SOMETH REMARKS
ON THE AVERAGE ORDER IN CYCLIC GROUPS

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Dedicated to Dr. János Fehér on his 70th anniversary

Abstract. The mean value of some arithmetical functions on thin sequences is investigated.

1. Notations. \( \mathcal{P} = \) set of primes, \( p \) denotes a general prime number, \( \pi(x) = \# \{ p \leq x \mid p \in \mathcal{P} \} \), \( \pi(x, k) = \# \{ p \leq x \mid p \in \mathcal{P}, \; p \equiv l \pmod{k} \} \). \( \varphi(n) \) = Euler’s totient function. Let \( \tau(n) \) = number of divisors of \( n \). Let

\[
\alpha(n) = \frac{1}{n} \sum_{d \mid n} d \varphi(d),
\]

\[
\beta(n) = \frac{\alpha(n)}{\varphi(n)}, \quad \gamma(n) = \frac{1}{\beta(n)}.
\]

I just read an interesting paper written by J. von zur Gathen, A. Knopfmacher, F. Luca, L. G. Lucht, I. Shparlinski [1]. They estimate the mean value of \( \alpha(n), \; \beta(n), \; \gamma(n) \). First they observe that

\[
\alpha(p^k) = \frac{p^{k+1}}{p+1} + \frac{1}{p^k(p+1)}, \quad \beta(p^k) = 1 + \frac{1}{p^k-1} \left( 1 + \frac{1}{p^k-1} \right)
\]

if \( p^k \) is a prime power, and that \( \alpha \) and \( \beta \) are multiplicative functions. Then they deduce that

\[
\frac{1}{x} \sum_{n \leq x} \alpha(n) - C_\alpha x = O \left( (\log x)^{\frac{3}{2}} (\log \log x)^{\frac{3}{4}} \right),
\]
if \( x \geq 3 \),
\[ C_\alpha = \frac{3\zeta(3)}{\pi^2}, \]
and that
\[
(1.5) \quad \left| \frac{1}{x} \sum_{n \leq x} \beta(n) - C_\beta \right| < \frac{1}{x} \prod_{p \in \mathcal{P}} \left( 1 + \frac{p + 2}{p^3 - p} \right),
\]
if \( x \geq 1 \),
\[ C_\beta = \frac{105\zeta(3)}{\pi^4}, \]
\[
(1.6) \quad \left| \frac{1}{x} \sum_{n \leq x} \gamma(n) - C_\gamma \right| \leq \frac{D}{x} \quad (x \geq 1),
\]
the positive constants \( C_\gamma, D \) are given explicitly.

In the second half of the paper they formulate some open questions, namely the existence of the asymptotic of
\[
\sum_{k \leq x} \frac{\alpha(2^k - 1)}{2^k - 1}, \quad \sum_{k \leq x} \beta(2^k - 1), \quad \sum_{p \leq x} \frac{\alpha(p - 1)}{p - 1}, \quad \sum_{p \leq x} \beta(p - 1).
\]

We shall show that this is true.

2. Let \( f(n) := \frac{\alpha(n)}{n} \). Then \( f \) is multiplicative, \( f(p^k) = \frac{p}{p^k - 1} + \frac{1}{p^k(p^k - 1)} \).
Let \( h \) be defined by \( f(n) = \sum_{d|n} h(d) \). Then
\[ h(p^k) = -\frac{p - 1}{p^{2k}} \quad (k = 1, 2, \ldots), \]
consequently
\[ |h(n)| \leq \frac{1}{n}. \]
We have
\[ \sum_{p \leq x} f(p - 1) = \sum_{d \leq x} h(d)\pi(x, d, -1) = \Sigma_1 + \Sigma_2 + \Sigma_3, \]
where in $\Sigma_1: d \leq (\log x)^A$, in $\Sigma_2: (\log x)^A < d \leq x^{\frac{2}{3}}$, in $\Sigma_3: x^{\frac{2}{3}} < d \leq x$.

From the Siegel-Walfisz theorem (see [1])

$$
\pi(x, k, l) = \frac{1}{\varphi(k)} \text{li} x + O(x e^{-c \sqrt{\log x}})
$$

uniformly as $(l, k) = 1$, $k \leq (\log x)^A$, we deduce that

$$
\Sigma_1 = (\text{li} x) \sum_{d \leq (\log x)^A} \frac{h(d)}{\varphi(d)} + O(x e^{-c \sqrt{\log x}}).
$$

Since $\pi(x, k, l) \leq \frac{ckx}{\varphi(k)}$ if $(l, k) = 1$, $k \leq x^{\frac{2}{3}}$, see [1], we have

$$
\Sigma_2 \ll \sum_{d \geq (\log x)^A} \frac{h(d)}{\varphi(d)} \text{li} x \ll (\text{li} x) \sum_{d \geq (\log x)^A} \frac{1}{d \varphi(d)} \ll \frac{x}{(\log x)^A}.
$$

Finally, since $\pi(x, d, 1) \leq \frac{x}{d}$, therefore $\Sigma_3 = O(\sqrt{x})$, say. We proved:

(2.1) $\frac{1}{\text{li} x} \sum_{p \leq x} \frac{\alpha(p-1)}{p-1} = C + O\left(\frac{1}{(\log x)^A}\right),$  

(2.2) $C = \sum_{d=1}^{\infty} \frac{h(d)}{d \varphi(d)}.$

On the same way we can deduce that

(2.3) $\frac{1}{\text{li} x} \sum_{p \leq x} \frac{\beta(p-1)}{p-1} = E + O\left(\frac{1}{(\log x)^A}\right),$  

where

(2.4) $E = \sum_{d=1}^{\infty} \frac{g(d)}{\varphi(d)},$

and $g$ is defined by $\beta(n) = \sum_{d|n} g(d)$. $g$ is multiplicative,

$$
g(p) = \beta(p) - 1 = \frac{1}{p(p-1)}, \quad g(p^k) = -\frac{1}{p^{2k}-1} \quad \text{if} \quad k \geq 2.
$$
3. Let $P(x)$ be a primitive, squarefree polynomial over \( \mathbb{Z}[x] \). Let $D = \text{discriminant of } P$. Then $D \neq 0$. Let $\rho(d)$ be the number of those residues $m \pmod{d}$, for which $P(m) \equiv 0 \pmod{d}$. Let furthermore $\tau(d)$ be those $m \pmod{d}$, for which $P(m) \equiv 0 \pmod{d}$, and $(m, d) = 1$.

If is known, that $\rho$ and $\tau$ are multiplicative, $\rho(p^\alpha) = \rho(p) \leq k$ if $p \nmid D$, and $\rho(p^\alpha) \leq C_1D^2$, if $p \mid D$ (see [3], and for a sharper estimate by Huxley, [4]). Furthermore, if $p \mid Dc_0$, $(c_0 = P(0)$, then $\rho(p^\alpha) = \tau(p^\alpha)$. Let

$$U_{x}(d) := \# \{ p \leq x \mid P(p) \equiv 0 \pmod{d} \}. $$

By using the Siegel-Walfisz theorem, and sieve estimates, we obtain that

$$U_{x}(d) = \frac{\kappa(d) \text{li } x}{\varphi(d)} + \mathcal{O}\left( \kappa(d)e^{-c\sqrt{\log x}} \right)$$

uniformly as $d \leq (\log x)^A$, and

$$U_{x}(d) \ll \frac{\kappa(d) \text{li } x}{\varphi(d)} \quad \text{if} \quad d \leq x^{\frac{1}{2}}.$$

We have

$$f(P(p)) = \frac{\alpha(P(p))}{P(p)} = \sum_{d \mid P(p)} h(d) = f_1(P(p)) + f_2(P(p)),$$

where

$$f_1(P(p)) = \sum_{d \leq x^{\frac{1}{2}} \mid P(p)} h(d), \quad \text{and} \quad f_2 := f - f_1.$$ 

Since $|h(d)| \leq \frac{1}{d^2}$, therefore $|f_2(P(p))| \leq \frac{\tau(P(p))}{x^{\frac{1}{2}}}$. Since $\tau(P(p)) \ll x^c$, we may assume that $f_2(P(p)) \ll x^{-\frac{1}{2}}$, say.

We have

$$\sum_{p \leq x} f(P(p)) = \sum_{p \leq x} f_1(P(p)) + \mathcal{O}(\sqrt{x}) =$$

$$= \sum_{d \leq x^{\frac{1}{2}}} h(d)U_{x}(d) + \mathcal{O}(\sqrt{x}).$$

From (3.1), (3.2), similarly as in Section 2 we deduce
Theorem 1. Let $P$ be as above. Then
\[
\sum_{p \leq x} \frac{\alpha(P(p))}{P(p)} = A_p \text{li} x + O \left( \frac{x}{(\log x)^2} \right),
\]
where
\[
A_p = \sum_{d=1}^{\infty} \frac{\kappa(d)}{\varphi(d)}.
\]

Similar theorems can be proved for
\[
\sum_{n \leq x} \frac{\alpha(P(n))}{P(n)} \sum_{n \leq x} \beta(P(n)) \sum_{p \leq x} \beta(P(p)).
\]

4. Now we consider
\[
\sum_{k \leq x} f(2^k - 1). \text{ Let } e(d) \text{ be the smallest } k \text{ for which } 2^k - 1 \equiv 0 \pmod{d}. \text{ It is clear that finite } k \text{ exists only if } d \text{ is odd. According to a theorem of Romanov [5], and Erdős-Turán}
\]
\[
\sum_{(d,2)=1} \frac{|\mu(d)|}{de(d)} < \infty
\]
(see Prachar [2], Ch. V., Lemma 8.3).

Since $d_1 \mid d_2$ implies that $e(d_1) \mid e(d_2)$, therefore (4.1) implies that
\[
\sum_{(d,2)=1} \frac{1}{de(d)} < \infty.
\]

We have
\[
\sum_{k \leq x} f(2^k - 1) = \sum_{d \leq x, (d,2)=1} h(d)\# \{ 2^k - 1 \equiv 0 \pmod{d}, \ k \leq x \} =
\]
\[
= \sum_{d \leq x} h(d) \left( \frac{x}{e(d)} + O(1) \right) + O \left( \sum_{e(d) \leq x} \frac{|h(d)|}{d} \right) =
\]
\[
= x \sum_{d \leq x} \frac{h(d)}{e(d)} + O \left( \sum_{d \leq x} |h(d)| \right) + O \left( x \sum_{d \geq x} \frac{|h(d)|}{d} \right) =
\]
\[
= Bx + O \left( x \sum_{d > x} \frac{|h(d)|}{e(d)} \right) + O(\log x),
\]
where

\( B = \sum_{(d, 2) = 1} \frac{h(d)}{e(d)}. \)  

From (4.2) and \(|h(d)| = \frac{1}{d}\) we obtain that \( B \) is convergent. If we use the estimate

\[ \#\{d \leq x \mid e(d) \leq (\log d)^2\} \ll \frac{x}{(\log x)^2} \]

due to Erdős and Turán (see Prachar, [2] Ch. V. (8.12)) we obtain that

\[ \sum_{2^j x < d < 2^{j+1} x} \frac{|h(d)|}{e(d)} \ll \frac{1}{2^j x \log 2^j x} \frac{2^{j+1} x}{(\log 2^j x)^2} + \frac{1}{\log^2 2^j x} \]

and so

\[ \sum_{d \geq x} \frac{|h(d)|}{e(d)} \ll \sum_{j=0}^{\infty} \frac{1}{(\log x + j \log 2)^2} \ll \frac{1}{\log x}. \]

Consequently the following assertion holds.

**Theorem 2.** We have

\[ \sum_{k \leq x} f(2^k - 1) = Bx + O\left(\frac{x}{\log x}\right). \]

Similarly we can prove that

\[ \sum_{k \leq x} \beta(2^k - 1) = Sx + O\left(\frac{x}{\log x}\right), \]

\[ S = \sum_{(d, 2) = 1} \frac{g(d)}{e(d)}. \]

**References**

Some remarks on the average order in cyclic groups


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