty} |c_k(f)| |\theta(k \cdot 2^{-n})| \leq \|f\|_p \sum_{k=0}^{\infty} |\theta(k \cdot 2^{-n})|.$$

Applying the generalized Minkowsky-inequality for V_n^θ by (22) we have

$$\begin{aligned} \|V_n^\theta f\|_p &\leq \left(\int_0^1 \left| 2^n \int_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t) dt \right|^p dx \right)^{1/p} \leq \\ &\leq \int_{\mathbb{F}} \left(\int_0^1 |2^n f(x+t)(\mathcal{F}\theta)(2^n t)|^p dx \right)^{1/p} dt = \\ &= \|f\|_p \int_{\mathbb{F}} |2^n \mathcal{F}\theta(2^n t)| dt = \|f\|_p \int_{\mathbb{F}} |(\mathcal{F}\theta)(t)| dt = \|f\|_p \cdot \|\mathcal{F}\theta\|_{L_{\mathbb{F}}^1}. \end{aligned}$$

Since for all fixed $n \in \mathbb{F}^\sharp$ the operators $V_n^\theta, \sigma_n^\theta : X_p \rightarrow X_p$ are bounded, and X_p is the closure of the trigonometric polynomials it is enough to prove our statements for the system $(u_k, k \in \mathbb{F}^\sharp)$. Let $f = u_k$ ($k \in \mathbb{F}^\sharp$). By the definition of σ_n^θ (35) and V_n^θ we get that

$$\begin{aligned} (\sigma_n^\theta f)(x) &= \theta(k \cdot 2^{-n}) u_k(x), \\ V_n^\theta(x) &= 2^n \int_{\mathbb{F}} u_k(x+t)(\mathcal{F}\theta)(2^n t) dt = 2^n \int_{\mathbb{F}} u_k(x) u_k(t)(\mathcal{F}\theta)(2^n t) dt = \\ &= 2^n u_k(x) \int_{\mathbb{F}} u_k(t)(\mathcal{F}\theta)(2^n t) dt = u_k(x) \int_{\mathbb{F}} u_k(2^{-n} t)(\mathcal{F}\theta)(t) dt = \\ &= u_k(x) \int_{\mathbb{F}} u_{k \cdot 2^{-n}}(t)(\mathcal{F}\theta)(t) dt = \theta(k \cdot 2^{-n}) u_k(x). \end{aligned}$$

We used the character property of the functions $(u_k, k \in \mathbb{F}^\sharp)$ and the inversion formulae (20). From this we can easy estimate

$$(38) \quad \|\sigma_n^\theta u_k - u_k\|_p \leq \|\theta(k \cdot 2^{-n}) u_k - u_k\|_p \leq |\theta(k \cdot 2^{-n}) - 1| \cdot \|u_k\|_p = |\theta(k \cdot 2^{-n}) - 1|.$$

Because of (30), $|\theta(k \cdot 2^{-n}) - 1| \rightarrow 0$ ($n \rightarrow \infty$) and so $\|\sigma_n^\theta u_k - u_k\|_X \rightarrow 0$ ($n \rightarrow \infty$). Using the Banach-Steinhaus theorem we get the statements of the theorem.

3.1. Estimations for the L^1 -norm of the Walsh-Fourier transform

First we investigate functions with compact support. It is known that if $\text{supp } \theta \subset [0, 1]$, then

$$\|\theta^\circ\|_1 = \sum_{k=0}^{\infty} |c_k^w(\theta)| \left(c_k^w(\theta) = \int_0^1 \theta(t) w_k(t) dt, \quad k \in \mathbb{N} \right).$$

We will use the estimation of the functions

$$J_n^{(1)}(x) := \int_0^x w_k(t) dt, \quad J_n^{(2)}(x) := \int_0^x J_n^{(1)}(t) dt \quad (x \in [0, 1], n \in \mathbb{N}).$$

It is known that

(39)

$$i) \quad J_{2^n+k}^{(1)} := w_k J_{2^n}^{(1)}, \quad J_{2^n+k}^{(1)}(1) := 0, \quad \|J_{2^n+k}^{(1)}\|_\infty \leq 2^{-n-2} \quad (n \in \mathbb{N}),$$

$$ii) \quad J_{2^n+2^m+k}^{(2)}(1) := 0, \quad \|J_{2^n+2^m+k}^{(2)}\|_\infty = 2^{-n-m-1} \quad (0 \leq k < 2^m, 0 \leq m < N)$$

(see Schipp-Wade-Simon [2]).

Using these estimations, we can prove the following

Lemma 1. *Assume that the function $\theta \in C^1[0, 1]$ is twice differentiable on the open interval $(0, 1)$ and $\theta'' \in L^1[0, 1]$. Then*

$$\|\theta^\circ\|_1 = \sum_{k=0}^{\infty} |c_k^w(\theta)| < N(\theta) := \|\theta\|_1 + \|\theta'\|_1 + \|\theta''\|_1 < \infty.$$

Proof. Let $\ell = 2^n + 2^m + k$ ($0 \leq k < 2^m, 0 \leq m < n$). Then $J_\ell^{(1)}(0) = J_m^{(1)}(1) = 0$, $J_\ell^{(2)}(0) = J_m^{(2)}(1) = 0$. Integrating by parts we get

$$\int_0^1 \theta(t) w_\ell(t) dt = - \int_0^1 \theta'(t) J_\ell^{(1)}(t) dt = -[\theta'(t) J_\ell^{(2)}(t)]_{t=0}^1 + \int_0^1 \theta''(t) J_\ell^{(2)}(t) dt,$$

and so

$$|c_\ell^w(\theta)| \leq \|\theta''\|_1 \|J_\ell^{(2)}\|_\infty \leq \|\theta''\|_1 2^{-n-m-1}.$$

If $\ell = 2^n$, then

$$\int_0^1 \theta(t) w_\ell(t) dt = - \int_0^1 \theta'(t) J_\ell^{(1)}(t) dt,$$

so

$$|c_\ell^w(\theta)| \leq \|\theta'\|_1 \|J_\ell^{(1)}\|_\infty \leq \|\theta'\|_1 2^{-n-2}.$$

From this follows that

$$\|\theta^\circ\|_1 \leq |c_0^w(\theta)| + \sum_{n=0}^{\infty} |c_{2^n}^w(\theta)| + \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} |c_{2^{n+2^m+k}}^w(\theta)| = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

From this follows that

(40)

$$i) \quad |\Sigma_1| \leq \|\theta\|_1,$$

$$ii) \quad |\Sigma_2| \leq \|\theta'\|_1/2,$$

$$iii) \quad |\Sigma_3| \leq \|\theta''\|_1 \sum_{n=0}^{\infty} 2^{-n-1} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} 2^{-m} = \|\theta''\|_1 \sum_{n=0}^{\infty} n 2^{-n-1} \leq \|\theta''\|_1.$$

In the general case we suppose that $\theta \in L^1(\mathbb{R}_+)$. Then for $x \in [k, k+1)$ ($k \in \mathbb{N}$)

$$\theta^\circ(x) := \int_0^\infty \theta(t) w_x(t) dt = \int_0^\infty \theta(t) w_{[x]}(t) w_{[t]}(x) dt = \sum_{\ell=0}^{\infty} w_\ell(x) \int_\ell^{\ell+1} \theta(t) w_k(t) dt.$$

Introduce the functions

$$\theta_\ell(t) := \theta(t + \ell) \quad (\ell \in \mathbb{N}, t \in [0, 1)),$$

the function θ° can be written in the form

$$\theta^\circ(x) = \sum_{\ell=0}^{\infty} w_\ell(x) c_k^w(\theta_\ell) \quad (x \in [k, k+1), k \in \mathbb{N}).$$

From this follows that

$$\|\theta^\circ\|_1 \leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |c_k^w(\theta_\ell)| < \sum_{\ell=0}^{\infty} N(\theta_\ell).$$

We proved the following

Lemma 2. *Assume that $\theta \in C^2(\mathbb{R}_+)$. Then*

$$\|\theta^\circ\|_1 \leq \sum_{\ell=0}^{\infty} N(\theta_\ell).$$

3.2. The Fourier-coefficients with respect UDMD system

By a dyadic interval of rank n in $[0, 1)$ we mean an interval of the form $[p/2^n, (p+1)/2^n)$, where $0 \leq p < 2^n$ and $n \in \mathbb{N}$. Given $a \in [0, 1)$ and $n \in \mathbb{N}$, there is one and only one interval of rank n which contains a . We denote it with $I_n(a)$. We denote the σ -algebra generated by the intervals $I_n(a)$ ($a \in [0, 1)$) by \mathcal{A}_n .

