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SOME REMARKS ON A PAPER WRITTEN BY J.-M. DESHOUILLERS AND F. LUCA

I. Kátai (Budapest, Hungary)

1. Let φ be the Euler's totient function, $\sigma(n)$ be the sum of divisors function, $\sigma_a(n) = \sum_{d|n} d^a$. Let $\mathcal{P}$ be the set of prime numbers, p a general element of $\mathcal{P}$. Let $p(n)$ be the smallest, $P(n)$ be the largest prime divisor of n, respectively.

Let $f(n) \quad (n \in \mathbb{N})$ be a positive function,

$$(1.1)-(1.2) \qquad s_N = s_N(f) = \left\{ \sum_{n=1}^{N} f(n) \right\}^{1/2}; \quad a_N = a_N(f) = \frac{1}{N}\sum_{n=1}^{N} f(n),$$

$$(1.3)-(1.4) \qquad g_N = g_N(f) = \left\{ \prod_{n=1}^{N} g(n) \right\}^{1/N}; \quad h_N = h_N(f) = \frac{N}{\sum_{n=1}^{N} \frac{1}{f(n)}}.$$

Deshouillers and Luca proved in [1], that all the sequences $s_N(\varphi)$, $a_N(\varphi)$, $g_N(\varphi)$, $h_N(\varphi)$ are dense mod 1.

Starting from the known relation

$$(E_N :=) \sum_{n \le N} \varphi(n) = \alpha N^2 + \mathcal{O}(N \log N), \quad \alpha = \frac{3}{\pi^2},$$

they deduced that

$$(1.5) \qquad s_{N+1}(\varphi) - s_N(\varphi) = \frac{\varphi(N+1)}{2\sqrt{\alpha}(N+1)} + o_N(1) \quad (N \to \infty),$$

furthermore, that

$$(1.6) \qquad a_{N+1}(\varphi) - a_N(\varphi) = \frac{\varphi(N+1)}{N+1} - \alpha + \mathcal{O}\left(\frac{\log N}{N}\right).$$

Let $\varphi_0(n) := \frac{\varphi(n)}{n}$.

They proved that for every $k \in \mathbb{N}$ and ε, satisfying

$$0 < \varepsilon < \frac{1}{2\sqrt{\alpha}} \min_{j=1,\dots,k} \varphi_o(j),$$

there exist infinitely many integers N such that

$$\varphi_0(N+j) \in \left[\frac{\sqrt{\alpha}\varepsilon}{2},\ 3\sqrt{\alpha}\varepsilon\right],$$

whence they deduced the assertion for $s_N(\varphi)$, $a_N(\varphi)$. Here we prove that similar assertion can be proved for several other arithmetical functions.

2. Let $f(n) > 0 \quad (n \in \mathbb{N})$,

$$E_N = E_N(f) = \sum_{n=1}^{N} f(n) = cN^\lambda + \mathcal{O}(\psi(N)), \tag{2.1}$$

where λ, c are positive constants, $\Psi(N) \to 0$ monotonically,

$$\frac{\psi(N)f(N)}{N^{\lambda-1}} \to 0 \quad (N \to \infty), \quad \frac{f(N)}{N^{\lambda-1}} = \mathcal{O}((\log N)^A),$$

A is an arbitrary constant.

We have

$$E_{N+1}^{1/\lambda} - E_N^{1/\lambda} =$$

$$=E_N^{1/\lambda}\left\{1 + \frac{f(N+1)}{E_N}\right\}^{1/\lambda} - E_N^{1/\lambda} =$$

$$=\frac{1}{\lambda} \cdot \frac{f(N+1)}{E_N^{1-1/\lambda}} + \mathcal{O}\left(\frac{f^2(N+1)}{E_N^{2-1/\lambda}}\right) =$$

$$=\frac{1}{\lambda} \frac{f(N+1)}{(cN^\lambda)^{1-1/\lambda}} + \mathcal{O}\left(\frac{f(N+1)\psi(N+1)}{N^{2\lambda-1}}\right) + \mathcal{O}\left(\frac{(\log N)^{2A}}{N}\right) =$$

$$=\frac{1}{\lambda} \frac{f(N+1)}{(cN^\lambda)^{1-1/\lambda}} + o_N(1).$$

That is

$$E_{N+1}^{1/\lambda} - E_N^{1/\lambda} = \frac{1}{\lambda \cdot c^{1-\lambda}} \cdot \frac{f(N+1)}{N^{\lambda-1}} + o_N(1) \qquad (N \to \infty). \tag{2.2}$$

Furthermore, after some computation we obtain

$$\frac{E_{N+1}}{(N+1)^{\lambda-1}} - \frac{E_N}{N^{\lambda-1}} = \frac{f(N+1)}{(N+1)^{\lambda-1}} - c(\lambda-1) + o_N(1) \qquad (N\to\infty). \tag{2.3}$$

3. Let $g(p^\alpha)$ be defined on the set of prime powers p^α, such that $g(p^\alpha)\to 0$ as $p^\alpha\to\infty$, and satisfying $g(p^\alpha)+p^\alpha>0$ for every p^α. We distinguish the cases $(A),(B),(C)$, where

$$(A) \qquad \sum_{g(p)<0} g(p) = -\infty, \qquad \sum_{g(p)>0} g(p) = \infty;$$

$$(B) \qquad \sum_{g(p)<0} g(p) = -\infty, \qquad \sum_{g(p)>0} g(p) < \infty;$$

$$(C) \qquad \sum_{g(p)<0} g(p) > -\infty, \qquad \sum_{g(p)>0} g(p) = \infty.$$

Let $u(n) := \prod_{p^\alpha || n} (1+g(p)+\ldots+g(p^\alpha))$.

Let $k\in\mathbb{N}$, $\mathcal{E}_{u,k}$ be the set of limit points of the sequence $(u(n+ +1),\ldots,u(n+k))$.

Theorem 1. *Let g and u as above. Then, in the case (A): if $\alpha_j\in[0,\infty]$, $j=1,\ldots,k$, then $(\alpha_1,\alpha_2,\ldots,\alpha_k)\in\mathcal{E}_{u,k}$; in the case (B): $(\alpha_1 u(1),\alpha_2 u(2),\ldots,\alpha_k u(k))\in\mathcal{E}_{u,k}$, if $\alpha_j\in[0,1]$, $j=1,\ldots,k$; in the case (C): $(\alpha_1 u(1),\ldots,\alpha_k u(k))\in\mathcal{E}_{u,k}$, if $\alpha_j\in[1,\infty]$, $j=1,\ldots,k$.*

Theorem 1 is an easy consequence of Theorem 2.6 in Halberstam - Richert [2], which we quote now as

Lemma 1. *Let $F_1(n),\ldots,F_k(n)$ be distinct irreducible polynomials with integer coefficients and write*

$$F(n)=F_1(n)\ldots F_k(n).$$

Let $\varrho(p)$ denote the number of solutions of the congruence

$$F(n)\equiv 0 \pmod p,$$

and assume that $\varrho(p)<p$ holds for all $p\in\mathcal{P}$.

Let u and x be real numbers such that

$$u\ge 1 \quad\text{and}\quad x^{1/u}\ge 2.$$

Then

(3.1)

$$\#\{n\le x \mid p(F_i(n))\ge x^{1/u} \quad \text{for} \quad i=1,\dots,k\}=$$
$$=x\prod_{p<x^{1/u}}\left(1-\frac{\varrho(p)}{p}\right)\{1+\mathcal{O}_F(\exp[-u(\log u-\log\log 3u-\log k-2)])+$$
$$+\mathcal{O}_F(\exp(-\sqrt{\log x}))\}.$$

Proof of the Theorem 1. We shall prove the assertion in the case (A) only. The proof in the case (B), and case (C) is similar.

Let $\alpha_1,\dots,\alpha_k$ be arbitrary nonnegative numbers, $\varepsilon>0$ be an arbitrary constant. Let $\beta_j=\frac{\alpha_j}{u(j)}\quad (j=1,\dots,k)$. It is clear that we can find a finite set of primes

$$\{p_1^{(1)},\dots,p_{s_1}^{(1)}\}=\mathcal{P}^{(1)}, \quad \text{such that} \quad k<p_1^{(1)}<\dots<p_{s_1}^{(1)} \quad \text{and}$$

$$\left|\prod_{j=1}^{s_1}\left(1+g\left(p_j^{(1)}\right)\right)-\beta_1\right|<\varepsilon.$$

Let $P_1:=p_1^{(1)}\dots p_{s_1}^{(1)}$. After then we can find a finite set of primes $\mathcal{P}_2=$ $=\{p_1^{(2)},\dots,p_{s_2}^{(2)}\}$ such that $k<p_1^{(2)}<\dots<p_{s_2}^{(2)},\quad \mathcal{P}_1\cap\mathcal{P}_2=\emptyset$, and

$$\left|\prod_{j=1}^{s_2}\left(1+g\left(p_j^{(2)}\right)\right)-\beta_j\right|<\varepsilon.$$

Let $P_2=p_1^{(2)}\dots p_{s_2}^{(2)}$. We can continue this procedure. Consequently we can find mutually coprime square free integers $P_1,\dots,P_k$ such that

$$|u(P_j)-\beta_j|<\varepsilon, \quad \min p(P_j)>k \qquad (j=1,\dots,k). \tag{3.2}$$

Let $\delta_x\to 0$ so slowly that

$$\frac{1}{\delta_x}\max_{p>x^{\delta_x}}|g(p^h)|\to 0 \quad (x\to\infty). \tag{3.3}$$

Let $Q=P_1\dots P_k$. We shall consider such integers N for which $N\equiv$ $\equiv 0(k!^2)$, and $N+l\equiv 0 \pmod{P_l}\quad (l=1,\dots,k)$. It holds if $N=k!^2m$, and $k!^2m+l\equiv 0 \pmod{P_l}\quad (l=1,2,\dots,k)$. Let $0\le r<Q$ be such a

residue for which $k!^2 r + l + \equiv 0 \pmod{P_l}$ $(l = 1, 2, \dots, k)$ satisfied. Then $N = Qk!^2 t + rk!^2$ $(t = 1, 2, \dots)$. Thus

$$N + l = \left[\frac{Qk!^2}{l} t + \left(\frac{rk!^2}{l} + 1\right)\right] l.$$

Let

$$F_l(t) := \frac{Qk!^2}{lP_l} t + \left(\frac{rk!^2}{l} + 1\right) \frac{1}{P_l},$$

$$F(t) = F_1(t) \dots F_k(t).$$

We shall show that $\varrho(p) < p$ holds for every prime p. It is obvious that $\varrho(p) = k$ if $p > k$. Assume that $p \le k$. Since $F_j(0) \equiv 1 \pmod{p}$ holds for every $j = 1, \dots, k$, therefore $\varrho(p) < p$.

Let us apply Lemma 1, with $u = 1/\delta_x$ and with our polynomial F. The error terms on the right hand side of (3.1) are tending to zero as $x \to \infty$. Furthermore $\varrho(p) = k$, if $p > k$, and so

$$\prod_{p < x^{\delta_x}} \left(1 - \frac{\varrho(p)}{p}\right) = c(k)(1 + o_x(1))[(\log x)\delta_x]^k.$$

Thus there exists such an N for which $N + l = l \cdot P_l \cdot m_l$ $(l = 1, 2, \dots, k)$ and $p(m_l) > x^{\delta_x}$. Consequently $u(N+l) = u(l)u(P_l)u(m_l)$ $(l = 1, 2, \dots, k)$. From (3.3) we obtain that $u(m_l) = \prod_{\substack{p^a \| N+l \\ x^{\delta_x} < p < x}} (1 + g(p) + \dots + g(p^a)) \to 1$ as $x \to \infty$.

Hence our theorem in the case (A) follows.

4. Let $\sigma_a(n) = \sum_{d|n} d^a$. Then $\sigma_{-a}(n) = \frac{\sigma_a(n)}{n^a}$. It is known that

(4.1) $$\sum_{n \le x} \sigma(n) = \frac{\pi^2}{12} x^2 + \mathcal{O}(x \log x) \qquad (G.L.Dirichlet),$$

(4.2) $$\sum_{n \le x} \sigma^2(n) = \frac{5}{6}\zeta(3) x^3 + \mathcal{O}(x^2 (\log x)^2) \qquad (S.Ramanujan),$$

(4.3) $$\sum_{n \le x} \sigma_a^2(n) = \frac{\zeta(2a+1) \cdot \zeta^2(a+1)}{(2a+1)\zeta(2a+2)} x^{2a+1} + \mathcal{O}(C_a(x)), \quad \text{if} \quad 0 < a.$$

Here $C_a(x) = \mathcal{O}(x^{2a+1/2})$. (Smaller error-term can be found in L. Tóth [4].)

Theorem 2. *Both of the sequences* $E_N^{1/\lambda}(f) \pmod 1$, $\frac{E_N(f)}{N^{\lambda-1}} \pmod 1$ $(N = 1, 2, \dots)$ *are everywhere dense, if*

$$(1) \quad f(n) = \sigma(n), \quad \lambda = 2,$$

$$(2) \quad f(n) = \sigma^2(n), \quad \lambda = 3,$$

$$(3) \quad f(n) = \sigma_a^2(n), \quad \lambda = 2a + 1 \quad (a > 0).$$

The assertion is a direct consequence of Theorem 1 and of (2.1), (2.2).

References

[1] **Deshouillers J.-M. and Luca F.**, On the distribution of some means concerning the Euler-function (manuscript)

[2] **Halberstam H. and Richert H.**, *Sieve methods,* Academic Press, London, 1974.

[3] **Ramanujan S.**, Some formulae in the analytic theory of numbers, *Messenger Math.,* **45** (1916), 81-84.

[4] **Tóth L.**, Generalizations of an asymptotic formula of Ramanujan, *Studia Univ. Babes-Bolyai,* **31** (1986), 9-15.

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I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu