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ON MEAN–VALUE THEOREMS FOR MULTIPLICATIVE FUNCTIONS IN ADDITIVE ARITHMETICAL SEMIGROUPS

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Dedicated to Professor Ferenc Schipp on the occasion of his 70th birthday
Dedicated to Professor Péter Simon on the occasion of his 60th birthday

Abstract. In this paper, the authors prove mean-value theorems for multiplicative functions in general additive arithmetical semigroups.

1. Introduction

Let (G, ∂) be an additive arithmetical semigroup. By definition G is a free commutative semigroup with identity element 1, generated by a countable subset P of primes and admitting an integer valued degree mapping $\partial : G \to \to \mathbb{N} \cup \{0\}$ which satisfies

(i) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in P$,

(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \geq 0$.

Obviously $G(0) = 1$ and G is countable. Putting

$$\pi(n) := \#\{p \in P : \partial(p) = n\}$$

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we obtain the identity

$$\sum_{n=0}^{\infty} G(n)t^n = \prod_{n=1}^{\infty}(1-t^n)^{-\pi(n)}.$$

In a monograph [10], Knopfmacher, motivated by earlier work of Fogels [3] on polynomial rings and algebraic function fields, developed the concept of an additive arithmetical semigroup satisfying the following axiom.

AXIOM $A^{\#}$**.** *There exist constants* $A>0,\ q>1,$ *and* ν *with* $0\le\nu<1$ *(all depending on* G*), such that*

$$G(n) = Aq^n + O(q^{\nu n}), \qquad \text{as } n\to\infty.$$

If G satisfies Axiom $A^{\#}$, then the generating function

$$\tilde{Z}(z) := \sum_{n=0}^{\infty} G(n)z^n \tag{1}$$

is holomorphic in the disc $|z|<q^{-\nu}$ up to a simple pole at $z=q^{-1}$. This means, that we have the representation

$$\tilde{Z}(z) = \frac{\tilde{H}(z)}{1-qz}, \tag{2}$$

where $\tilde{H}$ is holomorphic for $|z|<q^{-\nu}$ and takes the form

$$\tilde{H}(z) = A + (1-qz)\sum_{n=0}^{\infty}(G(n)-Aq^n)z^n.$$

Obviously $\tilde{H}(0)=1$ and $\tilde{H}(q^{-1})=A$.

$\tilde{Z}$ can be considered as the zeta-function associated with the semigroup (G,∂), and it has an Euler-product representation (cf. [10], Chapter 2)

$$\tilde{Z}(z) = \prod_{n=1}^{\infty}(1-z^n)^{-\pi(n)} \quad (|z|<q^{-1}).$$

The logarithmic derivative of $\tilde{Z}$ is given by

$$\frac{\tilde{Z}'(z)}{\tilde{Z}(z)} = \sum_{n=1}^{\infty}\left(\sum_{d|n} d\pi(d)\right)z^{n-1} =: \sum_{n=1}^{\infty}\lambda(n)z^{n-1}, \tag{3}$$

where the *von Mangoldt's* coefficients $\lambda(n)$ and the *prime element* coefficients $\pi(n)$ are related by

$$\lambda(n) = \sum_{d|n} d\pi(d)$$

and, because of the Möbius inversion formula

$$n\pi(n) = \sum_{d|n} \lambda(d)\mu\left(\frac{n}{d}\right).$$

Defining the von Mangoldt function $\Lambda : G \to \mathbb{R}$ by

$$\Lambda(b) = \begin{cases} \partial(p) & \text{if } b \text{ is a prime power } p^r \neq 1, \\ 0 & \text{otherwise,} \end{cases}$$

we see

$$\lambda(n) = \sum_{\substack{b\in G \\ \partial(b)=n}} \Lambda(b)$$

and

$$\partial(a) = \sum_{bd=a} \Lambda(b).$$

Chapter 8 of [10] deals with a theorem called the *abstract prime number theorem: If the additive arithmetical semigroup G satisfies Axiom $A^{\#}$, then*

$$\pi(n) = \frac{q^n}{n} + O\left(\frac{q^n}{n^\alpha}\right) \quad (n \to \infty),$$

or equivalently,

$$\lambda(n) = q^n + O\left(\frac{q^n}{n^{\alpha-1}}\right) \quad (n \to \infty), \tag{4}$$

is true for any $\alpha > 1$.

But this result is only valid if $\tilde{Z}(-q^{-1}) \neq 0$. In [8], Indlekofer et al. gave (in a more general setting) much sharper results valid also in the case $\tilde{Z}(-q^{-1}) = 0$. For instance, if $\tilde{Z}(-q^{-1}) = 0$, then Axiom $A^{\#}$ yields ($\varepsilon > 0$)

$$\frac{\lambda(n)}{q^n} = 1 - (-1)^n + O_\varepsilon\left(q^{n(\nu+\varepsilon-1)}\right). \tag{5}$$

In both cases, the Chebyshev inequality

$$\lambda(n) \ll q^n \tag{6}$$

holds, respectively.

Moving to the investigation of the mean-value properties of complex valued multiplicative functions f satisfying $|f(a)| \le 1$ for all $a \in G$, in [6] Indlekofer and Manstavičius proved analogues of the results of Delange, Wirsing and Halász. Here, as in the classical case, f is called *multiplicative* if $f(1) = 1$ and $f(ab) = f(a)f(b)$ whenever $a, b \in G$ are coprime, and the general aim is to characterize the asymptotic behaviour of the summatory function

$$M(n, f) := q^{-n} \sum_{\substack{a \in G \\ \partial(a)=n}} f(a) \qquad \text{as } \ n \to \infty. \tag{7}$$

The main results (Analogue of Halász's Theorem (see [6])) are

Proposition 1. *Suppose that G is an additive arithmetical semigroup satisfying Axiom $A^\sharp$ and let $f : G \to \mathbb{C}$ be a multiplicative function, $|f(a)| \le 1$. Then there exist a real constant $\tau_0 \in (-\pi, \pi]$ and a complex constant D such that*

$$M(n, f) = D \exp\left\{ i\tau_0 n + i \sum_{k=1}^{n} \operatorname{Im}(q^{-k} \sum_{\partial(p)=k} f(p) e^{-i\tau_0 k}) \right\} + o(1) \tag{8}$$

as $n \to \infty$.

Proposition 2. *Suppose that G is an additive arithmetical semigroup satisfying Axiom $A^\#$. In order that $M(n, f) = o(1)$ as $n \to \infty$, it is both necessary and sufficient that one of the following conditions is satisfied:*

(i) *for each $\tau \in (-\pi, \pi]$ the series*

$$\sum_{p \in P} q^{-\partial(p)} \left(1 - \operatorname{Re}(f(p) e^{-i\tau\partial(p)})\right) \tag{9}$$

diverges;

(ii) *there exists a unique $\tau = \tau_0 \in (-\pi, \pi]$ such that the series (9) converges for $\tau = \tau_0$ and*

$$\prod_{\partial(p) \le c} (1 + f(p)(q^{-1}e^{-i\tau_0})^{\partial(p)} + f(p^2)(q^{-1}e^{-i\tau_0})^{2\partial(p)} + \ldots) = 0. \tag{10}$$

The fundamental question arises: what conditions ensure such alternative asymptotic estimates (5), (6) and (10)? Can these assertions be established under rather loose conditions and hence, hold in principle for a much larger variety of additive arithmetical semigroups? One aim of the book [11] of Knopfmacher and Zhang is to give answers to these questions, and the authors proved, for instance, Chebyshev upper estimates, abstract prime number theorems and mean-value theorems for multiplicative functions. Zhang [13] showed assuming

$$\sum_{n=1}^{\infty} \sup_{n \le m} |G(m)q^{-m} - A| < \infty, \tag{11}$$

that the Chebyshev-type upper estimate (6) and the assertions of Propositions 1 and 2 hold.

Different kind of assumptions have been used by Indlekofer [5]. Assuming some mild conditions on the boundary behaviour of the function $\tilde{H}$ in (2) Chebyshev inequality and the prime number theorem could be proved.

In a similar way we formulate in this paper conditions on $\tilde{H}$ which lead to a proof of Propositions 1 and 2 (see Theorem 2). These conditions imply essentially the estimate

$$\sum_{n \le N} \left(\lambda(n) q^{-n}\right)^2 = O(N) \quad \text{as} \quad N \to \infty \tag{12}$$

which is much weaker than the Chebyshev inequality (6). Further we show that Theorem 2 superceeds all the corresponding results by Zhang (cf. § 2).

Putting $z = q^{-1}y$ in (2) we define $Z(y) := \tilde{Z}(q^{-1}y)$ and $H(y) := \tilde{H}(q^{-1}y)$ and obtain

$$Z(y) = \frac{H(y)}{1-y} \quad \text{for} \quad |y| < 1, \tag{13}$$

and shall assume that $H(y)$ is bounded in the disc $|y| < 1$ satisfying

$$\lim_{y \to 1^-} H(y) = A > 0. \tag{14}$$

To ease notational difficulties we restrict ourselves to *completely multiplicative* functions $f : G \to \mathbb{C}$ under the condition $|f| \le 1$. Then the generating function $\hat{F}$ of f is given by

$$\hat{F}(y) := \sum_{n=0}^{\infty} \sum_{\substack{a \in G \\ \partial(a) = n}} f(a) q^{-n} y^n = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{nq^n} y^n \right) \tag{15}$$

for $|y| < 1$, where

$$\lambda_f(n) = \sum_{\substack{p\in P, k\in\mathbb{N}\\ \partial(p^k)=n}} (f(p))^k \partial(p). \tag{16}$$

We investigate the behaviour of $M(n,f)$ as n tends to infinity, and deal with two alternating cases: the series (9) converges for some unique $\tau \in (-\pi, \pi]$ or diverges for all $\tau \in (-\pi, \pi]$. The proof follows the same lines as in [7] and [11] and is adapted to the given assumptions.

2. Results

Put $H(y) = \sum_{n=0}^{\infty} h(n) y^n$. Then the following holds.

Theorem 1. *Let $H(y)$ be continuous for $|y| \le 1$ and satisfy* (14)*. If*

$$\sum_{n=1}^{\infty} n^2 h^2(n) r^{2n} = O\left(\frac{1}{1-r}\right) \quad \text{as} \quad 0 < r < 1,\ r \to 1, \tag{17}$$

then

$$\sum_{n\le N} \left(\lambda(n) q^{-n}\right)^2 = O(N)$$

as $N \to \infty$.

For example, assume that $\sum_{n=1}^{\infty} n h^2(n) < \infty$. Put $S_n = \sum_{m=1}^{n} m h^2(m)$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2 h^2(n) r^{2n} &= \sum_{n=1}^{\infty} n(S_n - S_{n-1}) r^{2n} = \\
&= (1-r^2) \sum_{n=1}^{\infty} (n+1) S_n r^{2n} - \sum_{n=1}^{\infty} S_n r^{2n} \ll \\
&\ll (1-r^2) \sum_{n=1}^{\infty} (n+1) r^{2n} \ll \\
&\ll \frac{1-r^2}{(1-r^2)^2} \ll \\
&\ll \frac{1}{1-r},
\end{aligned}$$

and (17) holds in this case.

Now, as an obvious consequence of Theorem 1 we formulate

Corollary 1. *Let $H(y)$ be continuous for $|y| \leq 1$ and satisfy* (14)*. If*

(i) $$h(n) = O(n^{-1}) \text{ for all } n \in \mathbb{N},$$

or

(ii) $$H'(y) = O\left(|1-y|^{-1}\right) \text{ as } |y| \to 1,$$

or

(iii) $$\sum_{n=1}^{\infty} nh^2(n) < \infty,$$

then

$$\sum_{n \leq N} \left(\lambda(n)q^{-n}\right)^2 = O(N)$$

as $N \to \infty$.

Theorem 2. *Let G be an additive arithmetical semigroup satisfying*

$$\sum_{n \leq N} \left(\lambda(n)q^{-n}\right)^2 = O(N)$$

and let $H \in \mathbb{H}^\infty$ (i.e. H is bounded in $|y| < 1$) satisfy (14)*. Further, let f be a completely multiplicative function, $|f| \leq 1$. Then the following two assertions hold.*

(i) *If the series* (9) *diverges for each $\tau \in (-\pi, \pi]$, then*

$$M(n, f) = o(1)$$

as $n \to \infty$.

(ii) If the series (9) *converges for some $\tau = \tau_0 \in (-\pi, \pi]$, then*

$$M(n, f) = cL(n) + o(1)$$

as $n \to \infty$, where c is an appropriate real constant, and $L(y)$ is a slowly oscillating function.

Theorem 2 superceeds all the corresponding results of Zhang (cf. [11]). His assumption

$$\sum_{n=0}^{\infty} \left|G(n)q^{-n} - A\right| < \infty \tag{18}$$

(for example [11], Theorem 6.2.2, p.243) implies, since $h(n) = G(n)q^{-n} - G(n-1)q^{-n+1}$ the absolute convergence

$$\sum_{n=0}^{\infty} |h(n)| < \infty,$$

and thus H is continuous on the closed disc $\bar{D} = \{y : |y| \leq 1\}$. In the same way the assumption (loc. cit.)

$$\sum_{n=1}^{\infty} n \left(G(n)q^{-n} - A\right)^2 < \infty$$

leads to

$$\sum_{n=1}^{\infty} n|h(n)|^2 < \infty.$$

Similarly, the condition (see for example [11])

$$\sum_{n=0}^{\infty} \sup_{n \leq m} |G(m)q^{-m} - A| < \infty$$

yields $\sum_{n=0}^{\infty} |h(n)| < \infty$ and $h(n) = o(n^{-1})$.

This can easily be seen in the following way. Put $g(n) = G(n)q^{-n} - A$ and $\bar{g}(n) := \max_{m \geq n} |g(m)|$. Then $\bar{g}(n)$ is monotonically decreasing and $\sum_{n=1}^{\infty} |\bar{g}(n)| < \infty$. Thus

$$\begin{aligned} n\bar{g}(2n) &\leq \bar{g}(n+1) + \bar{g}(n+2) + \ldots + \bar{g}(2n) = \\ &= o(1) \end{aligned}$$

and, since $g(n) = -\sum_{m>n} h(m)$,

$$\sum_{n=0}^{\infty} |h(n)| < \infty,$$

$$\sum_{m \geq n} h(m) = o\left(\frac{1}{n}\right),$$

$$h(n) = o\left(\frac{1}{n}\right)$$

as $n \to \infty$.

Further, the assumption of the Chebyshev inequality $\lambda(n)q^{-n} = O(1)$ together with (18) (cf. [11], Theorem 6.3.1) is much stronger than the condition $\sum_{n\le N} \big(\lambda(n)q^{-n}\big)^2 = O(N)$ together with (14). This may also be illustrated by the following

Example 1. Let the function $\omega : \mathbb{N} \to \mathbb{N}_0$ be such that

$$\sum_{n=1}^{\infty} \frac{\omega^2(n)}{n} < \infty, \tag{19}$$

and let $q \ge 2$ be an integer. Then

$$\sum_{n\le N} \omega(n) = O(N) \quad \text{and} \quad \sum_{n\le N} \omega^2(n) = O(N).$$

Define integers $a_n, 0 \le a_n < n$ by

$$q^n + \omega(n)q^n \equiv a_n \bmod n \tag{20}$$

and put

$$\pi(n) = (q^n + \omega(n)q^n - a_n)\frac{1}{n} + 1 \tag{21}$$

for $n = 1, 2, \ldots$. Then, by (19), the corresponding Zeta-function $Z(y) = \tilde{Z}(q^{-1}y)$ satisfies (13) and (14). Since $\pi(n) \ll q^n$ we get

$$\lambda(n) = n\pi(n) + O(n \log n \cdot q^{\frac{n}{2}}),$$

and we obtain using (19)

$$\sum_{n\le N} \frac{\lambda^2(n)}{q^{2n}} = O(N)$$

as $N \to \infty$.

This example shows that the Chebyshev-type assumption $\lambda(N) \ll q^N$ is stronger than our assumption $\sum_{n\le N} \big(\lambda(n)q^{-n}\big)^2 = O(N)$ with $H \in \mathbb{H}^\infty$ in Theorem 2.

3. Proof of Theorem 1

Obviously,

$$y\frac{Z'(y)}{Z(y)} = \sum_{n=1}^{\infty} \lambda(n) q^{-n} y^n = \frac{y}{1-y} + y\frac{H'(y)}{H(y)}.$$

We show that, if $0 < r < 1$,

$$\int_{-\pi}^{\pi} \left| \frac{re^{i\tau} Z'(re^{i\tau})}{Z(re^{i\tau})} \right|^2 d\tau \ll \frac{1}{1-r}.$$

For this we use the following result which is due to Montgomery (see [2]).

Lemma 1. *Let the series*

$$A(y) = \sum_{n=0}^{\infty} a_n y^n \quad \text{and} \quad B(y) = \sum_{n=0}^{\infty} b_n y^n$$

converge for $|y| < R$. *Let* $|a_n| \le b_n$ *hold for* $n = 0, 1, 2, \dots$. *Then for* $0 < r < R$ *we have for any* τ_0

$$\int_{\tau_0}^{\tau_0+\eta} \left|A(re^{i\tau})\right|^2 d\tau \le 2 \int_{-\eta}^{\eta} \left|B(re^{i\tau})\right|^2 d\tau.$$

Proof. See, for example [11].

Note that $H(y)$ has no zeros in the open disc $D := \{y : |y| < 1\}$. Since $H(1) = A \neq 0$ and $H(y)$ is continuous for $y \in \bar{D} := \{y : |y| \le 1\}$ there exists some $\eta > 0$ such that $H(y) \neq 0$ for $y = re^{i\tau}$ with $0 \le r \le 1$ and $|\tau| \le \eta$. Fix $0 < \eta \le \pi$. Then

$$\int_{\tau_0}^{\tau_0+\eta} \left| \frac{re^{i\tau} Z'(re^{i\tau})}{Z(re^{i\tau})} \right|^2 d\tau \le 2 \int_{-\eta}^{\eta} \left| \frac{re^{i\tau} Z'(re^{i\tau})}{Z(re^{i\tau})} \right|^2 d\tau \ll$$

$$\ll 1 + \int_{-\eta}^{\eta} \left(\frac{1}{|1-re^{i\tau}|^2} + \left|H'(re^{i\tau})\right|^2 \right) d\tau \ll$$

$$\ll \int_{-\pi}^{\pi} \left(\frac{1}{|1-re^{i\tau}|^2} + \left|H'(re^{i\tau})\right|^2 \right) d\tau =$$
$$= \sum_{n=0}^{\infty} r^{2n} + \sum_{n=1}^{\infty} n^2 h^2(n) r^{2n} \ll$$
$$\ll \frac{1}{1-r}$$

and thus

$$\int_{-\pi}^{\pi} \left| \frac{re^{i\tau} Z'(re^{i\tau})}{Z(re^{i\tau})} \right|^2 d\tau = \sum_{n=1}^{\infty} \left(\lambda(n) q^{-n}\right)^2 r^{2n} =$$
$$= O\left(\frac{1}{1-r}\right).$$

Choosing $r = 1 - N^{-1}$ with $N > 1$ gives

$$\sum_{n=1}^{N} \left(\lambda(n) q^{-n}\right)^2 \leq \left(1 - \frac{1}{N}\right)^{-2N} \sum_{n=0}^{N} \left(\lambda(n) q^{-n}\right)^2 r^{2n} =$$
$$= O(N).$$

This ends the proof of Theorem 1.

4. Proof of Theorem 2

We have

$$nM(n,f) = q^{-n} \sum_{\substack{a \in G \\ \partial(a)=n}} f(a)\partial(a) =$$
$$= q^{-n} \sum_{\substack{a \in G \\ \partial(a)=n}} \sum_{bd=a} \Lambda(b) f(b) f(d) =$$
$$= \sum_{\partial(b) \leq n} \frac{\Lambda(b) f(b)}{q^{\partial(b)}} \sum_{\substack{d \\ \partial(d)=n-\partial(b)}} \frac{f(d)}{q^{n-\partial(b)}}.$$

Then

$$|nM(n,f)| \le \sum_{\partial(b)\le n} \frac{\Lambda(b)}{q^{\partial(b)}} |M(n-\partial(b),f)| \le$$

$$\le \sum_{m\le n} \frac{\lambda(m)}{q^m} |M(n-m,f)|.$$

Applying Cauchy-Schwarz's inequality we obtain

$$|nM(n,f)| \le \left(\sum_{m\le n} \frac{\lambda(m)^2}{q^{2m}}\right)^{\frac{1}{2}} \left(\sum_{m\le n} |M(m,f)|^2\right)^{\frac{1}{2}} =: \Sigma_1^{\frac{1}{2}} \cdot \Sigma_2^{\frac{1}{2}}. \tag{22}$$

Obviously

$$\Sigma_1^{\frac{1}{2}} \ll n^{\frac{1}{2}}$$

as $n \to \infty$. By definition

$$\hat{F}(y) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{q^n n} y^n\right), \tag{23}$$

$$Z(y) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(n)}{q^n n} y^n\right). \tag{24}$$

Putting $y = re^{i\tau}$ with $0 < r < 1$ and $\tau \in (-\pi, \pi]$ gives

$$\frac{|\hat{F}(y)|}{Z(|y|)} = \left|\exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n)e^{i\tau n} - \lambda(n)}{q^n n} r^n\right)\right| =$$

$$= |H_1(y)| \left|\exp\left(-\sum_{p\in P} \left(1 - f(p)e^{i\tau\partial(p)}\right) q^{-\partial(p)} r^{\partial}(p)\right)\right|,$$

where

$$H_1(y) := \exp\left(\sum_{n=1}^{\infty} \sum_{\substack{\partial(p^k)=n \\ p\in P, k\in\mathbb{N}, k\ge 2}} \frac{\partial(p)(f(p)^k e^{i\tau n} - 1)}{q^n n} r^n\right).$$

Obviously by Cauchy-Schwarz's inequality,

$$\sum_{n\le N} \frac{\lambda(n)}{q^n} = O(N).$$

Since $|f| \leq 1$, we conclude

$$\sum_{n=1}^{\infty} \sum_{\substack{\partial(p^k)=n \\ p\in P, k\in\mathbb{N}, k\geq 2}} \left| \frac{\partial(p)(f(p)^k e^{i\tau n} - 1)}{q^n n} r^n \right| \leq \sum_{n=1}^{\infty} 2 \sum_{\substack{\partial(p)\leq \frac{n}{2} \\ p\in P}} \frac{\partial(p)}{q^n n} r^n \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{2}{q^n n} \sum_{\substack{\partial(a)\leq \frac{n}{2} \\ a\in G}} \Lambda(a) r^n \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{2}{q^n n} \sum_{m\leq \frac{n}{2}} \lambda(m) r^n \ll$$

$$\ll \sum_{n=1}^{\infty} \frac{2}{q^n n} \frac{n}{2} q^{\frac{n}{2}} r^n \ll$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{q^{\frac{n}{2}}} r^n.$$

Thus $H_1(y)$ is holomorphic for $|y| < q^{1/2}$. Further, using (13) we get

$$|\hat{F}(y)| = \frac{\left|\hat{F}(y)\right|}{Z(|y|)} \cdot Z(|y|) =$$

$$= |H_1(y)| \cdot \left| \exp\left(-\sum_{p\in P} \left(1 - f(p)e^{i\tau\partial(p)}\right) q^{-\partial(p)} r^m \right) \right| \cdot \frac{H(r)}{1-r}.$$

Assume now that the series (9) diverges for every $\tau \in (-\pi, \pi]$. Then, for every τ

$$-\sum_{p\in P} \left(1 - \mathrm{Re} f(p)e^{i\tau\partial(p)}\right) q^{-\partial(p)} r^m \to -\infty$$

as $r \to 1$, and Dini's theorem shows

$$\exp\left(-\sum_{p\in P} \left(1 - \mathrm{Re} f(p)e^{i\tau\partial(p)}\right) q^{-\partial(p)} r^m \right) = o(1)$$

uniformly in $\tau \in (-\pi, \pi]$ as $r \to 1$. Since

$$\frac{H(r)}{1-r} \sim \frac{A}{1-r}$$

we get

$$\sum_{m=0}^{\infty} M(m,f)r^m e^{mi\tau} = o\left(\frac{1}{1-r}\right) \tag{25}$$

uniformly for $\tau \in (-\pi,\pi]$ as $r \to 1$. The remaining part of the proof of (i) may be found in [7]. For the sake of completeness we repeat the details. By Parseval's identity

$$W(r) := \sum_{m=0}^{\infty} |M(m,f)|^2 r^{2m} = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{m=0}^{\infty} M(m,f)r^m e^{mi\tau}\right|^2 d\tau.$$

Then, using the estimate (25)

$$W(r) = o\left((1-r)^{-\frac{1}{2}}\right)\int_0^{2\pi}\left|\sum_{m=0}^{\infty} M(m,f)r^m e^{mi\tau}\right|^{3/2} d\tau. \tag{26}$$

The Euler-product gives the representation

$$\hat{F}(y) = \sum_{m=0}^{\infty} M(m,f)y^m = H_2(y)\exp\left\{\sum_{k=1}^{\infty} q^{-k}\sum_{\partial(p)=k} f(p)y^k\right\}, \tag{27}$$

where $|y|<1$ and $H_2(y)$ is a regular function in the disc $|y|<1+c$ with some $c>0$. Observe that $H_2(y)\neq 0$ and $H_2(y)\asymp 1$ in the disc $|y|\le 1$. Hence using the same representation for the function fg, where $g(a)=(3/4)^{\Omega(a)}$ and $\Omega(a)$ denotes the number of all prime elements dividing a, we obtain

$$|\hat{F}(y)|^{3/4} \asymp \exp\left\{\sum_{k=1}^{\infty} q^{-k}\sum_{\partial(p)=k}\frac{3}{4}f(p)y^k\right\} \asymp \sum_{m=0}^{\infty} M(m,fg)y^m, \quad \text{for} \quad |y|<1. \tag{28}$$

Since $|M(m,fg)| \ll |M(m,g)|$, applying the Parseval's identity again we deduce

$$\begin{aligned}\int_0^{2\pi}|\hat{F}(re^{i\tau})|^{3/2}\,d\tau &\ll \int_0^{2\pi}\left|\sum_{m=0}^{\infty} M(m,fg)r^m e^{mi\tau}\right|^2 d\tau = \\ &= 2\pi\sum_{m=0}^{\infty}|M(m,fg)|^2 r^{2m} \ll \\ &\ll 2\pi\sum_{m=0}^{\infty}|M(m,g)|^2 r^{2m} = \\ &= \int_0^{2\pi}\left|\sum_{m=0}^{\infty} M(m,g)r^m e^{mi\tau}\right|^2 d\tau.\end{aligned}$$

The representation of type (27) and $H \in \mathbb{H}^\infty$ shows that

$$\int_0^{2\pi} \left| \sum_{m=0}^{\infty} M(m,g) r^m e^{mi\tau} \right|^2 d\tau \ll \int_0^{2\pi} |Z(re^{i\tau})|^{3/2} d\tau =$$

$$= \int_0^{2\pi} \left| \frac{H(re^{i\tau})}{1 - r^{e^{i\tau}}} \right|^{3/2} d\tau \ll \tag{29}$$

$$\ll \int_0^{2\pi} |1 - re^{i\tau}|^{-3/2} d\tau.$$

Hence

$$\int_0^{2\pi} |\hat{F}(re^{i\tau})|^{3/2}\, d\tau \ll \int_0^{2\pi} |1 - re^{i\tau}|^{-3/2}\, d\tau \ll \tag{30}$$

$$\ll \int_0^{1-r} (1-r)^{-3/2}\, d\tau + \int_{1-r}^{2\pi} \tau^{-3/2}\, d\tau \ll \frac{1}{(1-r)^{1/2}}. \tag{31}$$

Thus from (26) we have the estimate $W(r) = o\left(\frac{1}{1-r}\right)$. Our assumption (14) yields by Hardy-Littlewood's Tauberian Theorem (see [4], p.155, Theorem 96)

$$\sum_{m \le n} |M(m,f)|^2 = o(n).$$

It follows

$$nM(n,f) \le \Sigma_1^{\frac{1}{2}} \cdot \Sigma_2^{\frac{1}{2}} \ll o(n)$$

for $n \to \infty$. This ends the proof of assertion (i).

