A CHARACTERIZATION OF
THE IDENTITY FUNCTION
WITH FUNCTIONAL EQUATIONS

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Abstract. It is proved that if \( f, g : \mathbb{N} \rightarrow \mathbb{C} \) are completely multiplicative functions such that
\[
\begin{align*}
  f(p + 1) &= g(p) + 1 \\
  f(p + q^2) &= g(p) + g(q^2)
\end{align*}
\]
hold for all primes \( p \) and \( q \), then either
\[
\begin{align*}
  f(p + 1) &= f(p + q^2) = 0, \\ 
  g(\pi) &= -1
\end{align*}
\]
for all primes \( p, q, \pi \)
or
\[
\begin{align*}
  f(n) &= g(n) = n \quad \text{for all } n \in \mathbb{N}.
\end{align*}
\]

Let \( \mathbb{N} \) and \( \mathcal{P} \) denote the set of all positive integers and the set of all primes, respectively. An arithmetical function \( f : \mathbb{N} \rightarrow \mathbb{C} \) with the condition \( f(1) = 1 \) is said to be multiplicative if \( (n, m) = 1 \) implies
\[
f(nm) = f(n)f(m)
\]
and it is called completely multiplicative if this holds for all pairs of positive integers \( n \) and \( m \). In the following we denote by \( \mathcal{M} \) and \( \mathcal{M}^* \) the set of all integer-valued multiplicative and completely multiplicative functions, respectively.

In 1992, C. Spiro [9] showed that if a function \( f \in \mathcal{M} \) satisfies
\[
f(p + q) = f(p) + f(q) \quad \text{for all } p, q \in \mathcal{P},
\]
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then \( f(n) = n \) for all \( n \in \mathbb{N} \). In [3] the identity function was characterized as the multiplicative function \( f \) for which
\[
f(p + n^2) = f(p) + f(n^2)
\]
holds for all \( p \in \mathcal{P} \) and for all \( n \in \mathbb{N} \).

Recently, J.-C. Schlage-Puchta [8] improved this result by showing that if
\[
f(p + 1) = f(p) + 1 \quad \text{and} \quad f(p + q^2) = f(p) + f(q^2)
\]
are satisfied for some function \( f \in \mathcal{M} \) and for all \( p, q \in \mathcal{P} \), then \( f \) is the identity function.

For other results in this topic we refer to works [1]-[2] and [4]-[7].

We prove the following

\textbf{Theorem 1.} If \( f, g \in \mathcal{M}^* \) satisfy the following equations

\begin{enumerate}
\item \( f(p + 1) = g(p) + 1 \) and \( f(p + q^2) = g(p) + g(q^2) \) for all \( p, q \in \mathcal{P} \),
\item \( f(p + 1) = f(p + q^2) = 0 \) and \( g(\pi) = -1 \) for all \( p, q, \pi \in \mathcal{P} \),
\end{enumerate}

then either

\begin{enumerate}
\item \( f(2) = 0 \) and \( g(2) = -1 \)
\end{enumerate}

or

\begin{enumerate}
\item \( f(2) = g(2) = 2 \).
\end{enumerate}

\textbf{Proof.} First we deduce from (1) that
\[
(f(p) - g(p))(g(p) + 1) = 0 \quad \text{for all} \quad p \in \mathcal{P}.
\]
Indeed, the repeated use of (1) for the case \( p = q \) gives

\[
f(p)[g(p) + 1] = f(p)f(p + 1) = f(p^2 + p) = g(p^2) + g(p) = g(p)^2 + g(p),
\]

which proves (6). Consequently, in the case \( p = 2 \) we have either \( g(2) = -1 \) or \( g(2) \neq -1 \) and \( f(2) = g(2) \).

