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SOME REMARKS ON A RESULT OF F. LUCA AND I. SHPARLINSKI

I. Kátai (Budapest, Hungary)

Abstract. Luca and Shparlinski proved that $n^{1/\omega(n)} \pmod 1$ is uniformly distributed. Generalizations are proved here. $\omega(n)$ = number of prime divisors of n.

1. Introduction

Luca and Shparlinski investigated the sequences

$$n^{1/\omega(n)} \pmod 1, \quad n^{1/\Omega(n)} \pmod 1, \quad \left(\prod_{p|n}\right)^{1/\omega(n)} \pmod 1$$

and proved that counted them up to $(n \leq)\, x$, the discrepancy is $\frac{1}{(\log x)^{1+o(1)}}$. Here and in the sequel $\omega(n)$ is the number of prime divisors, $\Omega(n)$ is the number of prime power divisors of n, p runs over the set of primes.

The main idea of the proof of the first sequence is to write the integers n as $n = mp$, where p is the largest prime divisor of n, then for fixed m count the distribution of $m^{\frac{1}{\omega(m)+1}} p^{\frac{1}{\omega(m)+1}} \pmod 1$, where p runs over the set of primes p in the interval $P(m) < p < \frac{x}{m}$, where in general $P(n)$= the largest prime divisor of n.

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The distribution of

$$p^{\frac{1}{\omega(m)+1}} \mod \left(\frac{1}{m^{1/\omega(m)+1}}\right)$$

can be computed by using the short interval version of the theorem of Huxley [2] and Heath-Brown [3], according to

$$\pi\left(x+y\right)-\pi\left(x\right)=\frac{y}{\log x}\left(1+O\left(\frac{\left(\log\log x\right)^{4}}{\log x}\right)\right).$$

Similar assertions with some weaker error term can be proved by other "arithmetical" sequences. These depend on the following obvious Lemma 1 and nontrivial theorems for short interval version of sums of multiplicative functions. Let $\pi_k(x) = \sharp\{n \le x \mid \omega(n) = k\}$.

Lemma 1. *Let $\kappa\left(n\right) \quad (n = 1, 2, \ldots)$ be a sequence of real numbers, $0 < < \kappa\left(n\right) < 1$. Assume that for $X > X_0$, the set of integers $n \in [X, 2X)$ can be classified into disjoint sets $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_t$, $\mathcal{T}$ such that in the notation*

$$S_j\left(x\right)=\sum_{\substack{n\le x\\ n\in\mathcal{I}_j}}1,\quad T\left(x\right)=\sum_{\substack{n\le x\\ n\in\mathcal{T}}}1$$

the following conditions hold:

(1) $\kappa\left(n\right) = \kappa_k$ is fixed for $n \in \mathcal{I}_k$, $\quad k = 1, \ldots, t$.

(2) $T\left(2X\right) \ll X\delta\left(X\right)$, where $\delta\left(x\right)$ is a monotonically decreasing function defined in $1 < x < \infty$ such that $\delta\left(\sqrt{x}\right) \ll \delta\left(x\right)$,

(3)

$$\max_{\frac{\delta(X)}{\kappa_k}x^{1-\kappa_k}\le h\le x}\ \max_{x\in[X,2X]}\left|\frac{S_k\left(x+h\right)-S_k\left(x\right)}{h}-\frac{S_k\left(2X\right)-S_k\left(X\right)}{X}\right|\ll$$

$$\ll\frac{S_k\left(2X\right)}{X}\cdot\delta\left(X\right).$$

Then the discrepancy of the sequence $n^{\kappa(n)} \pmod 1 \quad (n = 1, \ldots, [x])$ is $\mathcal{O}\left(\delta\left(x\right)\right)$.

2. Short interval version of sums of some multiplicative function

By using the so called Hooley-Huxley contour for functions depending on Dirichlet L-function, Ramachandra [4] proved a general theorem:

Let S_1, S_2, S_3 be the sets of L-series, the derivatives, and the logarithms of L-series, respectively. $\log L(s,\chi)$ is defined by analytic continuation from the halfplane $\sigma = \text{Re } s > 1$; for some complex z, we define

$$L(s,\chi)^z = \exp(z \log L(s,\chi)).$$

Let $P_1(s)$ be any finite product (with complex exponents) of functions of S_1. Let $P_2(s)$ be any finite power product (with nonnegative integral exponents) of functions of S_2. Let also $P_3(s)$ denote any finite power product with nonnegative exponents of S_3. Let c_n be a sequence of complex numbers such that $|c_n| \ll n^{\varepsilon}$ for every $\varepsilon > 0$ and

$$\sum \frac{|c_n|}{n^{\sigma}} < \infty \qquad \text{for} \quad \sigma > 1/2.$$

Let $F_0(s) = \sum\limits_n \frac{c_n}{n^s}$. Furthermore, let

$$F_1(s) = P_1(s) P_2(s) P_3(s) F_0(s) = \sum \frac{g_n}{n^s},$$

and

$$E(x) = \sum_{n \le x} g_n.$$

Let $0 < r < 1/2$. We define the contour $C(r)$ by starting from the circle $\{s \| s-1| = r\}$, removing the point $1-r$, and proceeding on the remaining portion of the circle in the clockwise direction. Let $C_0 = C(r)$. Assume that r is so small that $F_1(s)$ has no singularities on the boundary and in interior of it, except, possibly, the place $s = 1$.

Let $N_\chi(\alpha, T)$ be the number of roots ϱ of $L(s,\chi)$ in the region $\text{Re } \varrho \geq$ $\geq \alpha, \quad |\text{Im } \varrho| \leq T$. Assume that

$$N_\chi(\alpha, T) = \mathcal{O}\left(T^{B(1-\alpha)} (\log T)^2\right)$$

holds for all characters occurring in P_1, P_2 and P_3.

Let $\varphi = 1 - 1/B + \varepsilon$.

Remark. According to Huxley's result, φ can be any constant greater than $\frac{7}{12}$.

Theorem of Ramachandra. *Let x be sufficiently large and $1 \le h \le x$. Let*

$$I(x,h) = \frac{1}{2\pi i} \int_0^h \left(\int_{C_0} F_1(s)(v+x)^{s-1}\, ds \right) dv.$$

Then

$$E(x+h) - E(x) = I(x,h) + \mathcal{O}_\varepsilon \left(h \cdot \exp\left(-(\log x)^{1/6} \right) + x^{\varphi} \right).$$

In [5] it was deduced from the above theorem that

$$\sum_{\substack{x \le n \le x+h \\ \omega(n)=k}} 1 = (1+o(1)) \frac{h}{x} \pi_k(x)$$

uniformly for any $k \le \log\log x + c_x \sqrt{\log\log x}$, where $c_x \to \infty$ sufficiently slowly, if $x^{\frac{7}{12}+\varepsilon} \le h \le x$. Sankaranarayanan and Srinivas [8] gave a version of Ramachandra's result in which the function $F_1(s)$ may depend on a parameter.

In our paper [7] we deduced the following theorem: Let $C_1 = C\left(\frac{1}{\log x}\right)$, and let L^-, L^+ be defined as the intervals of straightlines

$$L^- = \left[\left(1 - \frac{1}{r}\right) e^{-i\pi},\ \left(1 - \frac{1}{\log x}\right) e^{-i\pi} \right],$$

$$L^+ = \left[\left(1 - \frac{1}{\log x}\right) e^{i\pi},\ \left(1 - \frac{1}{r}\right) e^{i\pi} \right].$$

Let C^* be the contour going along L^- starting from $\left(1 - \frac{1}{r}\right) e^{-i\pi}$, then on C_1, and finally, on L^+.

Theorem (1 and 2, in [7]). *Assume that $F_1(s)$ satisfies the conditions stated in Ramachandra's theorem. Let $r > 0$ and $\varepsilon > 0$ be constants so that $\frac{7}{12} + \varepsilon < \frac{2}{3} - \frac{2r}{3}$, and that $U(s) := F_1(s)(s-1)^{-z}$ is analytic in the disc $|s-1| \le$ $\le r$. Consequently $U(s) = A_0 + A_1(s-1) + \ldots + A_k(s-1)^k + (s-1)^{k+1} V(s)$ holds for every fixed k, where $V(s)$ is bounded in $|s-1| \le r$.*

If $x^{\frac{7}{12}+\varepsilon} \le h \le x^{\frac{2}{3}-\frac{2r}{3}}$, *then*

$$\frac{E(x+h)-E(x)}{h} = \frac{1}{2\pi i}\int_{C^*} F_1(s)\, x^{s-1} ds + \mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right) =$$

$$= \sum_{l=0}^{k} A_l \frac{\Gamma(l-z)}{(\log x)^{l-z+1}} \cdot \frac{(-1)^{l+1}\sin\pi}{\pi} + \tag{2.1}$$

$$+\,\mathcal{O}\left(\frac{1}{(\log x)^{k+2-\mathrm{Re} z}}\right),$$

whenever Re $z \le k+1$.

3. Short interval versions of the theorems of Sathe and Selberg

Let furthermore

$$\pi_l(x) := \#\{n \le x | \omega(n) = l\}; \quad N_l(x) := \#\{n \le x | \Omega(n) = l\}.$$

Further we shall use the abbreviation: $x_1 = \log x$, $x_2 = \log x_1$, etc. By using our theorem for

$$F_1(s) = \sum \frac{z^{\omega(n)}}{n^s} = \zeta^z(s)\, h(s),$$

$$h(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^z \left(1 - \frac{z}{p^s - 1}\right)$$

($h(s)$ is regular in Re$s > 1/2$ for arbitrary $z \in \mathbb{C}$), and for

$$F_1(s) = \sum \frac{z^{\Omega(m)}}{m^s} = \zeta^z(s)\, h(s),$$

$$h(s) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z$$

($h(s)$ is regular in Re $s > 1/2$, whenever $|z| \le 2-\delta$, $\sigma > 0$).

We deduced the following assertions which will be quoted now as Lemma 2 and 3.

Lemma 2. *Let $c_1 > 0$, $\varepsilon > 0$ be arbitrary positive constants, r be another constant so that $\frac{7}{12} + \varepsilon \leq \frac{2}{3} - \frac{2r}{3}$. Then, for $|z| \leq c_1$, $x^{\frac{7}{12}+\varepsilon} \leq h \leq x^{\frac{2}{3}-\frac{2r}{3}}$ we have*

$$h^{-1} \sum_{x \leq m \leq x+h} z^{\omega(m)} = \varphi(z)(\log x)^{z-1} + \mathcal{O}\left((\log x)^{\mathrm{Re} z - 2}\right),$$

where

$$\varphi(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{z}{p-1}\right).$$

Lemma 3. *Let $|z| \leq 2 - \delta$, $0 < \delta < 1$ be an arbitrary constant, ε, r be as in Lemma 2, $x^{\frac{7}{12}+\varepsilon} \leq h \leq x^{\frac{2}{3}-\frac{2r}{3}}$. Then*

$$h^{-1} \sum_{x \leq m \leq x+h} z^{\Omega(m)} = G(z)(\log x)^{z-1} + \mathcal{O}\left((\log x)^{\mathrm{Re} z - 2}\right),$$

$$G(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

Lemma 4. *Let c be an arbitrary constant, $\varepsilon, r > 0$ be such constants for which $\frac{7}{12}\varepsilon < \frac{2}{3} - \frac{2r}{3}$ (i.e. $r < \frac{3}{4} - \frac{3}{2}\varepsilon$). Then, uniformly as $1 \leq k \leq$ $\leq cx_2$, $x^{\frac{7}{12}+\varepsilon} \leq h \leq x^{\frac{2}{3}-\frac{2r}{3}}$, we have*

$$\pi_k(x+h) - \pi_k(x) = \frac{h}{x_1} \cdot \frac{x_2^{k-1}}{(k-1)!} \left\{ \varphi\left(\frac{k-1}{x_2}\right) + \mathcal{O}\left(\frac{k}{x_2^2}\right) \right\}.$$

It is known (see e.g. in J. Kubilius [9], or in G. Tenenbaum [10]) that under the conditions stated for $|z|$ in Lemma 2 and 3,

$$\frac{1}{x} \sum_{m \leq x} z^{\Omega(m)} = G(z)(\log x)^{z-1} + \mathcal{O}\left((\log x)^{\mathrm{Re} z - 2}\right),$$

$$\frac{1}{x} \sum_{m \leq x} z^{\omega(m)} = \varphi(z)(\log x)^{z-1} + \mathcal{O}\left((\log x)^{\mathrm{Re} z - 2}\right),$$

consequently

$$\left| \frac{1}{h} \sum_{x \leq m \leq x+h} z^{\Omega(m)} - \frac{1}{x} \sum_{m \leq x} z^{\Omega(m)} \right| \ll (\log x)^{\mathrm{Re} z - 2},$$

$$\left| \frac{1}{h} \sum_{x \le m \le x+h} z^{\omega(m)} - \frac{1}{x} \sum_{m \le x} z^{\omega(m)} \right| \ll (\log x)^{\mathrm{Re} z - 2}$$

and by integrating with respect to z on the unit circle we obtain that

$$\max_k \left| \frac{\pi_k(x+h) - \pi_k(x)}{h} - \frac{\pi_k(x)}{x} \right| \ll \frac{1}{\log x}, \tag{3.1}$$

$$\max_k \left| \frac{N_k(x+h) - N_k(x)}{h} - \frac{N_k(x)}{x} \right| \ll \frac{1}{\log x}. \tag{3.2}$$

(3.1), (3.2) is proved under the condition $h \le x^{\frac{2}{3} - \frac{2r}{3}}$. By using the formula (17) and (19) in [10] Chapter II.6., page 205 we obtain that

$$\max_{x<y<2x} \left| \frac{\pi_k(x)}{x} - \frac{\pi_k(y)}{y} \right| \ll \frac{1}{\log x} \tag{3.3}$$

uniformly as $k \le Ax_2$, and that

$$\max_{x<y<2x} \left| \frac{N_k(x)}{x} - \frac{N_k(y)}{y} \right| \ll \frac{1}{\log x}, \tag{3.4}$$

uniformly as $k \le (2-\delta) x_2$. Hence we obtain easily

Lemma 5. *(3.1) is true uniformly as* $x^{\frac{7}{12}+\varepsilon} \le h \le x, \quad k \le Ax_2$; *(3.2) is true uniformly as* $x^{\frac{7}{12}+\varepsilon} \le h \le x,\ k \le (2-\delta) x_2$.

4. Main result

Theorem 4. *Let* $Q(n) = a_0 n^m + a_1 n^{m-1} + \ldots + a_m \in R[x]$ *be a polynomial,* $a_0 > 0$.

(1) Assume that for every $X \ge 2$ *for* $n \in [X, 2X]$ $\kappa(n)$ *depends only on* $\omega(n)$ *if* $\omega(n) \le e \log\log X$.

Assume furthermore that

$$\frac{5}{12m} - \varepsilon > \kappa(n) > \frac{1}{(\log X)^{1-\delta}} \qquad \textit{for} \quad n \in [X, 2X] \tag{4.1}$$

satisfying $\omega(n) \le e\log\log X$. *Then the discrepancy of the sequence* $Q(n)^{\kappa(n)} \pmod 1$ $(n \le x)$ *is* $\mathcal{O}\left(\frac{x_2}{x_1}\right)$.

(2) Assume that for every $X \ge 2$ *for* $n \in [X, 2X]$ $\kappa(n)$ *depends only on* $\Omega(n)$ *if* $\Omega(n) \le \beta\log\log X$, *where* $1 < \beta < 2$. *Assume furthermore that (4.1) holds if* $\Omega(n) \le \beta\log\log X$. *Then the discrepancy of* $Q(n)^{\kappa(n)}$ $\pmod 1$ $(n \le x)$ *is* $\mathcal{O}\left(\frac{1}{x_1} x_1^{\beta\log\frac{e}{\beta}}\right)$.

Remark. The proof is a straightforward consequence of (3.1),(3.2) since the sums

$$\begin{aligned}&\#\left\{n|n\in\mathcal{P}_k,\quad A<Q^{\kappa(n)}(n)<A+\Delta\right\}=\\ =&\#\left\{n|n\in\mathcal{P}_k,\quad Q^{-1}\left(A^{\frac{1}{\kappa(n)}}\right)<n<Q^{-1}\left((A+\Delta)^{\frac{1}{\kappa(n)}}\right)\right\}\end{aligned} \tag{4.2}$$

and similarly

$$\begin{aligned}&\#\left\{n|n\in\mathcal{N}_k,\ A<Q^{\kappa(n)}(n)<A+\Delta\right\}=\\ =&\#\left\{n|n\in\mathcal{N}_k,\quad Q^{-1}\left(A^{\frac{1}{\kappa(n)}}\right)<n<Q^{-1}\left((A+\Delta)^{\frac{1}{\kappa(n)}}\right)\right\}\end{aligned} \tag{4.3}$$

can be estimated by error $\mathcal{O}\left(\frac{1}{\log x}\right)$, uniformly as $\frac{1}{\log x} \le \Delta \le 1$, uniformly as $k \le ex_2$ in the case (4.2), and as $k \le \beta x_2$ in the case (4.3).

Furthermore, we observe that

$$\#\{n \le x|\omega(n) > ex_2\} \ll \frac{x}{x_1},$$

$$\#\{n \le x|\Omega(n) > \beta x_2\} \ll \frac{x}{x_1} x_1^{\beta\log\frac{e}{\beta}}.$$

The theorem follows from these observations easily. We omit the details.

5. Further remark

Let $\tau(n)$ be the number of divisors of n. Let K run over the set square-full numbers, $\mathcal{M}_K$ be the set of square-free integers m which are coprime to K. Every integer $n \in \mathbb{N}$ can be uniquely written as $n = Km$, where K is the square-full part and m is the square-free part of n.

Let $\mathcal{M}_{K,l}$ be the set of those integers n the square-full part of which is K, the square-free part is m, and $\omega(m) = l$. Let

$$M_{K,l}(x) = \#\{n \le x, n \in \mathcal{M}_{K,l}\}.$$

By using the method applied in [11] one can prove that

$$\left| \frac{M_{K,l}(x+h) - M_{K,l}(x)}{h} - \frac{M_{K,l}(x)}{x} \right| \ll \frac{1}{K(\log x)^{1/2}}$$

uniformly as

$$x^{\frac{7}{12}+\varepsilon} \le h \le x^{0,66}, \quad K \le x_1^4, \quad |x_2 - l| \le x_2^{3/4}.$$

Hence it follows

Theorem 2. *Let Q be as in Theorem (1). Then the sequence $Q(n)^{1/\tau(n)}$ is uniformly distributed* (mod 1).

We omit the proof.

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I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu