Annales Univ. Sci. Budapest., Sect. Comp. 32 (2010) 63-72

ON SETS CHARACTERIZED BY THE VALUES OF SOME MULTIPLICATIVE FUNCTIONS

I. Kátai (Budapest, Hungary)

Dedicated to Prof. Jean-Marie De Koninck on his 60th anniversary

Abstract. Asymptotic with good error term is given for $E_q(x) = n \leq \leq x \mid \varphi(n) \not\equiv 0 \pmod q$, where φ is Euler's totient function, q is an odd prime. Short interval version of $E_q(x)$ is given as well.

1. Introduction

Let $E_q(x) = \sharp\{n \leq x \mid \varphi(n) \not\equiv 0 \pmod q\}$, where q is a fixed odd prime number. Spearman and Williams proved in [8] that

$$E_q(x) = e(q)x(\log x)^{-\frac{1}{q-1}} + O\left(x \cdot (\log x)^{-\frac{q}{q-1}+\varepsilon}\right).$$

They count the constant $e(q)$ explicitly. In the proof they use the theorem of Wirsing [2] and that of Odoni [9]. By using the theorem of Levin-Fainleib [1], which is quoted now as Lemma 1 we can deduce immediately

Theorem 1. *Let q be an odd prime, $E_q(x)$ as earlier. Then for every positive integer N*

$$E_q(x) = x \sum_{l=0}^{N} A_l x_1^{-\frac{1}{q-1}-l} + O\left(x \cdot x_1^{-\frac{1}{q-1}-N-1}\right), \tag{1.1}$$

Mathematics Subject Classification: 11N25, 11N36

where $x_1 = \log x, A_0, \dots, A_N$ *are suitable constants, which may depend on* q, $A_0 \neq 0$, *furthermore the implied constant in the error term may depend on* q.

Lemma 1. *Let f(n) be a multiplicative function,* $\lambda_f(n)$ *be defined from the equation*

$$f(n)\log n = \sum_{d|n} f(d)\lambda_f\left(\frac{n}{d}\right). \tag{1.2}$$

Assume that

$$\sum_{n\le x}\frac{\lambda_f(n)}{n} = \tau x_1 + B + h(x), \tag{1.3}$$

$$h(x) = O(x_1^{-N-1}),$$

$$\prod_{p\le x}\left(1+\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r}\right) = O(x_1^A), \tag{1.4}$$

A is any positive constant,

$$\lim_{p\to\infty}\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r} = 0. \tag{1.5}$$

Then

$$m(x) := \sum_{n\le x} f(n) = \tag{1.6}$$

$$= x\sum_{1\le\nu<\mathrm{Re}\tau+N+1-A}\tau(\tau-1)\dots(\tau-\nu+1)c_\nu x_1^{\tau-\nu} + O(x\cdot x_1^{A-N-1-\varepsilon}),$$

where

$$c_\nu = \sum_{\lambda+\mu=\nu-1}(-1)^\lambda a_\mu \tag{1.7}$$

and

$$\nu a_\nu u = -\sum_{\lambda+\mu=\nu-1} a_\lambda b_\mu, \qquad b_0 = B,$$

$$b_\mu = \frac{(-1)^{\mu-1}}{(\mu-1)!}\int_1^\infty \frac{h(u)(\log u)^{\mu-1}}{u}du \qquad (\mu\ge 1). \tag{1.8}$$

2. Proof of Theorem 1

Let M be the set of those $m \in \mathbf{N}$ for which if $p|m$, p prime, then $p \not\equiv$ $\not\equiv 1 \pmod q$, and $p \neq q$. Let $A(x) = \sharp\{m \leq x, m \in M\}$. Let us define now the multiplicative function $f(n)$ so that $f(p^\alpha) = f(p) = 0$ if $p \equiv 1 \pmod 9$, or $p \equiv 9$; and $f(p^\alpha) = f(p) = 1$, if $p \not\equiv 1 (\mathrm{mod}\ 9)$, and $p \not\equiv 9$.

It is clear that $E_q(x) = A(x) + A\left(\frac{x}{a}\right)$. Therefore it is enough to prove (1.1) with $A(x)$ instead of $E_a(x)$. From (1.2) we can deduce that $\lambda_f(p^\alpha) = \lambda_f(p) =$ $= f(p) \log p$, and $\lambda_f(N) = 0$ if n has at least two distinct prime divisors. It is well-known, that (1.3) holds. Since $f(p) = O(1)$, therefore (1.4), (1.5) hold, consequently (1.6) is true. Thus Theorem 1 holds true.

3. A more general theorem

Let $D > 1$ be an integer, $s_1, \ldots, s_h$ be a collection of distinct residues mod D, $(s_i, D) = 1$ $(i = 1, \ldots, h)$. Let $f(n) \in \{0, 1\}$ be a multiplicative function defined for prime numbers p as follows:

$$f(p) = \begin{cases} 1 & \text{if } p (\mathrm{mod}\ D) \in \{s_1, \ldots, s_h\}, \\ 0 & \text{otherwise.} \end{cases}$$

Assume furthermore that the values of $f(p^\alpha)$ $(\alpha = 2, 3, \ldots)$ are fixed. As earlier, we can get that $\lambda_f(p) = f(p) \log p$, $\lambda_f(n) = 0$, if n has at least two prime factors and $\lambda_f(p^\alpha) = O(\log p)$.

Let us consider (1.2). We can prove that

$$|\lambda_f(p^\alpha)| \leq \alpha \cdot 2^\alpha \log p, \tag{3.1}$$

whenever $|f(p^\beta)| \leq 1$ for every prime power p^β. This is clear. Let $n = p^\alpha$. If $\alpha = 1$, then (3.1) is true. Let $\alpha \geq 2$. From (1.2),

$$\lambda_f(p^\alpha) = f(p^\alpha) \alpha \log p - \sum_{j=1}^{\alpha-1} \lambda_f(p^\beta) f(p^{\alpha-j}),$$

and so

$$|\lambda_f(p^\alpha)| \le \alpha(\log p) + \sum_{j=1}^{\alpha-1} |\lambda_f(p^j)|. \tag{3.2}$$

Assume that (3.1) holds for $p^j (j = 1, \dots, \alpha - 1)$. Then

$$|\lambda_f(p^\alpha)| \le (\log p) \left\{ \alpha + \sum_{j=1}^{\alpha-1} j \cdot 2^i \right\} \le \alpha \cdot 2^\alpha \cdot \log p.$$

Hence we obtain that

$$\sum_{\alpha\ge 2} \sum_{p>2} \frac{|\lambda_f(p^\alpha)|}{p^\alpha} < \infty. \tag{3.3}$$

Assume that

$$\sum_{\alpha=1}^{\infty} \frac{|\lambda_f(2^\alpha)|}{2^\alpha} < \infty. \tag{3.4}$$

Under these conditions we obtain that

$$\sum_{n\le x} \frac{\lambda_f(n)}{n} = \frac{h}{\varphi(D)} x_1 + B + h(x), \tag{3.5}$$

$h(x) = O(x_1^{-N-1})$, B a suitable constant. This is a consequence of the prime number theorem for arithmetical progressions mod D. (1.4), (1.5) hold as well. Thus the next theorem is true.

Theorem 2. *Let f be a multiplicative function, defined as above in §3. Then*

$$\sum_{n\le x} f(n) = x \left\{ \sum_{\nu=0}^{N} d_\nu x_1^{\tau-\nu} \right\} + O(x \cdot x_1^{\tau-N-1+\varepsilon}), \tag{3.6}$$

where $\tau = \frac{h-\varphi(D)}{\varphi(D)}$.

As special cases we obtain the following assertions.