Let φ_n ($n \in \mathbb{N}$) \mathcal{A}_n -measurable functions with $|\varphi_n| = 1$. Then

$$\phi_n := r_n \varphi_n \quad (n \in \mathbb{N})$$

is an UDMD sequence, where r_n is the n th Rademacher function. We denote by $\Psi = (\psi_m, m \in \mathbb{N})$ the product system of $\Phi = (\phi_n, n \in \mathbb{N})$:

$$\psi_m := \prod_{n=0}^{\infty} \phi_n^{m_n} \quad \left(m = \sum_{n=0}^{\infty} m_n 2^n \in \mathbb{N}, m_n \in \{0, 1\} \right).$$

It is called the UDMD product system.

In this section we will examine the order of the Fourier coefficients

$$\hat{f}(m) := \langle f, \psi_m \rangle \quad (m \in \mathbb{N})$$

of the function $f \in L^1[0, 1)$ with respect to the system Ψ .

Let

$$(41) \quad R_n(x) := \int_0^x r_n(t) dt \quad (x \in [0, 1), n \in \mathbb{N}).$$

Obviously R_n is linear on dyadic intervals with rank $n+1$, and

$$(42) \quad R_n(k2^{-n}) = 0, \quad R_n((k+1/2) \cdot 2^{-n}) = 2^{-(n+1)} \quad (k, n \in \mathbb{N}).$$

The basis of the estimation is the following

Lemma 3. *Assume that the function $f \in L^1[0, 1]$ is absolutely continuous, and let $\gamma_n := R_n \varphi_n$ ($n \in \mathbb{N}$). Then*

$$(43) \quad \hat{f}(m) = -\widehat{(f' \overline{\gamma_n})}(m') \quad (m = 2^n + m', 0 \leq m' < 2^n).$$

Proof. Let

$$\mathcal{J}_{m,0} := \psi_m, \quad \mathcal{J}_{m,\ell+1}(x) := \int_0^x \mathcal{J}_{m,\ell}(t) dt \quad (x \in [0, 1], m, \ell \in \mathbb{N}).$$

Integrating by parts we get that

$$(44) \quad \hat{f}(m) := \int_0^1 f(t) \overline{\mathcal{J}'_{m,1}(t)} dt = [f \overline{\mathcal{J}_{m,1}}]_0^1 - \int_0^1 f'(t) \overline{\mathcal{J}_{m,1}(t)} dt = - \int_0^1 f'(t) \overline{\mathcal{J}_{m,1}(t)} dt,$$

because $\int_0^1 \psi_m(t) dt = 0$ for all $m \in \mathbb{N}, m > 0$. It follows from the definition of ψ_m that if $m = 2^n + m', 0 \leq m' < 2^n, n \in \mathbb{N}$, then

$$\begin{aligned} \mathcal{J}_{m,1}(x) &= \int_0^x \psi_m(t) dt = \int_0^x \phi_n(t) \psi_{m'}(t) dt = \int_0^x r_n(t) \varphi_n(t) \psi_{m'}(t) dt = \\ &= \varphi_n(x) \psi_{m'}(x) \int_0^x r_n(t) dt = \varphi_n(x) \psi_{m'}(x) R_n(x) = \gamma_n(x) \psi_{m'}(x). \end{aligned}$$

Substitute this in (44), we get our statement.

With the aid of Lemma 3 we get

Theorem 2. *Let f be absolutely continuous function, and $f' \in L^2([0, 1])$. Then $\hat{f} \in \ell^1$.*

Proof. Using the Cauchy inequality, Lemma 3, the Parseval formula and (42) we get that

$$\begin{aligned} \sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)| &\leq 2^{n/2} \left(\sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)|^2 \right)^{1/2} = \\ &= 2^{n/2} \left(\sum_{m'=0}^{2^n-1} |\widehat{(f' \overline{\gamma_n})}(m')|^2 \right)^{1/2} \leq 2^{n/2} \|f' \overline{\gamma_n}\|_2 \leq 2^{-n/2-1} \|f'\|_2 \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently

$$\begin{aligned} \|\hat{f}\|_{\ell^1} &= \sum_{n=0}^{\infty} |\hat{f}(n)| = \\ &= |\hat{f}(0)| + \sum_{n=0}^{\infty} \sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)| \leq |\hat{f}(0)| + \sum_{n=0}^{\infty} 2^{-n/2-1} \|f'\|_2 < \infty. \end{aligned}$$

It is known that the systems $(w_n, n \in \mathbb{N})$ and $(v_n, n \in \mathbb{N})$ are UDMD product systems. So, Theorem 2 gives an other condition for the function θ in the dyadic case:

Corollary 1. *If the absolutely continuous function θ has compact support with $\text{supp } \theta \subset [0, 1]$, and $\theta' \in L^2[0, 1)$, then in the logical and in the arithmetic case $\hat{\theta} \in L^1[0, \infty)$.*

4. Examples

In this section we consider some known examples for the θ -summation. We will examine the conditions of Theorem 1, and in certain cases the conditions of Corollary 1. For the trigonometric case see [6].

1. The 2^n -th partial sum. If we use the function

$$(45) \quad \theta(x) = \chi_{[0,1)}(x),$$

where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0, 1)$, the θ -summation will yield the 2^n -th partial sum. The Walsh-Fourier transform of (45) is itself, and it is in $L^1[0, \infty)$, so the conditions of Theorem 1 are met.

θ is absolutely continuous with compact support and its derivative is in $L^2[0, 1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0, 1)$ and the conditions of Theorem 1 are fulfilled in the arithmetic and in the logical case, too.

2. The (C,1)-summation. If we use the function

$$(46) \quad \theta(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, \infty), \end{cases}$$

then the θ -summation results in the 2^n th subsequence of the (C,1)-sums. The function θ is of course in $L^1[0, \infty)$. It follows from Lemma 1 that θ° is in $L^1[0, \infty)$, so the conditions of Theorem 1 are satisfied.

On the other hand θ is absolutely continuous with compact support and its derivative is in $L^2[0, 1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0, \infty)$ and the conditions of Theorem 1 are fulfilled in the arithmetic and in the logical case, too.

3. De La Vallée-Poussin summation. If we use the function

$$(47) \quad \theta(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ 2(1-x) & \text{if } x \in [1/2, 1), \\ 0 & \text{if } x \in [1, \infty), \end{cases}$$

then the θ -summation results in the 2^n th subsequence of the De La Vallée-Poussin-sums. The function θ is of course in $L^1[0, \infty)$. It follows from Lemma 1 that θ° is in $L^1[0, \infty)$, so the conditions of Theorem 1 are fulfilled.

Furthermore θ is absolutely continuous function with compact support and its derivative is in $L^2[0, 1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0, \infty)$ and the conditions of Theorem 1 are satisfied in the arithmetic and in the logical case, too.

4. The Riesz-summation. If we use the function

$$(48) \quad \theta(x) = \begin{cases} (1-x^\gamma)^\alpha & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, \infty), \end{cases}$$

where $1 \leq \alpha < \infty$ and $0 < \gamma < \infty$, then the θ -summation results in the 2^n th subsequence of the Riesz-sums. The function θ is of course in $L^1[0, \infty)$. Since $\alpha \cdot \gamma \geq 1$, so $\theta'' \in L^1[0, 1)$ and so Lemma 1 holds.

Furthermore θ is absolutely continuous with compact support and its derivative is in $L^2[0, 1)$ if $\gamma > 1/2$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0, \infty)$ and the conditions of Theorem 1 are met in the arithmetic and in the logical case, too, if $\gamma > 1/2$.

