Annales Univ. Sci. Budapest., Sect. Comp. 33 (2010) 27-48

ABSOLUTE SUMMABILITY FACTORS OVER BANACH ALGEBRAS

Mati Abel (Tartu, Estonia)
Simson Baron (Ramat Gan, Israel)

Dedicated to Professors Ferenc Móricz, Ferenc Schipp and Péter Simon on their birthdays

Abstract. Let α be a nonnegative integer, A a unital Banach algebra, X a unital Banach A-algebra, $|T_A^\alpha|$ a method of absolute summability for X, defined by a normal series-to-series matrix over A, the inverse matrix of which has exactly $\alpha + 1$ non-zero diagonals, and B_A a method of summability defined by an infinite matrix over A. The cases, where T_A^α is a) a Riesz weighted means summability method over A and b) the product of Riesz weighted means summability method P_A and Q_A over A, are considered as application.

1. Introduction

1.1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, A an associative (not necessarily commutative) Banach algebra over $\mathbb{K}$ (for

---

Mathematics Subject Classification: Primary 40C05, 40D05, 40D15, 40F05; secondary 46H05, 46H25.

This research is partially supported by Estonian Science Foundation grant 7320, by Estonian Targent Financing Project SF 0180039s08, by the Gelbart Research Institute for Mathematical Sciences of Bar-Ilan Unviersity and the Minerva Foundation in Germany through the Emmy Noether Research Institute of Mathematics of Bar-Ilan University.

short, a Banach algebra) with norm $\|\cdot\|_A$ and X a left Banach A-module, i.e. a Banach space over $\mathbb{K}$ with norm $\|\cdot\|_X$ for which there has been defined a bilinear map (the multiplication over A) $(a,x)\to ax$ from $A\times X$ into X such that (cf. [6], p. 49, or [7], pp. 51 and 238)

1° $a(bx)=(ab)x$ for each $a,b\in A$ and $x\in X$;

2° $\|ax\|_X\le\|a\|_A\|x\|_X$ for each $a\in A$ and each $x\in X$;

3° if A has the unit element e_A, then $e_Ax=x=xe_A$ for each $x\in X$.

A left Banach A-module X is called a *left Banach A-algebra* if its underlying Banach space is a Banach algebra (see [1], p. 238, or [2]). In this case

$$\|x_1x_2\|_X\le\|x_1\|_X\|x_2\|_X$$

for each $x_1,x_2\in X$ and $ae_X=e_Xa$ for each $a\in A$ if X has the unit element e_X.

1.2. Let $\mathbb{N}_0=\{0,1,\dots\}$, A a Banach algebra, X a left Banach A-module, $x=(x_n)$, where $x_n\in X$ for each $n\in\mathbb{N}_0$,

$$c(X):=\{x:\exists\lim_{n\to\infty}x_n\in X\}$$

and

$$l(X):=\left\{x:\sum_{n=0}^{\infty}\|x_n\|_X<\infty\right\}.$$

The addition and the multiplication over $\mathbb{K}$ in $l(X)$ we define coordinatewise, the multiplication over A by $ax=(ax_n)$ for each $a\in A$ and each $x\in l(X)$ and the norm $\|x\|_{l(X)}$ of $x\in l(X)$ by

$$\|x\|_{l(X)}=\sum_{n=0}^{\infty}\|x_n\|_X.$$

Then

$$\|ax\|_{l(X)}\le\|a\|_A\|x\|_{l(X)}$$

for each $a\in A$ and each $x\in l(X)$. Taking this into account, we obtain that $l(X)$ is a left Banach A-module.

1.3. Let A be a Banach algebra with unit element e_A and (a_{nk}) a *normal matrix* over A that is, an infinite matrix over A for which $a_{nk}=\theta_A$ (the null element in A) if $k>n$ for each $k,n\in\mathbb{N}_0$ and a_{nn} is invertible in A for each

$n \in \mathbb{N}_0$. Hence every normal matrix over A has the inverse matrix (ξ_{nk}) such that

$$\sum_{k=\nu}^{n} a_{n\nu}\xi_{\nu k} = \delta_{nk},$$

where

$$\delta_{nk} = \begin{cases} e_A & \text{if } k = n, \\ \theta_A & \text{if } k \neq n \end{cases}$$

for each $k, n \in \mathbb{N}_0$.

1.4. Let A be a Banach algebra, X a left Banach A-module, (τ_{nk}) an infinite matrix over A, which defines a matrix transformation

$$T_n x = \sum_{k=0}^{\infty} \tau_{nk} x_k \tag{1}$$

of a series $\sum_k x_k$ (with terms x_k from X) to a sequence $(T_n x)$, and $(\overline{\tau}_{nk})$ is an infinite matrix over A, which defines a matrix transformation

$$\overline{T}_n x = \sum_{k=0}^{\infty} \overline{\tau}_{nk} x_k \tag{2}$$

of a series $\sum_k x_k$ to a series $\sum_n \overline{T}_n x$. We shall say that a normal series-to-series matrix $(\overline{\tau}_{nk})$ over A is a T_A^{α}*–matrix* if the inverse matrix $(\overline{\eta}_{nk})$ of $(\overline{\tau}_{nk})$, given by (2), has exactly $\alpha + 1$ (α is a nonnegative integer) non-zero diagonals that is, $\overline{\eta}_{nk} = \theta_A$ for $n < k$ and $n > k + \alpha$. The inverse matrix (η_{nk}) of the corresponding series-to-sequence matrix (τ_{nk}) has then $\alpha + 2$ nonzero diagonals because $\eta_{nk} = \overline{\eta}_{nk} - \overline{\eta}_{n,k+1}$ for each $n, k \in \mathbb{N}_0$ with $k \geq n$ (see [5], p. 56, formula (9.7)).

A series $\sum_k x_k$ is called to be

a) *summable by the method* T_A, defined by a matrix (τ_{nk}) over A (for short, T_A-*summable*) if $(T_n x) \in c(X)$ and

b) *absolutely summable by the method* $\overline{T}_A$, defined by a matrix $(\overline{\tau}_{nk})$ over A, (for short $|T_A|$-summable) if $(\overline{T}_n x) \in l(X)$.

Let $\varepsilon = (\varepsilon_n)$, where $\varepsilon_n \in A$ for each $\mathbb{N}_0$. If B_A is a method of summability over A, then the sequence ε *is called to be a summability factor of*

a) $(|T_A^\alpha|, B_A)$-*type* for X (for short, $\varepsilon \in (|T_A^\alpha|, B_A)$) if the series $\sum_k \varepsilon_k x_k$ is B_A-summable for each $|T_A|$-summable series $\sum_k x_k$ in X and

b) $(|T_A^\alpha|, |B_A|)$-*type* for X (for short, $\varepsilon \in (|T_A^\alpha|, |B_A|)$) if the series $\sum_k \varepsilon_k x_k$ is $|B_A|$-summable for each $|T_A|$-summable series $\sum_k x_k$ in X (see [3], p. 147).

In [3] have been proved the following generalizations of a classical Knopp-Lorentz theorem (see [8], p. 12, or [5], p. 34) and Hahn theorem (see [5], p. 25):

Proposition 1. *Let A be a Banach algebra, X a left Banach A-algebra with unit element e_X and $(\overline{\tau}_{nk})$ an infinite matrix over A. The matrix transformation* (2)*, defined by $(\overline{\tau}_{nk})$, maps $l(X)$ into $l(X)$ if and only if*

$$\sum_{n=0}^{\infty} \|\overline{\tau}_{nk} e_X\|_X = O(1). \tag{3}$$

Proof. See [3], pp. 149–150.

Proposition 2. *Let A be a Banach algebra, X a left Banach A-algebra with unit e_X and (τ_{nk}) an infinite matrix over A. The matrix transformation* (1)*, defined by (τ_{nk}), maps $l(X)$ into $c(X)$ if and only if*

1) $\|\tau_{nk} e_X\| = O(1)$;

2) $(\tau_{nk} e_X)$ *converges in X* for each $k \in \mathbb{N}_0$.

Proof. See [3], pp. 151–152.

1.5. In the paper [3], Theorems 4 and 5, have been found the necessary and sufficient conditions for elements ε_n of a Banach algebra A to be $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for a left Banach A-algebra X, where (see [3], pp. 147–148) P_A denotes the Riesz weighted means summability method over A (it is a T_A^1-method of summability) and B_A is a described summability method over A.

In the present paper we generalize these results giving the necessary and sufficient conditions for elements ε_n of a Banach algebra A to be $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors of summability for a left Banach A-algebra X in case, when an integer $a \geq 1$ and B_A is a described summability method over A. As an application, the $(|P_A|, B_A)$-factors, $(|P_A|, |B_A|)$-factors, $(|Q_A P_A|, B_A)$-factors and $(|Q_A P_A|, |B_A|)$-factors of summability for a left Banach A-algebra X are described in case when the matrix method B_A satisfies other conditions than in [3].

2. Main result

Before to describe the main result of this paper we give the necessary notations. For a given unital Banach algebra A method T_A^α of summability, defined by the matrix (τ_{nk}) over A, and a sequence $\varepsilon := (\varepsilon_n)$ in A let

$$D_n = \sup_k \|\tau_{n+k,n+k}\eta_{n+k,k}\|_A,$$

$$K\varepsilon_n = \sum_{k=n}^{n+\alpha}\sum_{\nu=k}^{n+\alpha} \varepsilon_\nu \eta_{\nu k},$$

where (η_{nk}) is the inverse matrix of (τ_{nk}).

If (η_{nk}) is the inverse matrix of (τ_{nk}) over A and $(\overline{\eta}_{nk})$ is the corresponding matrix (on the series to series form), then elements of these matrices are connected by

$$\overline{\eta}_{nk} = \sum_{\nu=k}^{n} \eta_{n\nu} \tag{4}$$

(see [5], formula (9.6)) for each $n \geq k$. By means of (4) we obtain

$$K\varepsilon_n = \sum_{\nu=n}^{n+\alpha} \varepsilon_\nu \overline{\eta}_{\nu n}. \tag{5}$$

Proposition 3. *Let α be a nonnegative integer, A a unital Banach algebra and X a left Banach A-algebra with unit element e_X. Let $|T_A^\alpha|$ be a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$, and B_A a series-to-sequence method of summability defined by a matrix (β_{nk}) over A. Let $\overline{\beta}_{nk} = = \beta_{nk} - \beta_{n-1,k}$. Then we have*

a) *If (β_{nk}) satisfies the condition*

$$\lim_{n\to\infty} \beta_{nk} = e_A \tag{6}$$

and elements ε_k of A are $(|T_A^\alpha|, B_A)$-factors for X, then

$$\|\beta_{nn}\varepsilon_n\tau_{nn}^{-1}e_X\|_X = O(1) \tag{7}$$

and

$$\|(K\varepsilon_n)e_X\|_X = O(1). \tag{8}$$

If, in addition $|T_A^\alpha|$ *preserves the absolute convergence*[1], *then also*

$$\|\varepsilon_n e_X\|_X = O(1). \tag{9}$$

b) *If* D_n *is finite for each* $n \in \{0, 1, \ldots, \alpha+1\}$ *and* B_A *is a normal method of summability which satisfies the conditions* (6),

$$\sum_{n=\nu}^{\infty} \|(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1}\|_A = O(1), \tag{10}$$

$$\|\beta_{kk}\beta_{k+1,k+1}^{-1}\|_A = O(1) \tag{11}$$

and

$$\sum_{n=k}^{\infty} \|\overline{\beta}_{nk}\|_A = O(1), \tag{12}$$

then the elements ε_k *of* A *are* $(|T_A^\alpha|, |B_A|)$*-factors for* X *if conditions* (7) *and* (8) *hold.*

Proof. a) Since every T_A^α-matrix is normal, there exists the inverse transformation

$$x_k = \sum_{\nu=0}^{k} \overline{\eta}_{k\nu}\overline{T}_\nu x \tag{13}$$

of (2). Therefore,

$$B_n(\varepsilon x) := \sum_{\nu=0}^{n} \beta_{n\nu}\varepsilon_\nu x_\nu = \sum_{\nu=0}^{n} \beta_{n\nu}\varepsilon_v \sum_{k=0}^{\nu} \overline{\eta}_{vk}\overline{T}_k x = \sum_{k=0}^{n}\left(\sum_{v=k}^{n} \beta_{nv}\varepsilon_v\overline{\eta}_{vk}\right)\overline{T}_k x$$

or

$$B_n(\varepsilon x) = \sum_{k=0}^{n} \gamma_{nk}\overline{T}_k x,$$

[1] That is, every sequence $x \in l(X)$ is $|T_A^\alpha|$-summable.

where

$$\gamma_{nk} = \sum_{\nu=k}^{k+\alpha} \beta_{n\nu}\varepsilon_\nu \overline{\eta}_{\nu k}$$

for $k \leq n$ because $\overline{\eta}_{nk} = \theta_A$ if $n > k + \alpha$. The matrix (γ_{nk}) transforms $(\overline{T}_n x) \in l(X)$ into $(B_n(\varepsilon x)) \in c(X)$ if and only if

$$\|\gamma_{nk} e_X\|_X = O(1) \tag{14}$$

and there exists

$$\lim_{n\to\infty} \gamma_{nk} e_X = \gamma_k \tag{15}$$

in X for each $k \in \mathbb{N}_0$ by Proposition 2. Hence,

$$\|\beta_{kk}\varepsilon_k \overline{\eta}_{kk} e_X\|_X = \|\gamma_{kk} e_X\|_X = O(1),$$

from which follows the condition (7) because $\overline{\eta}_{kk} = \tau_{kk}^{-1}$ for each $k \in \mathbb{N}_0$ (see [5], p. 57).

Since $\gamma_k = (K\varepsilon_k)e_X$ by the condition (6) in view of (5), then the condition (8) holds. In the particular case, when $|T_A^\alpha|$ preserves absolute convergence in X, then holds also (9) by Lemma 1 from [3], p. 152.

b) Let the elements $\varepsilon_n \in A$ satisfy the conditions (7) and (8). To show that $\varepsilon \in (|T_A^\alpha|, |B_A|)$, we have to show that the series $\sum_n \varepsilon_n x_n$ is $|B_A|$-summable for each $|T_A^\alpha|$-summable series $\sum_n x_n$ in X. For it we assume that $\sum_n x_n$ is a $|T_A^\alpha|$-summable series in X. Since every T_A^α-matrix is normal, there exists the inverse transformation (13) of (2). Then

$$\overline{B}_n(\varepsilon x) := \sum_{\nu=0}^{n} \overline{\beta}_{n\nu}\varepsilon_\nu x_\nu = \sum_{\nu=0}^{n} \overline{\beta}_{n\nu}\varepsilon_v \sum_{k=0}^{\nu} \overline{\eta}_{vk}\overline{T}_k x = \sum_{k=0}^{n}\left(\sum_{v=k}^{n} \overline{\beta}_{nv}\varepsilon_v\overline{\eta}_{vk}\right)\overline{T}_k x$$

or

$$\overline{B}_n(\varepsilon x) = \sum_{k=0}^{n} \overline{\gamma}_{nk}\overline{T}_k x,$$

where

$$\overline{\gamma}_{nk} = \sum_{\nu=k}^{n} \overline{\beta}_{n\nu}\varepsilon_\nu\overline{\eta}_{\nu k} = \sum_{\nu=k}^{k+\alpha} \overline{\beta}_{n\nu}\varepsilon_\nu\overline{\eta}_{\nu k}$$

for $k \geq n$ because again $\overline{\eta}_{nk} = \theta_A$ if $n > k + \alpha$. The matrix $(\overline{\gamma}_{nk})$ transforms $(\overline{T}_n x) \in l(X)$ into $(\overline{B}_n(\varepsilon x)) \in l(X)$ if and only if

$$\sum_{n=k}^{\infty} \|\overline{\gamma}_{nk} e_X\|_X = O(1) \tag{16}$$

by Proposition 1.

By partial summation we have

$$\overline{\gamma}_{nk} = \overline{\beta}_{n,n+1} \sum_{\nu=k}^{n} \varepsilon_\nu \overline{\eta}_{\nu k} + \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \sum_{l=k}^{\nu} \varepsilon_l \overline{\eta}_{lk} = \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \left(\sum_{l=k}^{k+\alpha} - \sum_{l=\nu+1}^{k+\alpha} \right) \varepsilon_l \overline{\eta}_{lk}.$$

Since $\overline{\beta}_{n,n+1} = 0$, then

$$\overline{\gamma}_{nk} = \left(\sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \right) K\varepsilon_k - \overline{\lambda}_{nk} = \overline{\beta}_{nk} K \varepsilon_k - \overline{\lambda}_{nk}$$

by (5), where

$$\overline{\lambda}_{nk} = \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk}.$$

We show first that

$$\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\| = O(1). \tag{17}$$

Indeed,

$$\begin{aligned}
\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\|_X &= \sum_{n=k}^{\infty} \left\| \sum_{\nu=k}^{n} (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X \leq \\
&\leq \sum_{n=k}^{\infty} \sum_{\nu=k}^{n} \left\| (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \right\|_A \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X = \\
&= \sum_{\nu=k}^{\infty} \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X \sum_{n=\nu}^{\infty} \left\| (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \right\|_A = \\
&= O(1) \sum_{\nu=k}^{\infty} \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X
\end{aligned}$$

by the assumption (10). Since the method B_A defined by (β_{nk}) satisfies the condition (11), there is a number $M > 1$ (which does not depend on k) such that $\|\beta_{kk}\beta_{k+1k+1}^{-1}\|_A \le M$ for each $k \in \mathbb{N}$. Therefore

$$\sum_{\nu=k}^{l} \|\beta_{\nu\nu}\beta_{ll}^{-1}\|_A \le \sum_{r=0}^{l-k} M^r \le (l-k+1)M^{l-k} \tag{18}$$

for each $k \le l \le k + a$. Taking this into account, we have

$$\begin{aligned}\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\|_X &= O(1) \sum_{\nu=k}^{\infty} \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \beta_{ll}^{-1} (\beta_{ll}\varepsilon_l \tau_{ll}^{-1})(\tau_{ll}\overline{\eta}_{lk}) e_X \right\|_X = \\ &= O(1) \sum_{\nu=k}^{\infty} \sum_{l=\nu}^{k+\alpha} \|\beta_{\nu\nu}\beta_{ll}^{-1}\|_A \, \|\beta_{ll}\varepsilon_l \tau_{ll}^{-1} e_X\|_X \, \|\tau_{ll}\overline{\eta}_{lk} e_X\|_X = \\ &= O(1) \sum_{l=k}^{k+\alpha} \|\beta_{ll}\varepsilon_l \tau_{ll}^{-1} e_X\|_X \, \|\tau_{ll} \sum_{s=k}^{l} \eta_{ls} e_X\|_X \sum_{\nu=k}^{l} \|\beta_{\nu\nu}\beta_{ll}^{-1}\|_A = \\ &= O(1) \sum_{l=k}^{k+\alpha} \|\beta_{ll}\varepsilon_l \tau_{ll}^{-1} e_X\|_X \sum_{s=k}^{l} \|\tau_{ll}\eta_{ls} e_X\|_X\end{aligned}$$