Theorem 3. *Let $D = q_1 \ldots q_r$ be an odd integer, $q_1, \ldots, q_r$ be distinct primes,*

$$f(p^\alpha) = f(p) = \begin{cases} 0 & \text{if } (p-1, D) \neq 1, \text{ or if } p|D. \\ 1 & \text{otherwise.} \end{cases}$$

Then for

$$A_D(x) := \sum_{n \leq x} f(n),$$

the relation (3.6) holds with $h = \prod_{j=1}^{r} (q_j - 2)$. Furthermore

$$E_D(x) := \sharp\{n \leq x \mid (\varphi(n), D) = 1\} = \sum_{\substack{\delta | D \\ (\varphi(\delta), D)=1}} A_D\left(\frac{x}{\delta}\right).$$

Theorem 4. *Let $D = q_1 \ldots q_r$ be an odd integer. Let $f(p^\alpha) = 1$ if and only if $p \neq 2, q_1, \ldots, q_r$ and $(p^\alpha + \ldots + p + 1, D) = 1$. Otherwise let $f(p^\alpha) = 0$. Let us extend f to $\mathbf{N}$ as a multiplicative function. Then, for $B_D(x) := \sum_{n \leq x} f(n)$ the relation (3.6) holds with $h = \prod(q-2)$. The conditions of Theorem 2 hold.*

Let $H_D(x) := \sharp\{n \leq x | (\sigma(n), D) = 1\}$. We have

$$H_D(x) = x \sum_{\nu=0}^{\infty} k_\nu x_1^{\tau-\nu} + O(x \cdot x_1^{\tau-N-1}),$$

where $\tau = \frac{h-\varphi(D)}{\varphi(D)}$, $k_0, \ldots, k_N$ be a suitable constants that depend on D.

Proof of Theorem 4. Since the conditions of Theorem 2 hold ($f(2^\alpha) = = 0 \quad (\alpha = 1, 2, \ldots)$ implies that $\lambda_f(2^\alpha) = 0$), we have an expansion for $B_D(x)$ written in (3.6). Let $k_\sigma = \{\alpha | (\sigma(2^\alpha), D) = 1\}$. Then

$$\sharp\{n \leq x \mid (\sigma(n), D) = 1\} = B_D(x) + \sum_{\alpha \in k_\sigma} B_D\left(\frac{x}{2^\alpha}\right).$$

Hence the assertion can be deduced immediately.

4. Short interval version by Ramachandra's theorem

We shall use the following theorem due to Ramachandra in [5]. Let $S_1,\ S_2\ , S_3$ be the sets of L-series, the derivatives, and the logarithms of L-series, respectively, $\log L(s,x)$ is defined by analytic continuation from the halfplane $r > 1$; for some complex z, we define

$$L(s,x)^z = \exp(z \log L(s,x)).$$

Let $P_1(s)$ be any finite power product (with complex exponents) of functions of S_1. Let $P_2(s)$ be any finite power product (with nonnegative integral exponents) of functions of S_2. Let also $P_3(s)$ denote any finite power product with nonnegative integral exponents of functions of S_3. Let c_n be a sequence of complex numbers such that $|c_n| \ll n^\varepsilon$ for every $\varepsilon > 0$ and

$$\sum \frac{|c_n|}{n^\sigma} < \infty \quad \text{for} \quad \sigma > \frac{1}{2}.$$

Let $F_0(s) = \sum \frac{c_n}{n^s}$. Furthermore, let

$$F_1(s) = P_1(s)P_2(s)P_3(s)F_0(s) = \sum \frac{g_n}{n^s} \tag{4.1}$$

and

$$E(x) = \sum_{n \le x} g_n.$$

Let $C_0 = C_0(r)$ be the counter defined as follows. We start from the circle $\{s||s-1| = r\}$, remove the point $s = 1 - r$, and proceeding on the remaining portion of the circle in the anticlockwise direction.

Assume that r is so small that $F_1(s)$ has no singularities on the boundary and in the interior of it, except, possibly, the place $s = 1$.

Let $C_1 = C\left(\frac{1}{\log x}\right)$, and let L^-, L^+ be defined as the intervals on straightlines

$$L^- = \left[\left(1 - \frac{1}{r}\right)e^{-i\pi},\ \left(1 - \frac{1}{\log x}\right)e^{-i\pi}\right],$$
$$L^+ = \left[\left(1 - \frac{1}{\log x}\right)e^{i\pi},\ \left(1 - \frac{1}{r}\right)e^{i\pi}\right].$$

Let C^* be the contour going along L^- starting from $\left(1 - \frac{1}{r}\right)e^{-i\pi}$, then on C_1, and finally, on L^+.

Let B be the constant occurring in the density result

$$N_\chi(\alpha, T) = O(T^{B(1-\alpha)}(\log T)^2)$$

which is valid for all characters occurring in P_1, P_2 and P_3. Let $\varphi = 1 - \frac{1}{B} + \varepsilon$ with arbitrary $\varepsilon > 0$.

Remark. According to Huxley's result, φ can be any constant greater than $\frac{7}{12}$.

Theorem of Ramachandra. *Let x be sufficiently large and $1 \le h \le x$. Let*

$$I(x,h) = \frac{1}{2\pi i} \int\limits_0^h \left(\int\limits_{C_0} F_1(s)(v+x)^{s-1} ds \right) dx. \tag{4.2}$$

Then

$$E(x+h) - E(x) = I(x,h) + O_\varepsilon \left(h \cdot \exp\left(-x_1^{\frac{1}{6}}\right) + x^\varphi \right).$$

In our paper [7] we proved the following theorems which are quoted now as Theorem A and B.

Theorem A. *Assume that $F_1(s)$ satisfies the conditions in Ramachandra's theorem. Let $r > 0$ and $\varepsilon > 0$ be sufficiently small constants, and let $x^{\frac{7}{12}+\varepsilon} \le$ $\le h \le x^{\frac{2}{3}-\frac{2r}{3}}$. Then*

$$\frac{E(x+h)-E(x)}{h} = \frac{1}{2\pi i} \int\limits_{C^*} F_1(s) x^{s-1} dx + O\left(\exp\left(-x_1^{\frac{1}{6}}\right)\right). \tag{4.3}$$

Let us assume that

$$F_1(s) = \frac{U(s)}{(s-1)^z}, \tag{4.4}$$

where $U(s)$ is analytic in the disc $|s-1| \le r$. Then for every fixed k,

$$U(s) = A_0 + A_1(s-1) + \ldots + A_k(s-1)^k + (s-1)^{k+1} V(s), \tag{4.5}$$

where $U(s)$ is bounded in $|s-1| \le r$. Since

$$\frac{1}{2\pi i} \int\limits_{C^*} x^{s-1}(s-1)^{a-z} ds = \frac{\Gamma(a-z)}{x_1^{a-z+1}} \cdot \frac{\sin \pi(a-z)}{\pi} + O(x^{-\frac{r}{2}}) \tag{4.6}$$

(for the proof, see Lemma 8 of [10]), we obtain

Theorem B. *Under the conditions stated above, we have*

(4.7)
$$\frac{1}{2\pi i}\int\limits_{C^*}\frac{U(s)}{(s-1)^z}x^{s-1}ds=\sum_{l=0}^{k}A_l\frac{\Gamma(l-z)}{x_1^{l-z+1}}\cdot\frac{(-1)^{l+1}\sin\pi z}{\pi}+O\left(\frac{1}{x_1^{k+2-\mathrm{Re}z}}\right),$$

whenever Re $z\le k+1$.

From the theorems we can deduce the short interval version of Theorems 1, 2, 3, 4. Earlier in our paper written jointly with DeKoninck [11] we used this method for short interval sums of $\tau(n)\omega(n)$. We shall prove this for Theorem 2 only. It is enough to prove that the function

$$F_1(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s} \tag{4.8}$$

($f(n)$ defined in §3) can be written in the "Ramachandra" form.

Let

$$\beta(\chi):=\frac{1}{\varphi(D)}\sum_{j=1}^{h}\bar{\chi}(s_j),$$

where χ runs over the Dirichlet characters mod D,

$$P_1(s)=\prod_{s}L(s,\chi)^{\beta(\chi)}. \tag{4.9}$$

Let $F_0(s)$ be defined by

$$F_1(s)=P_1(s)F_0(s). \tag{4.10}$$

We can see easily that $F_0(s)=\sum\frac{c_n}{n^s}$ with the condition $\sum\frac{|c_n|}{n^\sigma}<\infty$ for $\sigma>\frac{1}{2}$. Indeed,