5. Weierstrass summation. If we use the function

$$\theta(x) = e^{-x^\gamma} \quad (0 \leq x < \infty, \gamma > 0),$$

then the θ -summation results in the 2^n th subsequence of the Weierstrass-sums. The function θ is of course in $L^1[0, \infty)$. In this case we get by Lemma 2 and Lemma 1 that

$$\|\theta^\circ\|_{L^1} \leq \sum_{\ell=0}^{\infty} N(\ell) \leq \sum_{\ell=0}^{\infty} \left(K \cdot e^{-\ell^\gamma} + e^{-\ell^\gamma} + \frac{\gamma \ell^{\gamma-1}}{e^{\ell^\gamma}} \right) < \infty,$$

where K is a constant, which depends only on γ .

6. Abel summation. If we use the function

$$\theta(x) = e^{-\kappa x} \quad (0 \leq x < \infty, \kappa > 0)$$

then the limit of σ_n^θ is equal to the Weierstrass-sum. The function θ is of course in $L^1[0, \infty)$. In this case we get by Lemma 2 and Lemma 1 that

$$\|\theta^\circ\|_{L^1} \leq \sum_{\ell=0}^{\infty} N(\ell) \leq \sum_{\ell=0}^{\infty} \left(\frac{1}{\kappa} + 1 + \kappa\right) e^{-\ell \cdot \kappa} < \infty.$$

7. Norm-depending θ functions. Assume that $\theta(x) = \theta(\|x\|)$ ($x \in \mathbb{R}^+$). Then its L^1 -norm is

$$(49) \quad \|\theta\|_1 = \int_0^\infty |\theta(t)| dt = \sum_{k=-\infty}^{\infty} |\theta(2^k)| \cdot 2^k.$$

Its Walsh-Fourier transform is

$$\begin{aligned} \theta^0(x) &= \int_0^\infty \theta(t)w_x(t)dt = \int_0^1 \theta(t)w_x(t)dt + \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \theta(t)w_x(t)dt = \\ &= \int_0^1 \theta(t)w_x(t)dt + \sum_{k=0}^{\infty} \theta(2^k) \int_{2^k}^{2^{k+1}} w_x(t)dt = \\ (50) \quad &= \int_0^1 \theta(t)w_{[x]}(t)w_{[t]}(x)dt + \sum_{k=0}^{\infty} \theta(2^k) \sum_{\ell=0}^{2^k-1} \int_{2^{k+\ell}}^{2^{k+\ell+1}} w_{[x]}(t)w_{[t]}(x)dt = \\ &= \int_0^1 \theta(t)w_{[x]}(t)dt + \sum_{k=0}^{\infty} \theta(2^k) \sum_{\ell=0}^{2^k-1} w_{2^{k+\ell}}(x) \int_{2^{k+\ell}}^{2^{k+\ell+1}} w_{[x]}(t)dt \end{aligned}$$

for all $x \in \mathbb{R}^+$. The integral of the Walsh-functions are zero over every interval with length 1 except the w_0 , so

$$(51) \quad \theta^\circ(x) = \begin{cases} \theta^0([x]) & \text{if } x > 1, \\ \theta^0([x]) + \sum_{k=0}^{\infty} \theta(2^k) \sum_{\ell=0}^{2^k-1} w_{2^{k+\ell}}(x) & \text{if } x \in [0, 1), \\ \theta^\circ([x]) & \text{if } x > 1, \\ \theta^\circ(0) + \sum_{k=0}^{\infty} \theta(2^k)r_k(x)D_{2^k}(x) & \text{if } x \in [0, 1). \end{cases}$$

From this expression we can estimate the L^1 -norm of θ° :

$$\begin{aligned}
\|\theta^\circ\|_1 &= \int_0^\infty |\theta^\circ(t)| dt = \\
&= \int_0^1 \left| \theta^\circ(0) + \sum_{k=0}^\infty \theta(2^k) r_k(x) D_{2^k}(x) \right| dx + \int_1^\infty |\theta^\circ([x])| dx \leq \\
&\leq |\theta^\circ(0)| + \sum_{k=0}^\infty 2^k |\theta(2^k)| + \sum_{k=1}^\infty |\theta^\circ(k)| = \sum_{k=0}^\infty 2^k |\theta(2^k)| + \sum_{k=0}^\infty |\theta^\circ(k)| \leq \\
&\leq \|\theta\|_1 + \sum_{k=0}^\infty |\theta^\circ(k)|.
\end{aligned}$$

From this follows that, if θ is integrable ($\theta \in L^1$), and its Walsh-Fourier series is absolutely convergent, then its Walsh-Fourier transform is in L^1 , too.

If $\theta(x) = \theta(\|x\|)$ ($x \in \mathbb{R}^+$), then θ has the form

$$(52) \quad \theta(x) = \sum_{n=0}^\infty \alpha_n \frac{2 \cdot D_{2^n}(x) - D_{2^{n+1}}(x)}{2^{n+1}} \quad (x \in (0, 1)),$$

on $(0, 1)$, where the $(\alpha_n, n \in \mathbb{N})$ is a real sequence. If $\lim_{x \rightarrow 0^+} \theta(x) = \theta(0) = 1$, then the sequence $(\alpha_n, n \in \mathbb{N})$ is convergent, and its limit is 1 ($\lim_{n \rightarrow \infty} \alpha_n = 1$).

Let us count the Fourier-coefficients of θ .

$$\begin{aligned}
(53) \quad \theta(x) &= \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(2 \sum_{k=0}^{2^n-1} w_k(x) - \sum_{k=0}^{2^{n+1}-1} w_k(x) \right) = \\
&= \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(\sum_{k=0}^{2^n-1} w_k(x) - \sum_{k=2^n}^{2^{n+1}-1} w_k(x) \right) = \\
&= \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(\sum_{k=0}^{2^{n+1}-1} w_k(x) \widehat{\text{sgn}}(2^n - k) \right),
\end{aligned}$$

where

$$\widehat{\text{sgn}}(t) = \begin{cases} 1, & \text{if } t > 0, \\ -1, & \text{if } t \leq 0. \end{cases}$$

If we replace the order of the summation, we get for all $x \in [0, 1)$ that
(54)

$$\begin{aligned}
\theta(x) &= \sum_{k=0}^{\infty} w_k(x) \sum_{n \geq \log_2(k+1)-1, n \in \mathbb{N}} \frac{\alpha_n}{2^{n+1}} \cdot \widetilde{\text{sgn}}(2^n - k) = \\
&= w_0(x) \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{n+1}} + \\
&\quad + \sum_{j=0}^{\infty} \sum_{\ell=0}^{2^j-1} w_{2^j+\ell}(x) \sum_{n \geq \log_2(2^j+\ell+1)-1, n \in \mathbb{N}} \frac{\alpha_n}{2^{n+1}} \cdot \widetilde{\text{sgn}}(2^n - 2^j - \ell) = \\
&= w_0(x) \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{n+1}} + \sum_{j=0}^{\infty} \sum_{\ell=0}^{2^j-1} w_{2^j+\ell}(x) \left(\sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}} \right).
\end{aligned}$$

If $\alpha_n \rightarrow 1$ ($n \rightarrow \infty$), then $\gamma_n := 1 - \alpha_n \rightarrow 0$ ($n \rightarrow \infty$), and so

$$(55) \quad 2^j \sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}} = 2^j \sum_{n=j+1}^{\infty} \frac{\gamma_n}{2^{n+1}} - \frac{\gamma_j}{2^{j+1}}.$$

The condition of the absolute convergence is

$$\begin{aligned}
(56) \quad \sum_{k=0}^{\infty} |\theta^\circ(k)| &= \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty} \sum_{\ell=0}^{2^j-1} \left| \sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}} \right| = \\
&= \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty} 2^j \left| \sum_{n=j+1}^{\infty} \frac{\gamma_n}{2^{n+1}} - \frac{\gamma_j}{2^{j+1}} \right| \leq \\
&\leq \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty} 2^j \sum_{n=j}^{\infty} \frac{|\gamma_n|}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{|\gamma_n|}{2^{n+1}} \sum_{j=0}^n 2^j \leq \\
&\leq \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{n=0}^{\infty} |\gamma_n| < \infty.
\end{aligned}$$

That means, if $\sum_{n=0}^{\infty} |\gamma_n| < \infty$, then $\theta^\circ \in L^1[0, +\infty)$.

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