A CHARACTERIZATION OF THE IDENTITY FUNCTION WITH FUNCTIONAL EQUATIONS

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Abstract. It is proved that if $f,\ g : \mathbb{N} \to \mathbb{C}$ are completely multiplicative functions such that $f(p+1) = g(p) + 1$ and $f(p+q^2) = g(p) + g(q^2)$ hold for all primes p and q, then either

$$f(p+1) = f(p+q^2) = 0,\ g(\pi) = -1 \text{ for all primes } p,\ q,\ \pi$$

or

$$f(n) = g(n) = n \text{ for all } n \in \mathbb{N}.$$

Let $\mathbb{N}$ and $\mathcal{P}$ denote the set of all positive integers and the set of all primes, respectively. An arithmetical function $f : \mathbb{N} \to \mathbb{C}$ with the condition $f(1) = 1$ is said to be multiplicative if $(n, m) = 1$ implies

$$f(nm) = f(n)f(m)$$

and it is called completely multiplicative if this holds for all pairs of positive integers n and m. In the following we denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all integer-valued multiplicative and completely multiplicative functions, respectively.

In 1992, C. Spiro [9] showed that if a function $f \in \mathcal{M}$ satisfies

$$f(p+q) = f(p) + f(q) \quad \text{for all } p,\ q \in \mathcal{P},$$

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then $f(n) = n$ for all $n \in \mathbb{N}$. In [3] the identity function was characterized as the multiplicative function f for which

$$f(p + n^2) = f(p) + f(n^2) \text{ holds for all } p \in \mathcal{P} \text{ and for all } n \in \mathbb{N}.$$

Recently, J.-C. Schlage-Puchta [8] improved this result by showing that if

$$f(p + 1) = f(p) + 1 \text{ and } f(p + q^2) = f(p) + f(q^2)$$

are satisfied for some function $f \in \mathcal{M}$ and for all $p,\ q \in \mathcal{P}$, then f is the identity function.

For other results in this topic we refer to works [1]-[2] and [4]-[7].

We prove the following

Theorem 1. *If $f, g \in \mathcal{M}^*$ satisfy the following equations*

$$(1)\quad f(p + 1) = g(p) + 1 \quad \textit{and} \quad f(p + q^2) = g(p) + g(q^2) \quad \textit{for all} \quad p,\ q \in \mathcal{P},$$

then either

$$(2)\quad f(p + 1) = f(p + q^2) = 0 \quad \textit{and} \quad g(\pi) = -1 \quad \textit{for all} \quad p,\ q,\ \pi \in \mathcal{P},$$

or

$$(3)\quad f(n) = g(n) = n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Our proof of Theorem 1 will follow from the next Lemma 1 - Lemma 3.

Lemma 1. *Assume that $f,\ g \in \mathcal{M}^*$ satisfy* (1). *Then either*

$$(4)\quad f(2) = 0 \quad \textit{and} \quad g(2) = -1$$

or

$$(5)\quad f(2) = g(2) = 2.$$

Proof. First we deduce from (1) that

$$(6)\quad (f(p) - g(p))(g(p) + 1) = 0 \text{ for all } p \in \mathcal{P}.$$

Indeed, the repeated use of (1) for the case $p = q$ gives

$$f(p)[g(p)+1] = f(p)f(p+1) = f(p^2+p) = g(p^2)+g(p) = g(p)^2+g(p),$$

which proves (6). Consequently, in the case $p = 2$ we have either $g(2) = -1$ or $g(2) \neq -1$ and $f(2) = g(2)$.

Case I. $g(2) = -1$. Let $x := f(2)$

By using (1), we also have

$$g(3) = f(3+1)-1 = f(4)-1 = x^2-1, \quad g(7) = f(7+1)-1 = f(2)^3-1 = x^3-1$$

and

$$x^4 = f(2)^4 = f(16) = f(3^2+7) = g(3)^2+g(7) = (x^2-1)^2+(x^3-1),$$

which implies that

$$x^2(x-2) = 0. \tag{7}$$

On the other hand, by using (1) and the condition $g(2) = -1$, we have

$$f(3) = f(2+1) = g(2)+1 = 0, \quad g(5) = f(6)-1 = f(2)f(3)-1 = -1$$

and so

$$x^5 = f(2)^5 = f(32) = f(5^2+7) = g(5)^2+g(7) = 1+(x^3-1) = x^3,$$

which implies that

$$x^5 = x^3. \tag{8}$$

It is clear that (7) and (8) imply $x = 0$. The proof of (4) is completed.

Case II. $g(2) \neq -1$, $f(2) = g(2)$.

In this case, let

$$x = f(2) = g(2) \neq -1.$$

So, we conclude from (1) and the above relations that

$$f(3) = f(2+1) = g(2)+1 = x+1, \quad g(5) = f(6)-1 = f(2)f(3)-1 = x(x+1)-1$$

and

$$(x+1)^2 = f(3)^2 = f(9) = f(2^2+5) = g(2)^2 + g(5) = x^2 + x(x+1) - 1.$$

Therefore $(x+1)(x-2) = 0$, and so the condition $x \neq -1$ gives $x = 2$.

Thus (5) and Lemma 1 is proved.

Lemma 2. *Assume that* $f,\ g \in \mathcal{M}^*$ *satisfy* (1). *If*

$$f(2) = 0 \quad \textit{and} \quad g(2) = -1,$$

then (2) *is true, that is*

$$f(p+1) = f(p+q^2) = 0 \quad \textit{and} \quad g(\pi) = -1 \quad \textit{for all} \quad p,\ q,\ \pi \in \mathcal{P}.$$

Proof. Since $g(2) = -1$, we shall prove that

$$g(\pi) = -1 \quad \text{for all primes} \quad \pi \in \mathcal{P},\ \pi \neq 2.$$

Let $\pi \in \mathcal{P},\ \pi \neq 2$. We deduce from (1) and the fact $f(2) = 0$ that

$$0 = f(2) f\left(\frac{\pi+1}{2}\right) = f(\pi+1) = g(\pi) + 1,$$

which implies $g(\pi) = -1$. Therefore, we prove that $g(\pi) = -1$ for all primes $\pi \in \mathcal{P}$.

Finally, the last fact with (1) shows that

$$f(p+1) = g(p) + 1 = 0 \ \text{ and } \ f(p+q^2) = g(p) + g(q^2) = -1 + (-1)^2 = 0$$

hold for all $p,\ q \in \mathcal{P}$. Hence Lemma 2 is proved.

Lemma 3. *Assume that* $f,\ g \in \mathcal{M}^*$ *satisfy* (1). *If*

$$f(2) = g(2) = 2,$$

then

$$f(n) = g(n) = n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. It is clear that Lemma 3 will follow if we can prove that following:

If T is an integer such that $f(n) = g(n) = n$ for all $n < T$, then $f(T) = T$.

Assume first that T is not a prime number. We may thus write $T = AB$ with $1 < A \leq B < T$, in which case we have $f(T) = f(AB) = f(A)f(B) = AB = T$ and $g(T) = g(AB) = g(A)g(B) = AB = T$.

Now assume that $T \in \mathcal{P}$. Since $f(2) = g(2) = 2$, we may assume that $T \in \mathcal{P}$ and $T \geq 3$. Then $2 < T$ and $\frac{T+1}{2} < T$. Hence, we get from (1) and our assumptions that

$$T + 1 = 2\frac{T+1}{2} = f(2)f\left(\frac{T+1}{2}\right) = f(T+1) = g(T) + 1,$$

consequently

$$g(T) = T.$$

Finally, we infer from (6) that

$$0 = (f(T) - g(T))(g(T) + 1) = (f(T) - g(T))(T + 1),$$

which shows that $f(T) = g(T) = T$.

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