Case I. \( g(2) = -1 \). Let \( x := f(2) \)

By using (1), we also have

\[
g(3) = f(3 + 1) - 1 = f(4) - 1 = x^2 - 1, \quad g(7) = f(7 + 1) - 1 = f(2)^3 - 1 = x^3 - 1
\]

and

\[
x^4 = f(2)^4 = f(16) = f(3^2 + 7) = g(3)^2 + g(7) = (x^2 - 1)^2 + (x^3 - 1),
\]

which implies that

\[
(7) \quad x^2(x - 2) = 0.
\]

On the other hand, by using (1) and the condition \( g(2) = -1 \), we have

\[
f(3) = f(2 + 1) = g(2) + 1 = 0, \quad g(5) = f(6) - 1 = f(2)f(3) - 1 = -1
\]

and so

\[
x^5 = f(2)^5 = f(32) = f(5^2 + 7) = g(5^2) + g(7) = 1 + (x^3 - 1) = x^3,
\]

which implies that

\[
(8) \quad x^5 = x^3.
\]

It is clear that (7) and (8) imply \( x = 0 \). The proof of (4) is completed.

Case II. \( g(2) \neq -1, \ f(2) = g(2) \).

In this case, let \( x = f(2) = g(2) \neq -1 \).

So, we conclude from (1) and the above relations that

\[
f(3) = f(2 + 1) = g(2) + 1 = x + 1, \quad g(5) = f(6) - 1 = f(2)f(3) - 1 = x(x + 1) - 1
\]
and

\[(x+1)^2 = f(3)^2 = f(9) = f(2^2+5) = g(2)^2 + g(5) = x^2 + x(x+1) - 1.\]

Therefore \((x+1)(x-2) = 0\), and so the condition \(x \neq -1\) gives \(x = 2\).

Thus \((5)\) and Lemma 1 is proved.

**Lemma 2.** Assume that \(f, g \in M^*\) satisfy \((1)\). If

\[f(2) = 0 \text{ and } g(2) = -1,\]

then \((2)\) is true, that is

\[f(p+1) = f(p + q^2) = 0 \text{ and } g(\pi) = -1 \text{ for all } p, q, \pi \in \mathcal{P}.\]

**Proof.** Since \(g(2) = -1\), we shall prove that

\[g(\pi) = -1 \text{ for all primes } \pi \in \mathcal{P}, \pi \neq 2.\]

Let \(\pi \in \mathcal{P}, \pi \neq 2\). We deduce from \((1)\) and the fact \(f(2) = 0\) that

\[0 = f(2)f\left(\frac{\pi + 1}{2}\right) = f(\pi + 1) = g(\pi) + 1,\]

which implies \(g(\pi) = -1\). Therefore, we prove that \(g(\pi) = -1\) for all primes \(\pi \in \mathcal{P}\).

Finally, the last fact with \((1)\) shows that

\[f(p+1) = g(p) + 1 = 0 \text{ and } f(p + q^2) = g(p) + g(q^2) = -1 + (-1)^2 = 0\]

hold for all \(p, q \in \mathcal{P}\). Hence Lemma 2 is proved.

**Lemma 3.** Assume that \(f, g \in M^*\) satisfy \((1)\). If

\[f(2) = g(2) = 2,\]

then

\[f(n) = g(n) = n \text{ for all } n \in \mathbb{N}.\]

**Proof.** It is clear that Lemma 3 will follow if we can prove that following:

If \(T\) is an integer such that \(f(n) = g(n) = n\) for all \(n < T\), then \(f(T) = T\).

Assume first that \(T\) is not a prime number. We may thus write \(T = AB\) with \(1 < A \leq B < T\), in which case we have \(f(T) = f(AB) = f(A)f(B) = AB = T\) and \(g(T) = g(AB) = g(A)g(B) = AB = T\).
Now assume that $T \in \mathcal{P}$. Since $f(2) = g(2) = 2$, we may assume that $T \in \mathcal{P}$ and $T \geq 3$. Then $2 < T$ and $\frac{T+1}{2} < T$. Hence, we get from (1) and our assumptions that

$$T + 1 = 2 \frac{T + 1}{2} = f(2)f\left(\frac{T + 1}{2}\right) = f(T + 1) = g(T) + 1,$$

consequently

$$g(T) = T.$$

Finally, we infer from (6) that

$$0 = (f(T) - g(T))(g(T) + 1) = (f(T) - g(T))(T + 1),$$

which shows that $f(T) = g(T) = T$.

References

[1] Chung P.V., Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, Math. Slovaca, 46 (2-3) (1996), 165-171.
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