Next, we need the following

Lemma 2. *Let G be an additive arithmetical semigroup satisfying*

$$\sum_{n \le N} \left(\lambda(n) q^{-n}\right)^2 = O(N)$$

and let further f be a completely multiplicative function with $|f| \leq 1$ and suppose the series (9) converges for $\tau = 0$. Then we have uniformly for $y = r$ and $\frac{1}{2}(1-\eta) \leq 1-r \leq 1-\eta$,

$$\sum_{p\in P} \left|\eta^{\partial(p)} - r^{\partial(p)}\right| \frac{|1-f(p)|}{q^{\partial(p)}} = o(1) \tag{32}$$

as $\eta \to 1-$.

Proof. For every $M > 0$

$$\begin{aligned}\sum_{p\in P} \left|\eta^{\partial(p)} - r^{\partial(p)}\right| \frac{|1-f(p)|}{q^{\partial(p)}} &= \sum_{\substack{p\in P\\ \partial(p)\leq M}} \left|\eta^{\partial(p)} - r^{\partial(p)}\right| \frac{|1-f(p)|}{q^{\partial(p)}} + \\ &+ \sum_{\substack{p\in P\\ \partial(p)> M}} \left|\eta^{\partial(p)} - r^{\partial(p)}\right| \frac{|1-f(p)|}{q^{\partial(p)}} = \\ &= S_1 + S_2.\end{aligned}$$

Choose $\varepsilon > 0$. Since $|1-f(p)|^2 \leq 2(1-\mathrm{Re} f(p))$ for $|f(p)| \leq 1$ there exists $M_0 > 0$ such that for $M > M_0$

$$\begin{aligned}S_2^2 &\leq \sum_{\substack{p\in P\\ \partial(p)> M}} q^{-\partial(p)} \left|\eta^{\partial(p)} - r^{\partial(p)}\right|^2 \sum_{\substack{p\in P\\ \partial(p)> M}} \frac{|1-f(p)|^2}{q^{\partial(p)}} \leq \\ &\leq \sum_{\substack{p\in P\\ \partial(p)> M}} q^{-\partial(p)} r^{2\partial(p)} \left|\left(\frac{\eta}{r}\right)^{\partial(p)} - 1\right|^2 \varepsilon = \\ &= \varepsilon S_3.\end{aligned}$$

Observe

$$\begin{aligned}\left|\left(\frac{\eta}{r}\right)^{\partial(p)} - 1\right|^2 &\leq 2\partial(p) \log \frac{r}{\eta} \leq \\ &\leq 4\partial(p)(1-r).\end{aligned}$$

Then

$$\begin{aligned}S_3 &\leq 4(1-r) \sum_{p\in P} \frac{\partial(p)}{q^{\partial(p)}} r^{2\partial(p)} \leq \\ &\leq 4(1-r) \sum_{n=1}^{\infty} \frac{\lambda(n)}{q^n} r^{2n} = \\ &= 4(1-r) O\left(\frac{1}{1-r}\right) = \\ &= O(1)\end{aligned}$$

as $r \to 1$ since $\sum_{n \le N} \frac{\lambda(n)}{q^n} = O(N)$. Letting $r \to 1$ and $\eta \to 1$ gives $S_1 = o(1)$ and this ends the proof of Lemma 2.

We assume now that (9) converges for $\tau = 0$. Our aim is to prove first

$$NM(N,f) = \frac{1}{2\pi i} \int_{|y|=r} \frac{\hat{F}'(y)}{y^N} dy = cNL(N) + o(N), \tag{33}$$

where $0 < r < 1$ and c is an appropriate real constant. For this we will show the following estimate

$$\hat{F}(y) = \frac{c}{1-y} L\left(\frac{1}{1-|y|}\right) + o\left(\frac{1}{1-|y|}\right)$$

as $|y| \to 1-$.

For $|y| < 1$ we have

$$\hat{F}(y) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{q^n n} y^n\right),$$

$$Z(y) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(n)}{q^n n} y^n\right).$$

This yields

$$\frac{\hat{F}(y)}{Z(y)} = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n) - \lambda(n)}{q^n n} y^n\right) =$$

$$= H_3(y) \exp\left(\sum_{n=1}^{\infty} \sum_{\substack{p \in P \\ \partial(p)=n}} \frac{f(p)-1}{q^n} y^n\right),$$

where

$$H_3(y) := \exp\left(\sum_{n=1}^{\infty} \sum_{\substack{\partial(p^k)=n \\ p \in P, k \in \mathbb{N}, k \ge 2}} \frac{\partial(p)(f(p)^k - 1)}{q^n n} y^n\right).$$

Similarly as above $H_3(y)$ is holomorphic for $|y| < q^{1/2}$. Further

$$\exp\left(\sum_{n=1}^{\infty} \sum_{\substack{p\in P \\ \partial(p)=n}} \frac{f(p)-1}{q^n} y^n\right) = \exp\Big\{ -\sum_{p\in P} (y^{\partial(p)} - |y|^{\partial(p)}) q^{-\partial(p)} (1-f(p)) -$$

$$- \sum_{p\in P} |y|^{\partial(p)} q^{-\partial(p)} (1-\mathrm{Re} f(p)) \Big\} L\left(\frac{1}{1-|y|}\right),$$

where

$$L\left(\frac{1}{1-|y|}\right) := \exp\left(i \sum_{p\in P} |y|^{\partial(p)} q^{-\partial(p)} \mathrm{Im} f(p) \right).$$

Obviously, $|L| = 1$. Put $u := (1-|y|)^{-1}$. To show that $L(u)$ is a slowly oscillating function of u, it suffices to note that, for $\frac{1}{2}u \le v \le u$, by Lemma 2 with $\eta = 1 - v^{-1}$,

$$\frac{L(v)}{L(u)} = \exp\left(i \sum_{p\in P} \left(\eta^{\partial(p)} - |y|^{\partial(p)}\right) \frac{\mathrm{Im} f(p)}{q^{\partial(p)}} \right) =$$

$$= \exp(o(1))$$

as $u \to \infty$. Set

$$c_1 := H_1(1) \exp\left(-\sum_{p\in P} q^{-\partial(p)} (1 - \mathrm{Re} f(p)) \right).$$

Let M be a fixed positive real number. The function $\hat{F}(y)\left(L\left(\frac{1}{1-|y|}\right)Z(y)\right)^{-1}$ is holomorphic in the disc $|y| < 1$ and converges at the boundary point $y = 1$. Then, by Stolz's theorem (see [12], page 121) we conclude, putting $y = re^{i\tau}$,

$$\frac{\hat{F}(y)}{L\left(\frac{1}{1-|y|}\right) Z(y)} = c_1 + o_M(1),$$

as $|y| \to 1-$ uniformly for $|\tau| \le M(1-|y|)$. Hence

$$\hat{F}(y) = c_1 \frac{H(y)}{1-y} L\left(\frac{1}{1-|y|}\right) + o_M\left(\frac{1}{|1-y|}\right) \tag{34}$$

and

$$\left|\frac{\hat{F}(y)}{Z(|y|)}\right| =$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{\mathrm{Re}\lambda_f(m)e^{i\tau m} - \lambda(m)}{mq^m} r^m\right) =$$

$$= \exp\left(-\sum_{p\in P} q^{-\partial(p)} r^{\partial}(p)\left(1 - \mathrm{Re} f(p) e^{i\tau\partial(p)}\right)\right) \times$$

$$\times \exp\left(-\sum_{m=1}^{\infty} \sum_{\substack{p\in P, k\geq 2\\ \partial(p^k)=m}} \frac{1 - \mathrm{Re} f(p)^k e^{i\tau\partial(p)}}{mq^m} r^m\right) \ll$$

$$\ll \exp\left(-\sum_{p\in P} q^{-\partial(p)} r^{\partial}(p)\left(1 - \mathrm{Re} f(p) e^{i\tau\partial(p)}\right)\right).$$

This shows

$$\frac{|\hat{F}(y)|}{Z(|y|)|Z(\bar{y})|} = \left|\frac{\hat{F}(y)}{Z(|y|)}\right|^2 \cdot \frac{Z(|y|)}{|Z(\bar{y})|} \ll$$

$$\ll \exp\left(-2\sum_{p\in P} q^{-\partial(p)} r^{\partial}(p)\left(1 - \mathrm{Re} f(p) e^{i\tau\partial(p)}\right) + \right.$$

$$\left. + \sum_{p\in P} q^{-\partial(p)} r^{\partial}(p)\left(1 - \mathrm{Re}\, e^{-i\tau\partial(p)}\right)\right) \ll$$

$$\ll \exp\left(2\sum_{p\in P} q^{-\partial(p)} r^{\partial}(p)\left(1 - \mathrm{Re} f(p)\right)\right)$$

$$\ll 1,$$

since the series (9) converges for $\tau = 0$ and since

$$2\left(1 - \mathrm{Re}\, e^{-i\tau\partial(p)}\right) = \left|1 - e^{-i\tau\partial(p)}\right|^2 \leq$$

$$\leq 2|1 - f(p)|^2 + 2\left|f(p) - e^{-i\tau\partial(p)}\right|^2 \leq$$

$$\leq 4(1 - \mathrm{Re} f(p)) + 4\left(1 - \mathrm{Re}\, e^{i\tau\partial(p)}\right).$$

This implies for $|\tau| \geq M(1-|y|)$,

$$|\hat{F}(y)| \ll M^{-\frac{1}{2}} \frac{1}{1-|y|}. \tag{35}$$

Collecting (34) and (35) shows

$$\hat{F}(y) = \frac{c}{1-y} L\left(\frac{1}{1-|y|}\right) + o\left(\frac{1}{1-|y|}\right) \tag{36}$$

as $|y| \to 1-$.

Now we proceed similarly as in the proof of Theorem 6.2.2 in [11]. Set $r = 1 - \frac{1}{N}$. Let K be a large positive number and let N be chosen that $N \geq 2K^2$. We break the circle $y = re^{i\tau}$ into two arcs

$$A_0 := \left\{\tau : |\tau| \leq \frac{K}{N}\right\} \quad \text{and} \quad A_1 := \left\{\tau : \frac{K}{N} \leq |\tau| \leq \pi\right\}.$$

We estimate the integral on the left-hand side of (33) on each arc separately. This will show that the integral on A_0 produces the main term on the right-hand side of (33), whereas the integral on A_1 gives an o-term.

(i) Estimate of $\int\limits_{A_0}$.

Let $y \in A_0$ and consider the circle $|w-y| = \frac{1}{2N}$. In the range $1 - \frac{3}{2N} \leq$ $\leq |w| \leq 1 - \frac{1}{2N}$ we have

$$L\left(\frac{1}{1-|w|}\right) = L(N) + o(1).$$

Thus, by (36),

$$\begin{aligned} \hat{F}(w) &= \frac{c}{1-w} L\left(\frac{1}{1-|w|}\right) + o\left(\frac{1}{1-|w|}\right) = \\ &= \frac{c}{1-w} L(N) + o(N) \end{aligned}$$

on this circle. Now, by Cauchy's theorem

$$\hat{F}'(y) = \frac{c}{(1-y)^2} L(N) + o(N^2) \quad \text{if} \quad y \in A_0.$$

Then

$$\frac{1}{2\pi i}\int\limits_{A_0}\frac{\hat{F}'(y)}{y^N}dy=\frac{cL(N)}{2\pi i}\int\limits_{A_0}\frac{dy}{(1-y)^2y^N}+o(N^2)\frac{2K}{N}. \tag{37}$$

The integral on the right-hand side of (37) can be evaluated by using the residue theorem (for more details see the proof of Theorem 6.2.2 in [11]):

$$\frac{1}{2\pi i}\int\limits_{A_0}\frac{dy}{(1-y)^2y^N}=N+O(K^{-1}N). \tag{38}$$

(ii) Estimate of $\int\limits_{A_1}$.

If f is a completely multiplicative function with $|f(a)|\le 1$, then $\hat{F}(y)\neq 0$ for $|y|<1$. By Cauchy-Schwarz's inequality we have

$$\left|\int\limits_{A_1}\frac{\hat{F}'(y)}{y^N}dy\right|\le$$

$$\le\left(\int\limits_{A_1}\left|\frac{re^{i\tau}\hat{F}'(re^{i\tau})}{\hat{F}(re^{i\tau})}\right|^2 r^{-N}d\tau\right)^{1/2}\times\left(\int\limits_{A_1}|\hat{F}(re^{i\tau})|^2r^{-N}d\tau\right)^{1/2}. \tag{39}$$

Parseval's identity and the conditions $\sum\limits_{n\le N}\lambda^2(n)q^{-2n}=O(N)$ and $|f|\le 1$ yield

$$\begin{aligned}\frac{1}{2\pi}\int\limits_{A_1}\left|\frac{re^{i\tau}\hat{F}'(re^{i\tau})}{\hat{F}(re^{i\tau})}\right|^2 r^{-N}d\tau &= \le r^{-N}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|\frac{re^{i\tau}\hat{F}'(re^{i\tau})}{\hat{F}(re^{i\tau})}\right|^2d\tau=\\ &=r^{-N}\sum_{n=0}^{\infty}\left|\frac{\lambda_f(n)}{q^n}\right|^2r^{2n}\le\\ &\le r^{-N}\sum_{n=0}^{\infty}\left|\frac{\lambda(n)}{q^n}\right|^2r^{2n}\ll\\ &\ll\frac{1}{1-r^2}\\ &\le\frac{1}{1-r}=\\ &=N,\end{aligned}$$

since $r = 1 - \frac{1}{N}$. Further

$$\int_{A_1} |\hat{F}(re^{i\tau})|^2 r^{-N} d\tau \le \max_{\frac{K}{N} \le |\tau| \le \pi} |\hat{F}(re^{i\tau})|^{1/2} \int_{A_1} |\hat{F}(re^{i\tau})|^{3/2} r^{-N} d\tau. \tag{40}$$

By (36),

$$|\hat{F}(re^{i\tau})| \le \left| \frac{c}{1 - re^{i\tau}} \right| + o(N).$$

For $\frac{K}{N} \le |\tau| \le \pi$,

$$|1 - re^{i\tau}| \ge \left| 1 - \left(1 - \frac{1}{N}\right) e^{iK/N} \right| = r_N \gg \frac{K}{N}$$

and hence

$$\max_{\frac{K}{N} \le |\tau| \le \pi} |\hat{F}(re^{i\tau})|^{1/2} \le O(K^{-1/2} N^{1/2}) + o\left(N^{1/2}\right). \tag{41}$$

As we have seen in the proof of (i) (cf. (28)-(31)) we get

$$\int_{A_1} |\hat{F}(re^{i\tau})|^{3/2} r^{-N} d\tau \ll \int_{-\pi}^{\pi} |Z(re^{i\tau})|^{3/2} d\tau = \int_{-\pi}^{\pi} \left| \frac{H(re^{i\tau})}{1 - re^{i\tau}} \right|^{3/2} d\tau \ll N^{\frac{1}{2}}. \tag{42}$$

This implies

$$\left| \int_{A_1} \frac{\hat{F}'(y)}{y^N} dy \right| \ll N^{1/2} \left(\left[O(K^{-\frac{1}{2}} N^{\frac{1}{2}}) + o(N^{\frac{1}{2}}) \right] N^{\frac{1}{2}} \right)^{1/2} \le \tag{43}$$

$$\le O\left(K^{-\frac{1}{4}} N\right) + o(N). \tag{44}$$

Combining the estimates we finally arrive at

$$\frac{1}{2\pi i} \int_{|y|=r} \frac{\hat{F}'(y)}{y^N} dy = \frac{cL(N)}{2\pi i} \int_{A_0} \frac{dy}{(1-y)^2 y^N} + o(KN) + O(K^{-\frac{1}{4}} N)$$

$$= cNL(N) + o(KN) + O(K^{-\frac{1}{4}} N).$$

Choosing K large and letting N tend to infinity shows

$$\frac{1}{2\pi i} \int\limits_{|y|=r} \frac{\hat{F}'(y)}{y^N} dy = cNL(N) + o(N),$$

as $N \to \infty$. Thus by (33)

$$NM(N, f) = cL(N)N + o(N).$$

Finally, assume that (9) converges for $\tau = \tau_0 \neq 0$. Then for the completely multiplicative function $f(a)q^{-i\partial(a)\tau_0}$, (9) converges for $\tau = 0$ and the above arguments prove assertion (ii) of Theorem 2.

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