$$\log P_1(s)=\sum_{\chi(\mathrm{mod}\ D)}\beta(\chi)\log L(s,\chi)=-\sum_{\chi}\beta(\chi)\sum_{p}\log\left(1-\frac{\chi(p)}{ps}\right)=$$
$$=-\sum_{\chi}\beta(\chi)\left\{\sum_{p}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}\chi(p^\nu)}{\nu p^{\nu s}}\right\}=\sum_{p}\frac{f(p)}{p^s}+g(s),$$

where

$$g(s) = \sum_{p^\nu, \nu \geq 2} \frac{b_{\nu,p}}{p^{\nu s}}, \quad b_{\nu,p} \text{ is bounded.}$$

Furthermore,

$$\log F_1(s) = \sum_p \frac{f(p)}{p^s} + h(s),$$

$$h(s) = \sum_{p^\nu, \nu \geq 2} \frac{d_{\nu,p}}{p^{\nu s}}, \quad d_{\nu,p} \text{ is bounded.}$$

Thus

$$\log \frac{P_1(s)}{F_1(s)} = g(s) - h(s) = \sum_{p^\nu, \nu \geq 2} \frac{b_{\nu,p} - d_{\nu,p}}{p^{\nu,s}}.$$

Hence we obtain that

$$\exp(g(s) - h(s)) = \sum \frac{c_n}{n^s},$$

$$\sum_n \frac{|c_n|}{n^\sigma} < \infty \text{ for } \sigma > \frac{1}{2}.$$

We formulate our results as

Theorem 2'. *Let f be as in §3. Let* $x^{\frac{7}{12}+\varepsilon} \leq h < x$, *where* $\varepsilon > 0$, *arbitrary,* $\tau = \frac{h-\varphi(D)}{\varphi(D)}$. *Then*

$$\frac{1}{h} \sum_{x \leq n < x+h} f(n) = \sum_{\nu=0}^{N} d_\nu \cdot x_1^{\tau-\nu} + O(x_1^{\tau-N-1+\varepsilon}).$$

References

[1] **Левин Б.В. и Файнлейб А.С.,** Применение некоторых интегральных уравнений к вопросам теории чисел, *Успехи математических наук,* **22** (135) (3), 119-197. (*Levin B.V. and Fainleib A.S.,* Applications of some integral equations to problems in number theory, *Russian Math. Surveys,* **22** (3) (1967), 119-204.)

[2] **Wirsing E.,** Das asympotische Verhalten von Summen ber multiplikative Funtionen, *Math Ann.,* **143** (1961), 75-102.

[3] **Pappalardi F., Saidak F. and Shparlinski I.E.,** Square-free values of the Carmichael function, *J. Number Theory,* **103** (2003), 122-131.

[4] **Kátai I.,** Square-free values of Carmichael functions, *Mathematica Pannonica,* **16** (2) (2005), 199-203.

[5] **Ramachandra K.,** Some problems of analytic number theory, *Acta Arithmetica,* **31** (1967), 313-324.

[6] **Kátai I.,** A remark on a paper of Ramachandra, *Proc. of Number Theory Conf., Ootacamund, 1984,* ed. K.Alladi, Lecture Notes in Math. **1122**, Springer Verlag, 1984, 147-152.

[7] **Kátai I. and Subbarao M.V.,** Some remarks on a paper of Ramachandra, *Liet. Matem. Rink.,* **43** (2003), 497-506.

[8] **Spearman B.K. and Williams K.S.,** Values of the Euler phi function not divisible by a given odd prime, *Ark. Mat.,* **44** (2006), 166-181.

[9] **Odoni R.W.K.,** A problem of Rankin on sums of powers of cusp-form coefficients, *J. London Math. Soc.,* **44** (1991), 203-217.

[10] **Balasubramanian R. and Ramachandra K.,** On the theorem of integers n such that $nd(n) \leq x$, *Acta Arith.,* **49** (1988), 313-322.

[11] **DeKonick J.-M. and Kátai I.,** On the average of $d(n)\omega(n)$ and similar functions on short interval, *Annales Univ. Sci. Budapest. Sect. Comp.,* **25** (2005), 131-142.

(Received October 24, 2007)

I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter s. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu