REMARKS ON THE NUMBER SYSTEMS OF THE GAUSSIAN INTEGERS

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Abstract. In the first section I prove which $\alpha = A + i \quad (A \in \mathbb{Z})$ form a number system with the symmetric digit set in a new way. With the modification of the canonical coefficient set we can get new number systems, specially some with holes. In the second section I show that $(-A+ +i, -A, \{0, 1, \ldots, A(A^2 + 1) - 1\})$ is a simultaneous number system if $A > 1$.

1. Remarks on modified canonical number systems

Let $\alpha = A + Bi$, $t = N(\alpha)$, $\mathcal{A} =$ complete remainder set mod α. We say that $(\alpha, \mathcal{A})$ is a number system (in $\mathbb{Z}[i]$), if every $\beta \in \mathbb{Z}[i]$ can be written in finite form $\beta = b_0 + b_1\alpha + \ldots + b_k\alpha^k$, where $b_j \in \mathcal{A}$ $(j = 0, 1, \ldots, k)$. We say that α is the base, and $\mathcal{A}$ is the coefficient set. The number system $(\alpha, \mathcal{A})$ with $\mathcal{A} = \{0, 1, \ldots, t-1\}$ is called canonical. I. Kátai and J. Szabó [1] determined all possible bases of canonical number systems. Namely they proved that α is the base of a canonical number system if and only if $Re(\alpha) < 0$ and $Im(\alpha) = \pm 1$.

We are interested in the following question: let $\mathcal{A} = \{a_1, a_2 = a_1 + +1, \ldots, a_t = a_1 + (t-1)\}$, $a_1 \in \mathbb{Z}$. Determine all those bases α for which $(\alpha, \mathcal{A})$ is a number system.

It is clear that if α is such a base, then:

(1) $-t < a_1 \leq 0$,

(2) $B \neq 0$,

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(3) $|B| = 1$,

(4) if α is such a base, then $\overline{\alpha}$ is such a base as well.

I. Kátai determined in [2] all those $\alpha = A+i$, for which $(\alpha, \mathcal{A}_s)$ is a number system, where $\mathcal{A}_s = \left\{ -\left[\frac{A^2}{2}\right], \dots, \left[\frac{A^2+1}{2}\right] \right\}$.

I give a new proof for this assertion.

Theorem 1. *If $\alpha = A + i,\ A > 4$, then $(\alpha, \mathcal{A}_s)$ is a number system.*

Proof. Let us consider an arbitrary Gaussian integer $\beta = a + bi$. Whilst $\beta = a + bi = a - Ab + b(A + i)$ thus we can always write β in the form

$$\beta = a_0 + a_1(A + i) + a_2(A + i)^2 + \ldots + a_n(A + i)^n, \quad a_j \in \mathbb{Z}.$$

Consider $|a_0| + \ldots + |a_n|$ for all finite representations of β, and choose one for which $|a_0| + \ldots + |a_n|$ is minimal. Let $\tau(\beta) = |a_0| + |a_1| + \ldots + |a_n|$. We will use the following equations:

$$A^2 + 1 = 0 + 2A \cdot (A + i) - 1 \cdot (A + i)^2,$$
$$-(A^2 + 1) = 0 - 2A \cdot (A + i) + 1 \cdot (A + i)^2.$$

Consider the representation $\beta = a_0 + \ldots + a_n\alpha^n$, for which the sum is minimal. Let $a_0 = q_0 t + r_0,\ r_0 \in \mathcal{A}_s$. We can write $\beta = r_0 + \alpha\beta_1$, where

(1.1) $\beta_1 = a_1 + a_2\alpha + \ldots + a_{n-1}\alpha^{n-1}$, in the case $q_0 = 0$,

(1.2) $\beta_1 = (a_1 + q_0 \cdot 2A) + (a_2 - q_0)\alpha + a_3\alpha^2 + \ldots + a_{n-1}\alpha^{n-1}$, in the case $q_0 > 0$,

(1.3) $\beta_1 = (a_1 - |q_0| \cdot 2A) + (a_2 + |q_0|)\alpha + a_3\alpha^2 + \ldots + a_{n-1}\alpha^{n-1}$, in the case $q_0 < 0$.

Lemma 1. *Let $A > 4$. We have $\tau(\beta_1) \le \tau(\beta)$, and if $\tau(\beta_1) = \tau(\beta)$, then $a_0 = 0$, and so α divides β.*

Proof. $\tau(\beta_1)$ is smaller or equal to the sum of absolute values of the coefficients of the expansions of (1.1), (1.2), (1.3), respectively.

$$\tau(\beta) = |a_0| + |a_1| + |a_2| + \sum_{j>2} |a_j|.$$

A) If $q_0 = 0$, then $\tau(\beta_1) \le |a_1| + |a_2| + \sum_{j>2} |a_j|$, thus $\tau(\beta) \ge \tau(\beta_1)$ and equation holds if and only if $a_0 = 0$.

B) If $q_0 > 0$, then $\tau(\beta_1) \le |a_1| + 2Aq_0 + |a_2| + q_0 + \sum_{j>2} |a_j|$. Let us solve the inequality:

$$|a_0| + |a_1| + |a_2| \ge |a_1| + 2Aq_0 + |a_2| + q_0,$$
$$r_0 + (A^2+1) \cdot q_0 \ge (2A+1) \cdot q_0.$$

Using $r_0 + (A^2+1) \cdot q_0 \ge (A^2+1) \cdot q_0 - \left\lfloor \frac{A^2}{2} \right\rfloor$ we get:

$$(A^2+1) \cdot q_0 - \left\lfloor \frac{A^2}{2} \right\rfloor \ge (2A+1) \cdot q_0,$$
$$(A^2-2A) \cdot q_0 \ge \left\lfloor \frac{A^2}{2} \right\rfloor.$$

Thus $\tau(\beta_1) < \tau(\beta)$ holds in this case, if $A > 4$.

C) If $q_0 < 0$, then $\tau(\beta_1) \le |a_1| + 2Aq_0 + |a_2| + |q_0| + \sum_{j>2} |a_j|$. Let us solve the inequality:

$$|a_0| + |a_1| + |a_2| \ge |a_1| + 2A|q_0| + |a_2| + |q_0|,$$
$$|r_0 - (A^2+1) \cdot |q_0|| \ge (2A+1) \cdot |q_0|.$$

Using $|r_0 - (A^2+1) \cdot |q_0|| \ge (A^2+1) \cdot |q_0| - \left\lfloor \frac{A^2+1}{2} \right\rfloor$ we get

$$(A^2+1) \cdot |q_0| - \left\lfloor \frac{A^2+1}{2} \right\rfloor \ge (2A+1) \cdot |q_0|,$$
$$(A^2-2A) \cdot |q_0| \ge \left\lfloor \frac{A^2+1}{2} \right\rfloor.$$

Thus $\tau(\beta_1) < \tau(\beta)$.

Proof of Theorem 1. This is an immediate consequence of Lemma 1. Indeed: if $\beta = 0$, then it can be represented in $(\alpha, \mathcal{A}_s)$. Assume that $\beta \ne 0$. Let us consider the sequence $\tau(\beta), \tau(\beta_1), \tau(\beta_2) \dots$. If there is a ν for which $\tau(\beta_\nu) = 0$, then β can be represented. It remains the case when there is a ν for which $0 \ne \tau(\beta_\nu) = \tau(\beta_{\nu+1}) = \tau(\beta_{\nu+2}) = \dots$. But then $\beta_\nu = \alpha^k \beta_{\nu+k}$ $(k = 1, 2, \dots)$, consequently $\alpha^k | \beta_\nu$ for every k, which implies that $\beta_\nu = 0$. We are ready.

This proof works for the case $A < -4$, too. If $|A| \le 4$ we can determine whether $(\alpha, \mathcal{A}_s)$ is a number system by checking the Gaussian integers with absolute value less than

$$\frac{\left\lfloor \frac{A^2+1}{2} \right\rfloor}{|\alpha| - 1}.$$

In the cases $\alpha = 3+i,\ \alpha = 2+i,\ \alpha = 1+i$ and $\alpha = -2+i$ we can find periodic elements.

Lemma 2. *If $\alpha = A + i$ and $\exists a_j \in \mathcal{A}:\ A^2 - 2A + 2 | a_j$, then $(\alpha, \mathcal{A})$ is not a number system.*

Proof.

$$a_j + a_j\alpha + a_j\alpha^2 + \ldots = \frac{a_j}{1-\alpha},$$

$$\frac{a_j}{1-\alpha} = \frac{a_j}{1-(A+i)} = \frac{a_j}{(1-A)-i} \cdot \frac{(1-A)+i}{(1-A)+i} = \frac{a_j(1-A+i)}{A^2-2A+2} =$$

$$= \frac{a_j(1-A)}{A^2-2A+2} + \frac{a_j}{A^2-2A+2} i.$$

If $A^2 - 2A + 2 | a_j$, then $\frac{a_j}{1-\alpha} \in \mathbb{Z}[i]$ and its expansion would not be finite, thus $(\alpha, \mathcal{A})$ cannot be a number system.

Remark 1. If $-(A^2 - 2A + 2) < a_1$ and $a_t < A^2 - 2A + 2$, then $\{-2A + +1, -2A + 2, \ldots, 2A - 2, 2A - 1\} \subset \mathcal{A}$.

Theorem 2. *If $\alpha = A + i \quad (A > 3)$, and $\mathcal{A} = \{a_1, a_1 + 1, \ldots, a_1 + t - 1\}$, furthermore, $a_1 > -(A^2 - 2A + 2)$ and $a_1 + t - 1 < A^2 - 2A + 2$, then $(\alpha, \mathcal{A})$ is a number system.*

Proof. Let $z \in \mathbb{Z}[i]$ be an arbitrary Gaussian integer. By Theorem 1 we have the following finite expansion of z :

$$z = \sum_{j=0}^{n} a_j \alpha^j, \quad a_j \in \left\{ -\left\lfloor \frac{A^2}{2} \right\rfloor, \ldots, \left\lfloor \frac{A^2+1}{2} \right\rfloor \right\} = \mathcal{A}_s.$$

Let us examine $a_0, a_1, \ldots$ If $a_j \notin \mathcal{A}$ is realized for some j then let us change a_j, a_{j+1} and a_{j+2} as follows:

if $a_j + t \in \mathcal{A}$, then

$$a_j := a_j + t, \quad a_{j+1} := a_{j+1} - 2A, \quad a_{j+2} := a_{j+2} + 1;$$

if $a_j - t \in \mathcal{A}$, then

$$a_j := a_j - t, \quad a_{j+1} := a_{j+1} + 2A, \quad a_{j+2} := a_{j+2} - 1.$$

Depending on the modification of a_{n-1} and a_n we get 9 different values for (a_{n+1}, a_{n+2}) :

$$(0,0),\ (2A,-1),\ (-2A,1),\ (-1,0),\ (2A-1,-1),$$

$$(-2A-1,1),\ (1,0),\ (2A+1,-1),\ (-2A+1,1),$$

$0,\ 1,\ -1,\ 2A-1$ and $-2A+1$ are in $\mathcal{A}$.

If $2A \notin \mathcal{A}$, then $2A-t \in \mathcal{A}$ and whilst $2A - 1\cdot\alpha = 2A-t+(2A-1)\cdot\alpha - 1\cdot\alpha^2$ we get a finite expansion with appropriate digits. Similarly we get that $(2A+1) - 1\cdot\alpha = (2A+1-t)+(2A-1)\cdot\alpha-1\cdot\alpha^2$, and it is a proper expansion as well. Since $2A-t = -(A^2-2A+1) \in \mathcal{A}$, therefore $-2A-1 \in \mathcal{A}$ and so $-2A \in \mathcal{A}$.

If $2A \in \mathcal{A}$ and $2A+1 \notin \mathcal{A}$, then in the case $A=4$ we get that $\mathcal{A} = \mathcal{A}_s$, so $(\alpha, \mathcal{A})$ is a number system. In the case $A > 4$, since $2A+1-t = -A^2+2A \in \mathcal{A}$ it is also true that $-2A-1, -2A \in \mathcal{A}$, furthermore $(2A+1)-1\cdot\alpha = (2A+1-t)+(2A-1)\cdot\alpha-1\cdot\alpha^2$, so arbitrary Gaussian integer has a finite expansion with proper digits. The proofs are similar in the cases $-2A \notin \mathcal{A}$ respectively $-2A \in \mathcal{A}$ and $-2A-1 \notin \mathcal{A}$.

Theorem 3. *If* $\alpha = -A+i \quad (A>2)$, *and* $\mathcal{A} = \{a_1, a_1+1, \ldots, a_1+t-1\}$, *furthermore* $0 \in \mathcal{A}$, *then* $(\alpha, \mathcal{A})$ *is a number system.*

Proof. The proof is similar to the proof of Theorem 2.

Definition 1. We say that $\delta \in \mathbb{Z}[i]$ is a hole in $(\alpha, \mathcal{A})$ if $\delta \notin \Gamma_k$ and $\delta + \varepsilon \in \Gamma_k\ \forall\varepsilon \in \{\pm1, \pm i\}$ for some $k \in \mathbb{N}$, where $\Gamma_k = \{z \in \mathbb{Z}[i] \mid z = a_0 + a_1\alpha + \ldots + a_k\alpha^k,\ a_j \in \mathcal{A}\}$.

Theorem 4. *Let* $\alpha = -A+i$ *and* $\mathcal{A} = \{0, 1, 2, \ldots, A^2-2, A^2-1, 2A^2+1\}$, *where* $A \in \mathbb{Z},\ A > 2$. *Then* $(\alpha, \mathcal{A})$ *is a number system with hole.*

Proof. First we shall prove that $(\alpha, \mathcal{A})$ is a number system, i.e. that every Gaussian integer has a finite expansion with digits from $\mathcal{A}$. We can reach this by proceeding from the expansion with canonical digit set and making some modifications. These modifications do not affect the digits with smaller index than the modified digit and none of the digits becomes negative. A digit must be modified if it is greater than A^2-1. We need the following Lemmas.

Lemma 3.

$$A^2 = 2A^2+1+2A\alpha+\alpha^2,$$
$$A^2 = 2A^2+1-(A-1)^2\alpha-(2A-1)\alpha^2-\alpha^3.$$

Remark 2. If $A > 2$, then $2A < A^2$, and so $2A \in \mathcal{A}$.

Proof of the Lemma. The statements can be verified with simple calculations.

Lemma 4.

$$A^2+1=0+(A-1)^2\alpha+(2A-1)\alpha^2+\alpha^3,$$
$$A^2+1=0-2A\alpha-\alpha^2.$$

Proof. The statements can be verified with simple calculations.

If a digit is equal to A^2, then we change it to $2A^2+1$ and we change the other digits by Lemma 3. If a digit is greater than A^2, then we decrease it by A^2+1 and we change the other digits by Lemma 4. Applying these modifications the change of a digit affects at the most 3 more digits.

Lemma 5. *Throughout the procedure none of the digits become greater than* $2A^2+1$.

Proof. Each digit is at the most three times increased and the increment is at the most $(A-1)^2$, $2A-1$ and 1, so each digit is increased by at the most $(A-1)^2+2A-1+1=A^2+1$.

Let us consider the canonical expansion of an arbitrary Gaussian integer z :

$$z=a_0+a_1\alpha+a_2\alpha^2+\ldots+a_n\alpha^n,\qquad a_i\in\{0,1,\ldots,A^2\}.$$

Let $b_i:=a_i$ for $0\le i\le n$ and perform the following procedure:

First examine b_0. If $b_0<A^2$, then we do not change it. If $b_0=A^2$ and $b_1\ge(A-1)^2$ and $b_2\ge(2A-1)$ and $b_3\ge1$, then we do the following modifications:

$$b_0:=2A^2+1,\qquad b_1:=b_1-(A-1)^2,\qquad b_2:=b_2-(2A-1),\qquad b_3:=b_3-1.$$

If $b_0=A^2$ is realized but some of the other conditions not, then we do the following modifications:

$$b_0:=2A^2+1,\qquad b_1:=b_1+2A,\qquad b_2:=b_2+1.$$

After this we examine b_1. In this case $b_1>A^2$ is also possible. If $b_1\le A^2$, then we make the above modifications. If $b_1>A^2$ and $b_2\ge2A$ and $b_3\ge1$, then we do the following modifications:

$$b_1:=b_1-(A^2+1),\qquad b_2:=b_2-2A,\qquad b_3:=b_3-1.$$

If $b_1 > A^2$ is realized but some of the other conditions not, then we do the following modifications:

$$b_1 := b_1-(A^2+1), \qquad b_2 := b_2+(A-1)^2, \qquad b_3 := b_3+2A-1, \qquad b_4 := b_4+1.$$

After the n-th step of this procedure when b_n is modified to be in $\mathcal{A}$ we examine b_{n+1}, b_{n+2} and b_{n+3} ($b_i = 0$, if $i > n+3$). There are 3 cases:

1st case: $b_{n+1} < A^2$.

In this case $b_{n+2} < A^2$ and $b_{n+3} < A^2$ hold, because b_{n+2} and b_{n+3} were originally 0, and they were not increased by the modification of b_{n+1}.

2nd case: $b_{n+1} = A^2$.

In this case b_{n+2} is increased by $2A$ and b_{n+3} is increased by 1 with the modification of b_{n+1}, so none of them can increase over $A^2 - 1$.

3rd case: $b_{n+1} = A^2 + 1$.

It holds only if $b_{n-2} > A^2$, $b_{n-1} > A^2$ and $b_n > A^2$ stood. In this case $b_{n+1} = 2A$ and $b_{n+2} = 1$ hold so with the modification of b_{n+1} we get $b_{n+1} = 0$, $b_{n+2} = 0$ and $b_{n+3} = 0$.

The existence of hole follows from

Lemma 6.

$$0 + (A^2 - 2A - 1)(-A + i) + (A - 1)(-A + i)^2 = A^2 - i \cdot (A^2 + 1) + 1,$$

$$A^2 - 1 + (A^2 - 1)(-A + i) + (A)(-A + i)^2 = A^2 - i \cdot (A^2 + 1) - 1,$$

$$A - 1 + (A^2 - 2A)(-A + i) + (A - 1)(-A + i)^2 = A^2 - i \cdot (A^2 + 1) + i,$$

$$A^2 - A + (A^2 - 2)(-A + i) + (A)(-A + i)^2 = A^2 - i \cdot (A^2 + 1) - i,$$

$$2A^2+1+(2A-2)(-A+i)+(A^2-A+2)(-A+i)^2+(2A-1)(-A+i)^3+(-A+i)^4 = \\ = A^2 - i \cdot (A^2 + 1).$$

Proof of the Lemma. The equations can be verified with simple calculations.

Whilst $A^2 - 2A - 1$, $A - 1$, $A^2 - 1$, A, $A^2 - 2A$, $A^2 - A$, $A^2 - 2$, $2A - 2$, $A^2 - A + 2$ and $2A - 1$ are all less than A^2 and greater than 0, if $A > 2$, thus the length of the expansion of $A^2 - i \cdot (A^2 + 1)$ is 5, and the length of the expansion of its neighbours is 3, so $A^2 - i \cdot (A^2 + 1)$ is a hole.

2. A remark on simultaneous number systems

Definition 2. We call the triple $(\alpha, N, \mathcal{A})$ a simultaneous number system if arbitrary Gaussian integer z and rational integer n has a finite expansion in the form:

$$z = a_0 + a_1\alpha + \ldots + a_k\alpha^k,$$

$$n = a_0 + a_1N + \ldots + a_kN^k$$

such that $a_j \in \mathcal{A}$.

Theorem 5. *If $\alpha = -A + i$, $N = -A$, $\mathcal{A} = \{0, 1, \ldots, A(A^2 + 1) - 1\}$, $A \in \mathbb{Z}$ and $A > 2$, then $(\alpha, N, \mathcal{A})$ is a simultaneous number system.*

Statement 1. *For all $z = a + bi \in \mathbb{Z}[i]$ and $n \in \mathbb{Z}$ exist $a_0, a_1, a_2 \in \mathbb{Z}$, such that $z = a_0 + a_1\alpha + a_2\alpha^2$ and $n = a_0 + a_1N + a_2N^2$.*

Proof. Let us solve the following system of equations

$$a + bi = a_0 + a_1\alpha + a_2\alpha^2,$$
$$n = a_0 + a_1N + a_2N^2.$$

The solution is

$$a_0 = n + Ab - A^2a + A^2n,$$

$$a_1 = b - 2Aa + 2An,$$

$$a_2 = -a + n.$$

If a, b and n are integers, then a_0, a_1 and a_2 are integers as well.

Statement 2. *-1 can be written as the sum of the powers of α and N with the same positive coefficients.*

Proof. Let us solve the following system of equations

$$-1 = b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3,$$
$$-1 = b_0 + b_1N + b_2N^2 + b_3N^3.$$

The solution is

$$b_0 = -1 + b_3A + b_3A^3,$$

$$b_1 = b_3 + 3b_3A^2,$$

$$b_2 = 3b_3A,$$

$$b_3 = b_3.$$

With the choice $b_3 = 1$ we get

$$\begin{aligned} b_0 &= A^3 + A - 1, \\ b_1 &= 3A^2 + 1, \\ b_2 &= 3A, \\ b_3 &= 1. \end{aligned}$$

If $A > 1$ then $A^3 + A - 1, 3A^2 + 1$ and $3A$ are positive, so the statement is true.

Corollary 1. For an arbitrary $z = a + bi \in \mathbb{Z}[i]$ and $n \in \mathbb{Z}$ exist $c_0, c_1, \dots, c_k \in \mathbb{N}$ such that $z = \sum_{j=0}^{k} c_j \alpha^j$ and $n = \sum_{j=0}^{k} c_j N^j$.

Statement 3.

$$A(A^2+1) = \\ = 0 + (A^3 - 3A^2 + A - 1)\alpha + (3A^2 - 3A + 1)\alpha^2 + (3A-1)\alpha^3 + \alpha^4,$$

$$A(A^2+1) = \\ = 0 + (A^3 - 3A^2 + A - 1)N + (3A^2 - 3A + 1)N^2 + (3A-1)N^3 + N^4.$$

Proof. The equations can be verified with simple calculations.

Statement 4. $A^3 - 3A^2 + A - 1,\quad 3A^2 - 3A + 1$ *and* $3A - 1$ *are positive if* $A > 2$.

Proof. The statement can be verified with simple calculations.

Proof of Theorem 5. Let z be an arbitrary Gaussian integer and let n be an arbitrary rational integer. Let us choose from the expansions

$$z = \sum_{j=0}^{k} e_j \alpha^j,$$

$$n = \sum_{j=0}^{k} e_j N^j,$$

where $e_j \in \mathbb{N}$ the one for which $\tau(z, n) = \sum_{j=0}^{k} e_j$ is minimal. Dividing e_0 by $A(A^2+1)$ we get

$$e_0 = q_0(A(A^2+1)) + r_0, \quad 0 \le r_0 < A(A^2+1).$$

So

$$z = r_0 + (e_1 + q_0(A^3 - 3A^2 + A - 1))\alpha + (e_2 + q_0(3A^2 - 3A + 1))\alpha^2 +$$
$$+ (e_3 + q_0(3A - 1))\alpha^3 + (e_4 + q_0)\alpha^4 + e_5\alpha^5 + \ldots + e_k\alpha^k,$$
$$n = r_0 + (e_1 + q_0(A^3 - 3A^2 + A - 1))N + (e_2 + q_0(3A^2 - 3A + 1))N^2 +$$
$$+ (e_3 + q_0(3A - 1))N^3 + (e_4 + q_0)N^4 + e_5N^5 + \ldots + e_kN^k.$$

This new expansion has the same weight as the previous one and $r_0 \in \mathcal{A}$.

$$z = r_0 + z_1\alpha,$$
$$n = r_0 + n_1N,$$
$$\tau(z_1, n_1) = \tau(z, n) - r_0.$$

So $\tau(z_1, n_1)$ is less than $\tau(z, n)$ or they are equal and $r_0 = 0$. Continuing this procedure we get that for some $l \in \mathbb{N}$ $\quad \tau(z_l, n_l) = 0$, so we get a finite expansion of (z, n) with digits from $\mathcal{A}$ or for some $m \in \mathbb{N}$ $r_j = 0$ holds if $j \geq m$. In this latter case from $z_m = r_m + z_{m+1}\alpha$ we get that $\alpha | z_m$. From this and $r_{m+1} = 0$ we get that $\alpha^2 | z_m \ldots$ Whilst z_m is divisible by an arbitrary power of α, thus $z_m = 0$, so we got a finite expansion of (z, n) with digits from $\mathcal{A}$.

Theorem 6. $(-2+i, -2, \{0, 1, \ldots, 9\})$ *is a simultaneous number system.*

Proof. We shall prove that no nontrivial periodic element exists. If $(z, n) \neq (0, 0)$ is a periodic element, then the following inequalities hold:

$$|z| \leq \frac{9}{\sqrt{5} - 1},$$
$$|n| \leq \frac{9}{2 - 1}.$$

We can prove by simple computation, that such (z, n) does not exist.

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