by (4) and (18). Now, by the condition (7), we have

$$\begin{aligned}\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\|_X &= O(1) \sum_{s=k}^{k+\alpha} \sum_{l=s}^{k+\alpha} \|\tau_{ll}\eta_{ls}\|_A = O(1) \sum_{s=k}^{k+\alpha} \sum_{l=0}^{k+\alpha-s} \|\tau_{l+s,l+s}\eta_{l+s,s}\|_A = \\ &= O(1) \sum_{s=k}^{k+\alpha} \sum_{l=0}^{k+\alpha-s} D_l = O(1) \sum_{l=0}^{\alpha} (\alpha+1-l) D_l = O(1)\end{aligned}$$

because $D_0, D_1, \ldots, D_{\alpha-1}$ and D_α are finite.

Next we show that

$$\sum_{n=k}^{\infty} \|\overline{\beta}_{nk}(K\varepsilon_k) e_X\|_X = O(1). \tag{19}$$

Since B_A satisfies the condition (12), then by the condition (8) we have

$$\sum_{n=k}^{\infty} \|\overline{\beta}_{nk}(K\varepsilon_k) e_X\|_X \le \sum_{n=k}^{\infty} \|\overline{\beta}_{nk}\|_A \|(K\varepsilon_k) e_X\|_X = O(1) \sum_{n=k}^{\infty} \|\overline{\beta}_{nk}\|_A = O(1).$$

Hence,

$$\sum_{n=k}^{\infty}\|\overline{\gamma}_{nk}e_X\|_X \leq \sum_{n=k}^{\infty}\|\beta_{nk}(K\varepsilon_k)e_X\|_X + \sum_{n=k}^{\infty}\|\overline{\lambda}_{nk}e_X\|_X = O(1)$$

by (16) and (17). Consequently, ε_n are $(|T_A^\alpha|, |B_A|)$-factors.

Theorem 1. *Let α be a nonnegative integer, A a unital Banach algebra, X a left Banach A-algebra with unit element e_X, $|T_A^\alpha|$ a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$ over A, and B_A a method of summability defined by a matrix (β_{nk}) over A. If (β_{nk}) satisfies the conditions* (6), (10), (11) *and* (12) *and D_n is finite for each $n \in \{0, 1, \ldots, \alpha\}$, then elements ε_n of A are $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors of summability for X if and only if the conditions* (7) *and* (8) *hold.*

Proof. Since $B_A \supset |B_A|$ (that is, every $|B_A|$-summable series is also B_A-summable), then conditions, necessary for $(|T_A^\alpha|, B_A)$-factors of summability for X, are necessary for $(|T_A^\alpha|, |B_A|)$-factors of summability for X also and conditions, sufficient for $(|T_A^\alpha|, |B_A|)$-factors of summability for X, are sufficient for $(|T_A^\alpha|, B_A)$-factors of summability for X also. Therefore, Theorem 1 holds by Proposition 3.

In particular case, when $A = \mathbb{R}$ or $A = \mathbb{C}$, Theorem 1 has been proved in [4], Theorem 3, and when T_A is the Riesz weighted means summability method over Banach algebra A, the Theorem 1 is proved in [3], Theorems 3 and 5.

Corollary 1. *Let α be a nonnegative integer, A a unital Banach algebra, $|T_A^\alpha|$ a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$ over A, and B_A a method of summability defined by a matrix (β_{nk}) over A. If D_n is finite for each $n \in \{0, 1, \ldots, \alpha\}$ and (β_{nk}) is normal and satisfies the conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors for A if and only if*

$$\|\beta_{nn}\varepsilon_n\tau_{nn}^{-1}\|_A = O(1)$$

and

$$\|K\varepsilon_n\|_A = O(1).$$

3. Applications to the Riesz weighted means summability method over Banach algebras

1. Let A be a Banach algebra with unit element e_A and (p_n) be such a sequence of elements of A for which

$$P_n = p_0 + \ldots + p_n$$

is invertible in A for each $n \in \mathbb{N}_0$. The *Riesz weighted means summability method* P_A (which transforms a series to sequence) is defined by the matrix (τ_{nk}), where

$$\tau_{nk} = \begin{cases} e_A - P_n^{-1}P_{k-1} & \text{if } k \le n, \\ \\ \theta_A & \text{if } k > n, \end{cases}$$

and the Riesz weighted means summability method $|P_A|$ (which transforms a series to a series) is defined by the matrix $(\overline{\tau}_{nk})$, where

$$\overline{\tau}_{nk} = P_{n-1}^{-1} p_n P_n^{-1} P_{k-1}$$

for each $k, n \in \mathbb{N}_0$ with $k \le n$ (see [3], p. 147–148). Moreover, if all elements p_n are also invertible in A, then we can speak about the inverse matrix $(\overline{\eta}_{nk})$ of $(\overline{\tau}_{nk})$, where

$$\overline{\eta}_{nk} = \begin{cases} p_n^{-1}P_n & \text{if } k = n, \\ \\ -p_{n-1}^{-1}P_{n-2} & \text{if } k = n-1, \\ \\ \theta_A & \text{if } k < n-1 \text{ or } k > n \end{cases}$$

for each $k, n \in \mathbb{N}_0$. Taking this and the equality $\eta_{nk} = \overline{\eta}_{nk} - \overline{\eta}_{n,k+1}$ into account, we see that elements of the inverse matrix (η_{nk}) of (τ_{nk}) have the form

$$\eta_{nk} = \begin{cases} p_n^{-1}P_n & \text{if } k = n, \\ \\ -(p_{n-1}^{-1} + p_n^{-1})P_{n-1} & \text{if } k = n-1, \\ \\ -p_{n-1}^{-1}P_{n-2} & \text{if } k = n-2, \\ \\ \theta_A & \text{if } k < n-2 \text{ or } k > n \end{cases}$$

for each $n, k \in \mathbb{N}_0$. Therefore, in the present case

$$D_0 = \sup_k \|\tau_{kk}\eta_{kk}\|_A = \sup_k \|(e_A - P_k^{-1}P_{k-1})p_k^{-1}P_k\|_A =$$
$$= \sup_k \|P_k^{-1}(P_k - P_{k-1})p_k^{-1}P_k\|_A = \sup_k \|P_k^{-1}p_k p_k^{-1}P_k\|_A = 1,$$

$$D_1 = \sup_k \|\tau_{k+1,k+1}\eta_{k+1,k}\|_A = \sup_k \|(e_A - P_{k+1}^{-1}P_k)[-(p_k^{-1} + p_{k+1}^{-1})P_k]\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}(P_{k+1} - P_k)[(p_k^{-1} + p_{k+1}^{-1})P_k]\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}p_{k+1}(p_k^{-1} + p_{k+1}^{-1})P_k)\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}p_{k+1}p_k^{-1}P_k + P_{k+1}^{-1}P_k)\|_A \leq$$
$$\leq \sup_k \|P_{k+1}^{-1}p_{k+1}p_k^{-1}P_k\|_A + 1 + \sup_k \|P_{k+1}^{-1}p_{k+1}\|_A,$$

$$D_2 = \sup_k \|\tau_{k+2,k+2}\eta_{k+2,k}\|_A = \sup_k \|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_k\|_A \leq$$
$$\leq \sup_k \|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}\|_A \left(1 + \sup_k \|P_{k+1}^{-1}p_{k+1}\|_A\right)$$

and if $n \geq 3$ then $D_n = 0$. Hence, if

$$\|P_n^{-1}p_n\|_A = O(1) \tag{20}$$

and

$$\|P_{n+1}^{-1}p_{n+1}p_n^{-1}P_n\|_A = O(1), \tag{21}$$

then D_1 and D_2 are finite. Moreover,

$$K\varepsilon_n = \sum_{\nu=n}^{n+1} \varepsilon_\nu \overline{\eta}_{\nu n} = \varepsilon_n p_n^{-1}P_n - \varepsilon_{n+1}p_n^{-1}P_{n-1} = (\Delta\varepsilon_n)p_n^{-1}P_n + \varepsilon_{n+1}. \tag{22}$$

Taking this into account, we have

Theorem 2. *Let A be a unital Banach algebra, (p_n) a sequence in A such that p_n and P_n are invertible in A for each $n \in \mathbb{N}_0$ and X a left Banach A-algebra with unit element e_X. Let P_A the Riesz weighted means summability method over A. Let B_A be a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21) *and*

$$\sum_{n=k}^{\infty} \|P_{n-1}^{-1}p_n P_n^{-1}P_{k-1}\|_A = O(1) \tag{23}$$

have been satisfied and B_A is normal and satisfies conditions (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for X if and only if hold* (9)*,*

$$\|\beta_{nn}\varepsilon_n P_n^{-1} p_n e_X\|_X = O(1) \tag{24}$$

and

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X = O(1). \tag{25}$$

Remark 1. By Proposition 1 the method B_A (resp. P_A) preserves the absolute convergence if and only if (12) (resp. (23)) is satisfied.

Proof of Theorem 2. If $\varepsilon \in (|P_A|, B_A)$ and $\varepsilon \in (|P_A|, |B_A|)$, then condition (9) holds by Proposition 3 (because $|P_A|$ preserves the absolute convergence by Proposition 1) and

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X + \varepsilon_{n+1} e_X\|_X = O(1) \tag{26}$$

hold by Theorem 1 and the equality (22). Since

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X \le \|(\Delta\varepsilon_n) p_n^{-1} P_n e_X + \varepsilon_{n+1} e_X\|_X + \|\varepsilon_{n+1} e_X\|_X,$$

the condition (25) holds by (9) and (26).

Let now the elements ε_n of A satisfy the conditions (9), (24) and (25). Then the condition (7) of Theorem 1 holds. Since

$$\|(K\varepsilon_n) e_X\|_X \le \|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X + \|\varepsilon_{n+1} e_X\|_X$$

by the equality (22), the condition (8) of Theorem 1 holds by the conditions (9) and (25). Consequently, the elements $\varepsilon \in (|P_A|, B_A)$ and $\varepsilon \in (|P_A|, |B_A|)$ by Theorem 1.

Corollary 2. *Let A be a unital Banach algebra, (p_n) a sequence in A such that p_n and P_n are invertible in A for each $n \in \mathbb{N}_0$. Let P_A be the Riesz weighted means summability method over A and B_A a method of summability, defined by a matrix (β_{nk}) over A. If the conditions* (20), (21) *and* (23) *have been satisfied and (β_{nk}) is a normal matrix which satisfies the conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for A if and only if*

$$\|\varepsilon_n\|_A = O(1), \tag{27}$$

(28) $$\|\beta_{nn}\varepsilon_n P_n^{-1} p_n\|_A = O(1)$$

and

(29) $$\|(\Delta\varepsilon_n) p_n^{-1} P_n\|_A = O(1).$$

2. Let A be a Banach algebra with unit element e_A. Let (p_n) and (q_n) be two sequences of elements of A for which

$$P_n = p_0 + \ldots + p_n \quad \text{and} \quad Q_n = q_0 + \ldots + q_n$$

are invertible in A for each $n \in \mathbb{N}_0$. The method $(QP)_A$ of summability over A (which first transforms the sequence $x = (x_n)$ to the sequence $y = (y_n)$ and then the sequence (y_n) to the sequence (z_n)) we define by the matrix transformations

(30) $$z_n = \sum_{k=0}^{n} t_{nk} x_k,$$

where

$$t_{nk} = \begin{cases} Q_n^{-1} \left(\sum\limits_{i=k}^{n} q_i P_i^{-1} \right) p_k & \text{if } k \leq n, \\ \theta_A & \text{if } k > n. \end{cases}$$

Then (by the formula (8.5) from [5], p. 51) elements of the corresponding matrix (τ_{nk}) of this method of summability (which transforms series to sequence) has the form

$$\tau_{nk} = \begin{cases} Q_n^{-1} \sum\limits_{i=k}^{n} q_i (e_A - P_i^{-1} P_{k-1}) & \text{if } k \leq n, \\ \theta_A & \text{if } k > n, \end{cases}$$

and the summability method $|(QP)_A|$ (which transforms a series to a series) we define by the matrix $(\overline{\tau}_{nk})$, where[2]

$$\overline{\tau}_{nk} = \begin{cases} Q_{n-1}^{-1} q_n Q_n^{-1} Q_{k-1} - \overline{\Delta} \left(Q_n^{-1} \sum\limits_{i=k}^{n} q_i P_i^{-1} P_{k-1} \right) & \text{if } k \leq n, \\ \theta_A & \text{if } k > n, \end{cases}$$

[2] Here and later on $\overline{\Delta} a_{nk} = a_{nk} - a_{n-1,k}$, where $a_{nk} \in A$ for each $k, n \in \mathbb{N}$, and $\overline{\Delta} a_n = a_n - a_{n-1}$, where $a_n \in A$ for each $n \in \mathbb{N}$.

because $\overline{\tau}_{nk} = \overline{\Delta}\tau_{nk}$ for each $n, k \in \mathbb{N}_0$ (see [5], p. 50, formula (8.2)).

Transforming the equation (30), we have

$$\overline{\Delta}(Q_n z_n) = \overline{\Delta}\left[\sum_{k=0}^{n} q_k P_k^{-1}\left(\sum_{i=0}^{k} p_i x_i\right)\right] = q_n P_n^{-1}\left(\sum_{i=0}^{n} p_i x_i\right)$$

and

$$\overline{\Delta}\left(P_n q_n^{-1}\overline{\Delta}(Q_n z_n)\right) = p_n x_n.$$

Therefore

$$\begin{aligned} x_n &= p_n^{-1}\left[P_n q_n^{-1}\overline{\Delta}(Q_n z_n) - P_{n-1} q_{n-1}^{-1}\overline{\Delta}(Q_{n-1} z_{n-1})\right] = \\ &= p_n^{-1}\left[P_n q_n^{-1}(Q_n z_n - Q_{n-1} z_{n-1}) - P_{n-1} q_{n-1}^{-1}(Q_{n-1} z_{n-1} - Q_{n-2} z_{n-2})\right] = \\ &= \left[p_n^{-1} P_n q_n^{-1} Q_n\right] z_n - \left[p_n^{-1} P_n q_n^{-1} Q_{n-1} + p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-1}\right] z_{n-1} + \\ &+ \left[p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-2}\right] z_{n-2}. \end{aligned}$$

Hence, elements of the inverse matrix (ξ_{nk}) of (t_{nk}) we

$$\xi_{nk} = \begin{cases} p_n^{-1} P_n q_n^{-1} Q_n & \text{if } k = n, \\ -\left[p_n^{-1} P_n q_n^{-1} Q_{n-1} + p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-1}\right] & \text{if } k = n-1, \\ p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-2} & \text{if } k = n-2, \\ \theta_A & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

For finding η_{nk}, we calculate

$$\overline{\Delta} x_n =$$

$$\begin{aligned} &= \left[p_n^{-1} P_n q_n^{-1} Q_n\right] z_n - \\ &- \left[p_n^{-1} P_n q_n^{-1} Q_{n-1} + p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-1} + p_{n-1}^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-1}\right] z_{n-1} + \\ &+ \left[p_n^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-2} + p_{n-1}^{-1} P_{n-2} q_{n-2}^{-1} Q_{n-2} + p_{n-1}^{-1} P_{n-1} q_{n-1}^{-1} Q_{n-2}\right] z_{n-2} - \\ &- \left[p_{n-1}^{-1} P_{n-2} q_{n-2}^{-1} Q_{n-3}\right] z_{n-3}. \end{aligned}$$

Consequently, the elements of the inverse matrix (η_{nk}) of (τ_{nk}) are

$$\eta_{nk} = \begin{cases} p_n^{-1}P_nq_n^{-1}Q_n & \text{if } k=n, \\ -\left[p_n^{-1}P_nq_n^{-1}Q_{n-1} + p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1} + \right. \\ \left. + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right] & \text{if } k=n-1, \\ p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2} + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2} + \\ +p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2} & \text{if } k=n-2, \\ -p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-3} & \text{if } k=n-3, \\ \theta_A & \text{if } k<n-3 \text{ or } k>n. \end{cases}$$

Now by (4) we have that

$$\overline{\eta}_{nn} = p_n^{-1}P_nq_n^{-1}Q_n,$$

$$\overline{\eta}_{n,n-1} =$$

$$= \left(p_n^{-1}P_nq_n^{-1}Q_n - p_n^{-1}P_nq_n^{-1}Q_{n-1}\right) -$$
$$- \left(p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1} + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right) =$$
$$= p_n^{-1}P_n - \left(p_n^{-1}P_nq_{n-1}^{-1}Q_{n-1} - q_{n-1}^{-1}Q_{n-1} + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right) =$$
$$= p_n^{-1}P_n\left(e_A - q_{n-1}^{-1}Q_{n-1}\right) + \left(e_A - p_{n-1}^{-1}P_{n-1}\right)q_{n-1}^{-1}Q_{n-1} =$$
$$= -p_n^{-1}P_nq_{n-1}^{-1}Q_{n-2} - p_{n-1}^{-1}P_{n-2}q_{n-1}^{-1}Q_{n-1},$$

$$\overline{\eta}_{n,n-2} =$$

$$= \left[-p_n^{-1}(P_{n-1}+p_n)q_{n-1}^{-1}Q_{n-2}\right] + \left[-p_{n-1}^{-1}P_{n-2}q_{n-1}^{-1}Q_{n-2} - q_{n-1} + \right.$$
$$\left. + p_{n-1}^{-1}(P_{n-2}+p_{N-1})q_{n-1}^{-1}Q_{n-2}\right] + p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2} =$$
$$= -q_{n-1}^{-1}Q_{n-2} - p_{n-1}^{-1}P_{n-2} + q_{n-1}^{-1}Q_{n-2} + p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2} =$$
$$= p_{n-1}^{-1}P_{n-2}(q_{n-2}^{-1}Q_{n-2} - e_A) = p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-3}$$

and $\overline{\eta}_{n,n-2} = \theta_A$ for each $k \leq n-3$. Then $(QP)_A$ is a T_A^2-matrix.

Hence,

$$K\varepsilon_n = \sum_{\nu=n}^{n+2} \varepsilon_\nu \overline{\eta}_{\nu n} =$$

$$= \varepsilon_n p_n^{-1} P_n q_n^{-1} Q_n - \varepsilon_{n+1}[p_{n+1}^{-1} P_n q_n^{-1} Q_n + q_n^{-1}(Q_n - q_n) - p_{n+1}^{-1} P_n +$$
$$+ p_n^{-1} P_n q_n^{-1} Q_n - q_n^{-1} Q_n] + \varepsilon_{n+2}[p_{n+1}^{-1} P_n q_n^{-1} Q_n - p_{n+1}^{-1} P_n] =$$
$$= (\Delta\varepsilon_n) p_n^{-1} P_n q_n^{-1} Q_n - (\Delta\varepsilon_{n+1}) p_{n+1}^{-1} P_n q_n^{-1} Q_n + (\Delta\varepsilon_{n+1})(p_{n+1}^{-1} P_{n+1} - e_A) +$$
$$+ \varepsilon_{n+1} =$$
$$= \Delta(\Delta\varepsilon_n p_n^{-1}) P_n q_n^{-1} Q_n + (\Delta\varepsilon_{n+1}) p_{n+1}^{-1} P_{n+1} + \varepsilon_{n+2}.$$

To find D_1, D_2 and D_3, we calculate

$$\tau_{k+1,k+1}\eta_{k+1,k} =$$

$$= Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} \left[p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_k + p_{k+1}^{-1} P_k q_k^{-1} Q_k + p_k^{-1} P_k q_k^{-1} Q_k\right] =$$
$$= -Q_{k+1}^{-1}(Q_{k+1} - q_{k+1}) - Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1}(P_{k+1} - p_{k+1}) q_k^{-1} Q_k -$$
$$- Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} p_k^{-1} P_k q_k^{-1} Q_k =$$
$$= -e_A + Q_{k+1}^{-1} q_{k+1} + Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k + Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} q_k^{-1} Q_k -$$
$$- Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} p_k^{-1} P_k q_k^{-1} Q_k,$$

$$\tau_{k+2,k+2}\eta_{k+2,k} =$$

$$= Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} \left[p_{k+2}^{-1} P_{k+1} q_{k+1}^{-1} Q_k + p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_k +\right.$$
$$\left. + p_{k+1}^{-1} P_k q_k^{-1} Q_k\right] =$$
$$= Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1}(P_{k+2} - p_{k+2}) q_{k+1}^{-1}(Q_{k+1} - q_{k+1}) +$$
$$+ Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{q+1}^{-1}(Q_{k+1} - q_{k+1}) +$$
$$+ Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1}(P_{k+1} - p_{k+1}) q_k^{-1} Q_k =$$
$$= Q_{k+2}^{-1} q_{k+2} q_{k+1}^{-1} Q_{k+1} - Q_{k+2}^{-1} q_{k+2} - Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} q_{k+1}^{-1} Q_{k+1} +$$
$$+ (Q_{k+2}^{-1} q_{k+2})(P_{k+2}^{-1} p_{k+2}) + Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_{k+1} -$$
$$- (Q_{k+2}^{-1} q_{k+2})(P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1}) +$$
$$+ (Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_{k+1})(Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k) -$$
$$- (Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} q_{k+1}^{-1} Q_{k+1})(Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k)$$

and

$$\tau_{k+3,k+3}\eta_{k+3,k} =$$

$$= -Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} \left[p_{k+2}^{-1}(P_{k+2} - p_{k+2}) q_{k+1}^{-1}(Q_{k+1} - q_{k+1})\right] =$$
$$= -\left(Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} p_{k+2}^{-1} P_{k+2} q_{k+2}^{-1} Q_{k+2}\right)\left(Q_{k+2}^{-1} q_{k+2} q_{k+1}^{-1} Q_{k+1}\right) +$$
$$+ (Q_{k+3}^{-1} q_{k+3})(P_{k+3}^{-1} p_{k+3} p_{k+2}^{-1} P_{k+2}) +$$
$$+ Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} Q_{k+2}^{-1} q_{k+2} q_k^{-1} Q_k q_{k+1}^{-1} Q_{k+1} - (Q_{k+3}^{-1} q_{k+3})(P_{k+3}^{-1} p_{k+3}).$$

Therefore

$$\|\tau_{k+1,k+1}\eta_{k+1,k}\|_A \leq$$

$$\leq 1 + \|Q_{k+1}^{-1}q_{k+1}\|_A + \|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A + \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}p_k^{-1}P_kq_k^{-1}Q_k\|_A,$$

$$\|\tau_{k+2,k+2}\eta_{k+2,k}\|_A \leq$$

$$\leq \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \|Q_{k+2}^{-1}q_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}\|_A + \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1})\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A$$

and

$$\|\tau_{k+3,k+3}\eta_{k+3,k}\|_A \leq$$

$$\leq \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}q_{k+2}^{-1}Q_{k+2}\|_A\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}\|_A\|P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}q_{k+2}^{-1}Q_{k+2}\|_A\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}\|_A\|P_{k+3}^{-1}p_{k+3}\|_A.$$

Hence

$$D_1 \leq$$

$$\leq 1 + \|Q_{k+1}^{-1}q_{k+1}\|_A + \|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A + \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}p_k^{-1}P_kq_k^{-1}Q_k\|_A,$$

$$D_2 \leq$$

$$\leq \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \|Q_{k+2}^{-1}q_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A$$

and

$$\begin{aligned}D_3 \leq &\|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}q_{k+2}^{-1}Q_{k+2}\|_A\,\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A+\\&+\|Q_{k+3}^{-1}q_{k+3}\|_A\,\|P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}\|_A+\\&+\|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}q_{k+2}^{-1}Q_{k+2}\|_A\,\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A+\\&+\|Q_{k+3}^{-1}q_{k+3}\|_A\|P_{k+3}^{-1}p_{k+3}\|_A.\end{aligned}$$

Consequently, D_k is finite for each $k \leq 3$ if the methods P_A and Q_A satisfy the following conditions:

$$\|Q_n^{-1}q_n\| = O(1), \tag{31}$$

$$\|Q_{n+1}^{-1}q_{n+1}q_n^{-1}Q_n\|_A = O(1), \tag{32}$$

$$\|Q_{n+1}^{-1}q_{n+1}P_{n+1}^{-1}p_{n+1}q_n^{-1}Q_n\|_A = O(1) \tag{33}$$

and

$$\|Q_{n+1}^{-1}q_{n+1}P_{n+1}^{-1}p_{n+1}p_n^{-1}P_nq_n^{-1}Q_n\|_A = O(1). \tag{34}$$

In the particular case, when A is a commutative Banach algebra, then conditions (33) and (34) are superfluous, because they hold by conditions (20), (31) and (32). Taking this into account, we have

Theorem 3. *Let A be a unital Banach algebra, (p_n) and (q_n) sequences in A such that p_n, q_n, P_n and Q_n are invertible in A for each $n \in \mathbb{N}_0$, X a left Banach A-algebra with unit element e_X. Let P_A and Q_A two Riesz weighted means summability methods over A such that $|(QP)_A| \supset |P_A|$ and B_A a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21), (23), (31)–(34),

$$\sum_{n=k}^{\infty}\|Q_{n-1}^{-1}q_nQ_n^{-1}Q_{k-1}\|_A = O(1) \tag{35}$$

and

$$\sum_{n=k}^{\infty}\left\|\overline{\Delta}\left(Q_n^{-1}\sum_{i=k}^{n}q_iP_i^{-1}P_{k-1}\right)\right\|_A = O(1) \tag{36}$$

have been satisfied (in case, when A is commutative, then (20), (21), (23), (31), (32), (35) *and* (36)*) and B_A is normal and satisfies conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|(QP)_A|, B_A)$-factors and $(|(QP)_A|, |B_A|)$-factors of summability for X if and only if* (9), (24), (25) *and*

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1} Q_n e_X\|_X = O(1) \tag{37}$$

have been satisfied.

Proof. If $\varepsilon \in (|(QP)_A|, B_A)$ and $\varepsilon \in (|(QP)_A|, |B_A|)$, then conditions (8) and (9) hold by Proposition 3 because $|(QP)_A|$ preserves the absolute convergence by (35) and (36) (see Proposition 1). Since every $|P_A|$-summable series in X is $|Q_A P_A|$-summable also by $|(QP)_A| \supset |P_A|$, the method $|P_A|$ preserves the absolute convergence by the condition (23) (see [3], Corollary 3), then the condition (24) and (25) hold. Moreover,

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1} Q_n e_X\|_X \leq$$

$$\leq \|(K\varepsilon_n)e_X\|_X + \|(\Delta\varepsilon_{n+1})p_{n+1}^{-1}P_{n+1}e_X\| + \|\varepsilon_{n+2}e_X\|.$$

Therefore, the condition (37) holds by conditions (8), (9) and (25).

Let now elements ε_n of A satisfy the conditions (9), (25) and (37). Since

$$\|(K\varepsilon_n)e_X\|_X \leq \|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1} Q_n e_X\|_X + \|(\Delta\varepsilon_{n+1})p_{n+1}^{-1}P_{n+1}e_X\|_X + \\ + \|\varepsilon_{n+2}e_X\|_X,$$

then the condition (8) has been satisfied by (9), (25) and (37). Hence, we have $\varepsilon \in (|(QP)_A|, B_A)$ and $\varepsilon \in (|(QP)_A|, |B_A|)$ by Theorem 1.

Corollary 3. *Let A be a unital Banach algebra, (p_n) and (q_n) sequences in A such that p_n, q_n, P_n and Q_n are invertible in A for each $n \in \mathbb{N}_0$, P_A and Q_A two Riesz weighted means summability methods over A and B_A a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21), (23), (31)–(34), (35) *and* (36) *(in case, when A is commutative, then conditions* (20), (21), (23), (31), (32), (35) *and* (36)*) have been satisfied and B_A is normal and satisfies conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|(QP)_A|, B_A)$-factors and $(|(QP)_A|, |B_A|)$-factors of summability for X if and only if* (27), (28), (29) *and*

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1} Q_n\|_A = O(1)$$

are fulfilled.

Remark 2. In the particular case, when A is the field of real or complex numbers, Corollary 2 (see [3], Corollary 6) and Corollary 3 (see [4], p. 177) are known.

Remark 3. The condition (10) is satisfied, for example, for $B_A = Q_A$, if Q_A conserves the absolute convergence, that is iff (35) is satisfied.

Indeed, $(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1} = -Q_{n-1}^{-1}q_nQ_n^{-1}q_\nu Q_\nu q_\nu^{-1} = -Q_{n-1}^{-1}q_nQ_n^{-1}Q_\nu$, and

$$\sum_{n=\nu}^{\infty}\|(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1}\|_A \leq \|e_A\|_A + \sum_{n=\nu+1}^{\infty}\|Q_{n-1}^{-1}q_nQ_n^{-1}Q_\nu\|_A = O(1).$$

References

[1] **Абель М.,** Фиксированные идеалы в топологических модуль–алгебрах, *Eesti Tead. Akad. Toimetised. Füüs. Mat.,* **34** (1985), 237–247. (*Abel, M.,* Fixed ideals in topological module-algebras, *Eesti Tead. Akad. Toimetised. Füüs. Mat.,* **34** (1985), 237–247. (in Russian))

[2] **Abel, M.,** Topological bimodule-algebras, *Acta Univ. Oulu. Ser. A. Sci. Rerum Natur.,* **408** (3) (2004), 25–42.

[3] **Abel, M. and Baron, S.,** Summability factors for absolute Riesz weighted summability methods over Banach algebras, *J. Anal.,* **5** (1997), 145–156.

[4] **Барон С.,** Теоремы о множителях суммируемости для методов A^α, *Tartu Riikl. Ül. Toimetised,* **253** (1970), 165–178. (*Baron, S.,* Theorems on the summability for A^α-methods, *Tartu Riikl. Ül. Toimetised,* **253** (1970), 165–178. (in Russian))

[5] **Барон С.,** *Введение в теорию суммируемости рядов,* Valgus, Tallinn, 1977. (Baron, S., *Introduction to the theory of summability of series,* Valgus, Tallinn, 1977. (in Russian))

[6] **Bonsal, F.F. and Duncan, J.,** *Complete normed algebras,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **80**, Springer-Verlag, New York-Heidelberg, 1973.

[7] **Dales, H.G.,** *Banach algebras and automatic continuity,* London Mathematical Society Monographs. New Series **24**, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[8] **Knopp, K. and Lorentz, G.G.,** Beiträge zur absoluten Limitierung, *Arch. Math.,* **2** (1949), 10–16.

Mati Abel
Institute of Mathematics
University of Tartu
Liivi 2–614.
50409 Tartu, Estonia
mati.abel@ut.ee

Simson Baron
Department of Mathematics
Bar-Ilan University
Ramat Gan 52900, Israel
baron@macs.biu.ac.il