SOME REMARKS ON A PAPER WRITTEN BY J.-M. DESHOUILLES AND F. LUCA

I. Kátai (Budapest, Hungary)

1. Let \( \varphi \) be the Euler’s totient function, \( \sigma(n) \) be the sum of divisors function, \( \sigma_a(n) = \sum_{d|n} d^a \). Let \( \mathcal{P} \) be the set of prime numbers, \( p \) a general element of \( \mathcal{P} \). Let \( p(n) \) be the smallest, \( P(n) \) be the largest prime divisor of \( n \), respectively.

Let \( f(n) \ (n \in \mathbb{N}) \) be a positive function,

(1.1) - (1.2) \[ s_N = s_N(f) = \left( \sum_{n=1}^{N} f(n) \right)^{1/2}; \quad a_N = a_N(f) = \frac{1}{N} \sum_{n=1}^{N} f(n), \]

(1.3) - (1.4) \[ g_N = g_N(f) = \left( \prod_{n=1}^{N} g(n) \right)^{1/N}; \quad h_N = h_N(f) = \frac{N}{\sum_{n=1}^{N} \frac{1}{f(n)}}. \]

Deshouillers and Luca proved in [1], that all the sequences \( s_N(\varphi), a_N(\varphi), g_N(\varphi), h_N(\varphi) \) are dense mod 1.

Starting from the known relation

\[ (E_N :=) \sum_{n \leq N} \varphi(n) = \alpha N^2 + \mathcal{O}(N \log N), \quad \alpha = \frac{3}{\pi^2}, \]

they deduced that

(1.5) \[ s_{N+1}(\varphi) - s_N(\varphi) = \frac{\varphi(N + 1)}{2\sqrt{\alpha(N + 1)}} + o_N(1) \quad (N \to \infty), \]

furthermore, that

(1.6) \[ a_{N+1}(\varphi) - a_N(\varphi) = \frac{\varphi(N + 1)}{N + 1} - \alpha + \mathcal{O}\left(\frac{\log N}{N}\right). \]
Let \( \varphi_0(n) := \frac{\varphi(n)}{n} \).

They proved that for every \( k \in \mathbb{N} \) and \( \varepsilon \), satisfying
\[
0 < \varepsilon < \frac{1}{2 \sqrt{\alpha}} \min_{j=1, \ldots, k} \varphi_0(j),
\]
there exist infinitely many integers \( N \) such that
\[
\varphi_0(N + j) \in \left[ \frac{\sqrt{\alpha \varepsilon}}{2}, 3 \sqrt{\alpha \varepsilon} \right],
\]
whence they deduced the assertion for \( s_N(\varphi), a_N(\varphi) \). Here we prove that similar assertion can be proved for several other arithmetical functions.

2. Let \( f(n) > 0 \) \( (n \in \mathbb{N}) \),

\[
E_N = E_N(f) = \sum_{n=1}^{N} f(n) = c N^\lambda + \mathcal{O}(\psi(N)),
\]
where \( \lambda, c \) are positive constants, \( \Psi(N) \to 0 \) monotonically,
\[
\frac{\psi(N)f(N)}{N^{\lambda-1}} \to 0 \quad (N \to \infty), \quad \frac{f(N)}{N^{\lambda-1}} = \mathcal{O}((\log N)^A),
\]
\( A \) is an arbitrary constant.

We have
\[
E_{N+1}^{1/\lambda} - E_N^{1/\lambda} =
\]
\[
= E_N^{1/\lambda} \left\{ 1 + \frac{f(N+1)}{E_N^{1/\lambda}} \right\}^{1/\lambda} - E_N^{1/\lambda} =
\]
\[
= \frac{1}{\lambda} \cdot \frac{f(N+1)}{E_N^{\lambda-1/\lambda}} + \mathcal{O} \left( \frac{f^2(N+1)}{E_N^{2\lambda-1/\lambda}} \right) =
\]
\[
= \frac{1}{\lambda} \cdot \frac{f(N+1)}{(cN^\lambda)^{1-1/\lambda}} + \mathcal{O} \left( \frac{(N+1)^2 \psi(N+1)}{N^{2\lambda-1}} \right) + \mathcal{O} \left( \frac{(\log N)^{2A}}{N} \right) =
\]
\[
= \frac{1}{\lambda} \cdot \frac{f(N+1)}{(cN^\lambda)^{1-1/\lambda}} + o_N(1).
\]

That is
\[
E_{N+1}^{1/\lambda} - E_N^{1/\lambda} =
\]
\[
= \frac{1}{\lambda} \cdot \frac{f(N+1)}{N^{\lambda-1}} + o_N(1) \quad (N \to \infty).
\]
Furthermore, after some computation we obtain

\[
\frac{E_{N+1}}{(N+1)^{\lambda-1}} - \frac{E_N}{N^{\lambda-1}} = \frac{f(N+1)}{(N+1)^{\lambda-1}} - c(\lambda-1) + o_{\lambda}(1) \quad (N \to \infty).
\]

3. Let \( g(p^\alpha) \) be defined on the set of prime powers \( p^\alpha \), such that \( g(p^\alpha) \to 0 \) as \( p^\alpha \to \infty \), and satisfying \( g(p^\alpha) + p^\alpha > 0 \) for every \( p^\alpha \). We distinguish the cases (A), (B), (C), where

(A) \[ \sum_{g(p)<0} g(p) = -\infty, \quad \sum_{g(p)>0} g(p) = \infty; \]

(B) \[ \sum_{g(p)<0} g(p) = -\infty, \quad \sum_{g(p)>0} g(p) < \infty; \]

(C) \[ \sum_{g(p)<0} g(p) > -\infty, \quad \sum_{g(p)>0} g(p) = \infty. \]

Let \( u(n) := \prod_{p^\alpha || n} (1 + g(p) + \ldots + g(p^\alpha)) \).

Let \( k \in \mathbb{N} \), \( \mathcal{E}_{u,k} \) be the set of limit points of the sequence \( (u(n + 1), \ldots, u(n+k)) \).

Theorem 1. Let \( g \) and \( u \) as above. Then, in the case (A): if \( \alpha_j \in [0, \infty], \ j = 1, \ldots, k \), then \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathcal{E}_{u,k} \); in the case (B): \( (\alpha_1 u(1), \alpha_2 u(2), \ldots, \alpha_k u(k)) \in \mathcal{E}_{u,k} \), if \( \alpha_j \in [0,1], \ j = 1, \ldots, k \); in the case (C): \( (\alpha_1 u(1), \ldots, \alpha_k u(k)) \in \mathcal{E}_{u,k} \), if \( \alpha_j \in [1, \infty], \ j = 1, \ldots, k \).

Theorem 1 is an easy consequence of Theorem 2.6 in Halberstam - Richert [2], which we quote now as

Lemma 1. Let \( F_1(n), \ldots, F_k(n) \) be distinct irreducible polynomials with integer coefficients and write

\[ F(n) = F_1(n) \ldots F_k(n). \]

Let \( g(p) \) denote the number of solutions of the congruence

\[ F(n) \equiv 0 \pmod{p}, \]

and assume that \( g(p) < p \) holds for all \( p \in \mathcal{P} \).

Let \( u \) and \( x \) be real numbers such that

\[ u \geq 1 \quad \text{and} \quad x^{1/u} \geq 2. \]
Then
\[(3.1)\]
\[
\#\{n \leq x \mid p(F_i(n)) \geq x^{1/u} \text{ for } i = 1, \ldots, k\} = \\
x \prod_{p < x^{1/u}} \left(1 - \frac{g(p)}{p}\right) \{1 + O_F(\exp[-u(\log u - \log \log 3u - \log k - 2)]) + \\
+ O_F(\exp(-\sqrt{\log x}))\}.
\]

Proof of the Theorem 1. We shall prove the assertion in the case (A) only. The proof in the case (B), and case (C) is similar.

Let \(\alpha_1, \ldots, \alpha_k\) be arbitrary nonnegative numbers, \(\varepsilon > 0\) be an arbitrary constant. Let \(\beta_j = \frac{\alpha_j}{u^j}\) \((j = 1, \ldots, k)\). It is clear that we can find a finite set of primes \(\{p^{(1)}_1, \ldots, p^{(1)}_{s_1}\} = P^{(1)}\), such that \(k < p^{(1)}_1 < \ldots < p^{(1)}_{s_1}\) and
\[
\left| \prod_{j=1}^{s_1} \left(1 + g\left(p^{(1)}_j\right)\right) - \beta_1 \right| < \varepsilon.
\]

Let \(P_1 := p^{(1)}_1 \ldots p^{(1)}_{s_1}\). After then we can find a finite set of primes \(P_2 = \{p^{(2)}_1, \ldots, p^{(2)}_{s_2}\}\) such that \(k < p^{(2)}_1 < \ldots < p^{(2)}_{s_2}\), \(P_1 \cap P_2 = \emptyset\), and
\[
\left| \prod_{j=1}^{s_2} \left(1 + g\left(p^{(2)}_j\right)\right) - \beta_2 \right| < \varepsilon.
\]

Let \(P_2 = p^{(2)}_1 \ldots p^{(2)}_{s_2}\). We can continue this procedure. Consequently we can find mutually coprime square free integers \(P_1, \ldots, P_k\) such that
\[(3.2)\]
\[
|u(P_j) - \beta_j| < \varepsilon, \quad \min p(P_j) > k \quad (j = 1, \ldots, k).
\]

Let \(\delta_x \to 0\) so slowly that
\[(3.3)\]
\[
\frac{1}{\delta_x} \max_{x^{1/u} < p < x} |g(p^k)| \to 0 \quad (x \to \infty).
\]

Let \(Q = P_1 \ldots P_k\). We shall consider such integers \(N\) for which \(N \equiv 0(k!)\), and \(N + l \equiv 0 \pmod{P_l}\) \((l = 1, \ldots, k)\). It holds if \(N = k!^2m,\) and \(k!^2m + l \equiv 0 \pmod{P_l}\) \((l = 1, 2, \ldots, k)\). Let \(0 \leq r < Q\) be such a
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residue for which $k!^2 r + r l + 1 \equiv 0 \pmod{P_l}$ $(l = 1, 2, \ldots, k)$ satisfied. Then $N = Qk!^2 t + r k!^2 (t = 1, 2, \ldots)$. Thus

$$N + l = \left[ \frac{Qk!^2}{l} t + \left( \frac{r k!^2}{l} + 1 \right) \right] l.$$ 

Let

$$F_i(t) := \frac{Qk!^2}{l P_l} t + \left( \frac{r k!^2}{l} + 1 \right) \frac{1}{P_l},$$

$$F(t) = F_1(t) \ldots F_k(t).$$

We shall show that $\varrho(p) < p$ holds for every prime $p$. It is obvious that $\varrho(p) = k$ if $p > k$. Assume that $p \leq k$. Since $F_j(0) \equiv 1 \pmod{p}$ holds for every $j = 1, \ldots, k$, therefore $\varrho(p) < p$.

Let us apply Lemma 1, with $u = 1/\delta x$ and with our polynomial $F$. The error terms on the right hand side of (3.1) are tending to zero as $x \to \infty$.

Furthermore $\varrho(p) = k$, if $p > k$, and so

$$\prod_{p < x^\delta} \left( 1 - \frac{\varrho(p)}{p} \right) = c(k)(1 + o_1(1))[(\log x)\delta x]^k.$$

Thus there exists such an $N$ for which $N + l = l \cdot P_l \cdot m_l$ $(l = 1, 2, \ldots, k)$ and $p(m_l) > x^{\delta_k}$. Consequently $u(N + l) = u(l)u(P_l)u(m_l)$ $(l = 1, 2, \ldots, k)$. From (3.3) we obtain that $u(m_l) = \prod_{p^n|N+l\atop x^\delta < p < x} (1+g(p)+\cdots+g(p^n)) \to 1$ as $x \to \infty$.

Hence our theorem in the case (A) follows.

4. Let $\sigma_a(n) = \sum_{d|n} d^a$. Then $\sigma_{-a}(n) = \frac{\tau_a(n)}{n^a}$. It is known that

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x) \quad (G.L.\text{Dirichlet}),$$

$$\sum_{n \leq x} \sigma^2(n) = \frac{5}{6} \zeta(3)x^3 + O(x^2(\log x)^2) \quad (S.\text{Ramanujan}),$$

$$\sum_{n \leq x} \sigma_a^2(n) = \frac{\zeta(2a+1) \cdot \zeta^2(a+1)}{(2a+1) \zeta(2a+2)} x^{2a+1} + O(C_a(x)), \quad \text{if } 0 < a.$$
Here $C_{a}(x) = O(x^{2a+1/2})$. (Smaller error-term can be found in L. Tóth [4].)

**Theorem 2.** Both of the sequences $E_{N}^{1/\lambda}(f) \pmod{1}$, $\frac{E_{N}(f)}{N^{\lambda-1}} \pmod{1}$ ($N = 1, 2, \ldots$) are everywhere dense, if

1. $f(n) = \sigma(n)$, $\lambda = 2$,
2. $f(n) = \sigma^{2}(n)$, $\lambda = 3$,
3. $f(n) = \sigma_{a}^{2}(n)$, $\lambda = 2a + 1$ ($a > 0$).

The assertion is a direct consequence of Theorem 1 and of (2.1), (2.2).

**References**

[1] Deshouillers J.-M. and Luca F., On the distribution of some means concerning the Euler-function (manuscript)


(Received July 2, 2008)