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A GENERALIZED BILINEAR SPLINE APPROXIMATION

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Abstract. In this paper a new method of spline approximation is given which is continuous and applicable for robust estimators. The model computations have shown that the method is suitable for cartography problems. Because of the good characteristic of the method, e.g. fast convergence, it seems to be widely applicable in engineering problems.

1. Introduction

The estimation of the main parameters describing samples with a huge number of data has of great importance in engineering applications and during the analyzation of cartography. One of the simple methods is the application of regression theory. In this model one chooses a possible regression curve based on the geometric location of data points which procedure implies that the solution is not unique. There are N points (x_i, y_i, f_i) with $i = 1, 2, \dots, N$, and a regression function $g(x, y)$. To find the solution for $g(x, y)$ one solves the equation

$$(1) \qquad \sum_{i=1}^{N} (f_i - g(x_i, y_i))^2 = \min_g$$

with the method of least squares.

Another possible method is the application of interpolation or approximation functions. During interpolation one reconstructs the function $f(x, y)$ from its values $f_1, f_2, \dots, f_N$ given in the discrete points (x_1, y_1), $(x_2, y_2), \dots,$ (x_N, y_N). Again, in general, the solution is not unique. One chooses a suitable

Mathematics Subject Classification: 41A15, 65D04 polynomial which best approximates the analyzed function on the domain in a given respect. Undesirable oscillations and huge computational capacity required to handle polynomials with high degree are the characteristics of these approximation methods which can be avoided by the use of spline interpolation. In this case one looks for the l times continuously differentiable function $g(x,y)$ which is the solution of the equations

$$\delta(g) = \iint_T \left(\partial_x^l g\right)^2 + \left(\partial_y^l g\right)^2 dT = \min_{W_2^l} . \tag{2}$$

Here W_2^l denotes the space of l times continuously differentiable and square-integrable functions. When $l < N$ the solution is unique, see the proof of Sard [1] who uses $(2l-1)$ degree smoothly connected polynomials defined on intervals. In most cases one is dealing with the solution with $l = 1$ or $l = 2$. When the values of $f(x,y)$ are modified by errors in the points (x_i, y_i) one looks for the best approximating solution in a given respect, namely with the composition of the theory of regression and interpolation curves,

$$\delta(g) = \iint_T \left(\partial_x^l g\right)^2 + \left(\partial_y^l g\right)^2 dT + \sum_{i=1}^{N} p_i (f_i - g(x_i, y_i))^2 = \min_{W_2^l}, \tag{3}$$

where the positive weighting numbers $\{p_i\}_{i=1}^N$ are capable of smoothing the solution when the errors are known.

2. The problem

Here we generalize the description outlined above to the case when there are more points given than the spline approximation polynomials we chose. The weights are fixed by the method of least squares as in the case of robust estimators and the spline approximation is obtained as a result of an iteration process.

Let us consider a rectangular domain $T = [X_0, X_m] \times [Y_0, Y_n]$ and take a division $X_0 < X_1 < X_2 < \ldots < X_m$, of the x and $Y_0 < Y_1 < Y_2 < \ldots < Y_n$ of the y axes and the sample $(x_{ijk}, y_{ijk}, f_{ijk})$, $k = 1, 2, \ldots, K_{ij}$ which are given in the domains $[X_{i-1}, X_i] \times [Y_{j-1}, Y_j]$ for $i = 1, 2, \dots, m,\ j = 1, 2, \dots, n$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} K_{ij} = N$. Using this notation we solve the variation problem for $l = 1$

$$\delta(g) = \lambda \int\int_T (\partial_x g)^2 + (\partial_y g)^2 \, dT + \sum_{i=1}^{N} p_i (f_i - g(x_i, y_i))^2 = \min_{W_2^l} \tag{4}$$

with the generalized Lagrange multiplicator λ and positive weighting numbers p_i. The solution of the variation problem results piecewise bilinear polynomials

$$g(x, y) = \{g_{ij}(x, y), \quad X_{i-1} \le x \le X_i, \quad Y_{j-1} \le y \le Y_j\}, \tag{5}$$

which are continuous in zeroth order and

$$g_{ij}(x, y) = a_{ij} + b_{ij}(x - X_{i-1}) + c_{ij}(y - Y_{j-1}) + d_{ij}(x - X_{i-1})(y - Y_{j-1}), \tag{6}$$

where $i = 1, 2, \dots, m,\ j = 1, 2, \dots, n$.

Using this solution the problem

$$\begin{aligned}
&\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K_{ij}} p_{ijk} (g(x_{ijk}, y_{ijk}) - f_{ijk})^2 + \\
&\quad + 2 \sum_{i=1}^{m-1} \sum_{j=1}^{n} [\Gamma_{ij} (g_{ij}(X_i, Y_{j-1}) - g_{i+1j}(X_i, Y_{j-1})) + \\
&\quad + \Delta_{ij} (g_{ij}(X_i, Y_j) - g_{i+1j}(X_i, Y_j))] + \\
&\quad + 2 \sum_{i=1}^{m} \sum_{j=1}^{n-1} [\Xi_{ij} (g_{ij}(X_{i-1}, Y_j) - g_{ij+1}(X_{i-1}, Y_j)) + \\
&\quad + \Theta_{ij} (g_{ij}(X_i, Y_j) - g_{ij+1}(X_i, Y_j))] \equiv \\
&\equiv Func \to \min
\end{aligned} \tag{7}$$

is solved the help of Lagrange multiplicators as a maximization problem.

As we can see the number of unknowns $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ is $4mn$, and the number of multiplicators $\Gamma_{ij},\ \Delta_{ij},\ \Xi_{ij},\ \Theta_{ij}$ is $2(m-1)n + 2m(n-1)$. For the solution we need $8mn - 2m - 2n$ linearly independent equations, which are determined by the solution of the analytic maximization problem. Since the number of unknowns equals with the number of equations of the linear system the problem can be solved in principle. The difficulty in finding the solution is that we have $8mn - 2m - 2n$ unknowns. As we will see later those can be determined as the solution of a system of linear equations. The computation of the weights p_{ijk} means additional difficulties. The system of linear equations is solved with unit weights first, then with the help of the solutions and the application of the methods of robust estimators the weights are redefined. These steps are repeated until the condition of iteration holds.

3. The equations

For further convenience we introduce $h_{1i} = x_i - x_{i-1}$ and $h_{2j} = y_j - y_{j-1}$. Using these quantities for the partial derivatives of the function $Func$ by Γ_{ij}, Δ_{ij}, Ξ_{ij}, and Θ_{ij} the following relations hold

$$a_{ij} + h_{1i}b_{ij} - a_{i+1j} = 0, \tag{8}$$

$$a_{ij} + h_{1i}b_{ij} + h_{2j}c_{ij} + h_{1i}h_{2j}d_{ij} - a_{i+1j} - h_{2j}c_{i+1j} = 0 \tag{9}$$

for every $i = 1, 2, \ldots, m-1$; $j = 1, 2, \ldots, n$ and

$$a_{ij} + h_{2j}c_{ij} - a_{ij+1} = 0, \tag{10}$$

$$a_{ij} + h_{1i}b_{ij} + h_{2j}c_{ij} + h_{1i}h_{2j}d_{ij} - a_{ij+1} - h_{1i}b_{ij+1} = 0 \tag{11}$$

for every $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n-1$. For a simpler notation of the partial derivatives we use the following quantities

$$U_{ijst} = \sum_{k=1}^{K_{ij}} p_{ijk}(x_{ijk} - X_{i-1})^s (y_{ijk} - Y_{j-1})^t, \tag{12}$$

$$V_{ijst} = \sum_{k=1}^{K_{ij}} p_{ijk} f_{ijk}(x_{ijk} - X_{i-1})^q (y_{ijk} - Y_{j-1})^r \tag{13}$$

for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and $s, t = 0, 1, 2$; $q, r = 0, 1$. Moreover

$$\Phi_{ijst} = U_{ijst}a_ij + U_{ijs+1t}b_ij + U_{ijst+1}c_ij + U_{ijs+1t+1}d_ij - V_{ijst} \tag{14}$$

for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and $s, t = 0, 1$.

In this way the partial derivatives of $Func$ by $a_{ij},\ b_{ij},\ c_{ij}$ and d_{ij} are

$$\Phi_{1100} + \Gamma_{11} + \Xi_{11} = 0, \tag{15}$$

$$\Phi_{i100} + \Gamma_{i1} - \Gamma_{i-11} + \Xi_{i1} = 0, \tag{16}$$

$$\Phi_{m100} - \Gamma_{m-11} + \Xi_{m1} = 0, \tag{17}$$

$$\Phi_{1j00} + \Gamma_{1j} + \Xi_{1j} - \Xi_{1j-1} = 0, \tag{18}$$

$$\Phi_{ij00} + \Gamma_{ij} - \Gamma_{i-1j} + \Xi_{ij} - \Xi_{ij-1} = 0, \tag{19}$$

$$\Phi_{mj00} - \Gamma_{m-1j} + \Xi_{mj} - \Xi_{mj-1} = 0, \tag{20}$$

$$\Phi_{1n00} + \Gamma_{1n} - \Xi_{1n-1} = 0, \tag{21}$$

$$\Phi_{in00} + \Gamma_{in} - \Gamma_{i-1n} - \Xi_{in-1} = 0, \tag{22}$$

$$\Phi_{mn00} - \Gamma_{m-1n} - \Xi_{mn-1} = 0 \tag{23}$$

for $i = 2, \ldots, m-1;\ j = 2, \ldots, n-1$ and

$$\Phi_{i110} + h_{1i}\Gamma_{i1} + \Theta_{i1} = 0, \tag{24}$$

$$\Phi_{m110} + \Theta_{m1} = 0, \tag{25}$$

$$\Phi_{ij10} + h_{1i}\Gamma_{ij} + \Theta_{ij} - \Theta_{ij-1} = 0, \tag{26}$$

$$\Phi_{mj10} + \Theta_{mj} - \Theta_{mj-1} = 0, \tag{27}$$

$$\Phi_{in10} + h_{1i}\Gamma_{in} - \Theta_{in-1} = 0, \tag{28}$$

$$\Phi_{mn10} - \Theta_{mn-1} = 0 \tag{29}$$

for $i = 1, 2, \ldots, m-1;\ j = 2, \ldots, n-1$ and

$$\Phi_{1j01} + \Delta_{1j} + h_{2j}\Xi_{1j} = 0, \tag{30}$$

$$\Phi_{ij01} + \Delta_{ij} - \Delta_{i-1j} + h_{2j}\Xi_{ij} = 0, \tag{31}$$

$$\Phi_{mj01} - \Delta_{m-1j} + h_{2j}\Xi_{mj} = 0, \tag{32}$$

$$\Phi_{1n01} + \Delta_{1n} = 0, \tag{33}$$

$$\Phi_{in01} + \Delta_{in} - \Delta_{i-1n} = 0, \tag{34}$$

$$\Phi_{mn01} - \Delta_{m-1n} = 0 \tag{35}$$

for $i = 2, \ldots, m-1;\ j = 1, 2, \ldots, n-1$ and

$$\Phi_{ij11} + h_{1i}\Delta_{ij} + h_{2j}\Theta_{ij} = 0, \tag{36}$$

$$\Phi_{mj11} + h_{2j}\Theta_{mj} = 0, \tag{37}$$

$$\Phi_{in11} + h_{1i}\Delta_{in} = 0, \tag{38}$$

$$\Phi_{mn11} = 0 \tag{39}$$

for $i = 1, \ldots, m-1;\ j = 1, 2, \ldots, n-1$.

3.1. The elimination of the muliplicators $\Gamma_{ij},\ \Delta_{ij},\ \Xi_{ij},\ \Theta_{ij}$

From the sum of Eqs. (15)-(23) we have

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\Phi_{st00} = 0. \tag{40}$$

From the sum of Eqs. (25), (27) and (29)

$$\sum_{t=1}^{n}\Phi_{mt10} = 0, \tag{41}$$

and from the sum of Eqs. (33)-(35)

$$\sum_{s=1}^{m} \Phi_{sn01} = 0. \tag{42}$$

From the sum of Eqs. (24), (26) and (28)

$$h_{1i} \sum_{t=1}^{n} \Gamma_{it} + \sum_{t=1}^{n} \Phi_{it10} = 0, \quad i = 1, \dots, m-1, \tag{43}$$

and from the sum of Eqs. (15), (16), (18), (19), (21), (22)

$$\sum_{t=1}^{n} \Gamma_{it} + \sum_{s=1}^{i} \sum_{t=1}^{n} \Phi_{st10} = 0, \quad i = 1, \dots, m-1. \tag{44}$$

From the difference of Eqs. (43) and (44)

$$\sum_{t=1}^{n} \left(\Phi_{it10} - h_{1i} \sum_{s=1}^{i} \Phi_{st00} \right) = 0, \quad i = 1, \dots, m-1. \tag{45}$$

Similarly the sum of Eqs. (15)-(20) and (30)-(32) results

$$\sum_{s=1}^{m} \left(\Phi_{sj01} - h_{2j} \sum_{t=1}^{j} \Phi_{st00} \right) = 0, \quad j = 1, \dots, n-1. \tag{46}$$

From the sum of Eqs. (25), (27)

$$\Theta_{mj} + \sum_{t=1}^{j} \Phi_{mt10} = 0, \quad j = 1, \dots, n-1. \tag{47}$$

From the difference of Eqs. (37) and (47)

$$\Phi_{mj11} - h_{2j} \sum_{t=1}^{j} \Phi_{mt10} = 0, \quad j = 1, \dots, n-1. \tag{48}$$

Similarly from the sum of Eqs. (33), (34) and (38) results

$$\Phi_{in11} - h_{1i} \sum_{s=1}^{i} \Phi_{sn01} = 0, \quad j = 1, \dots, m-1. \tag{49}$$

From the sum of Eqs. (15), (16), (18), (19) we have

$$\sum_{t=1}^{j} \Gamma_{it} + \sum_{s=1}^{i} \Xi_{it} + \sum_{s=1}^{i}\sum_{t=1}^{j} \Phi_{st00} = 0, \tag{50}$$

and from the sum of Eqs. (24), (26)

$$\Theta_{ij} + h_{1i}\sum_{t=1}^{j} \Gamma_{it} + \sum_{t=1}^{j} \Phi_{it10} = 0, \tag{51}$$

and from the sum of Eqs. (30), (31)

$$\Delta_{ij} + h_{2j}\sum_{s=1}^{i} \Xi_{sj} + \sum_{s=1}^{i} \Phi_{sj10} = 0 \tag{52}$$

for every $i = 1, \ldots, m-1;\ j = 1, \ldots, n-1$. The sum of Eqs. (36), (50), (51) and (52) results

$$\Phi_{ij11} - h_{1i}\sum_{s=1}^{i} \Phi_{sj01} - h_{2j}\sum_{t=1}^{j} \Phi_{it10} + h_{1i}h_{2j}\sum_{s=1}^{i}\sum_{t=1}^{j} \Phi_{st00} = 0 \tag{53}$$

for $i = 1, \ldots, m-1;\ j = 1, \ldots, n-1$.

4. The algorithm

The main steps of the iteration are:

1. Choose suitable values for the initial weights $p_{ijk},\ i = 1, \ldots, m;\ j = 1, \ldots, n;\ k = 1, \ldots, K_{ij}$. At first use unit weights.

2. Compute the coefficients $b_{ij},\ c_{ij}$ and d_{ij}.

In the first step compute

$$b_{ij} = \frac{1}{h_{1i}}(a_{i+1j} - a_{ij}) \quad \text{and} \quad d_{ij} = \frac{1}{h_{1i}}(c_{i+1j} - c_{ij}) \tag{54}$$

for $i = 1, \ldots, m-1;\ j = 1, \ldots, n$;

$$c_{ij} = \frac{1}{h_{2j}}(a_{ij+1} - a_{ij}) \quad \text{and} \quad d_{ij} = \frac{1}{h_{2j}}(b_{ij+1} - b_{ij}) \tag{55}$$

for $i = 1, \ldots, m;\ j = 1, \ldots, n-1$;

$$d_{ij} = \frac{1}{h_{1i}h_{2j}}(a_{i+1j+1} - a_{i+1j} - a_{ij+1} + a_{ij}) \tag{56}$$

for $i = 1, \ldots, m-1;\ j = 1, \ldots, n-1$;

$$d_{in} = \frac{1}{h_{1i}}(c_{i+1n} - c_{in}) \tag{57}$$

for $i = 1, \ldots, m-1$;

$$d_{mj} = \frac{1}{h_{2j}}(b_{mj+1} - b_{mj}) \tag{58}$$

for $j = 1, \ldots, n-1$ from (8)-(11).

In the second step from (39) we have

$$\begin{aligned} d_{mn} &= d_{mn}(a_{mn}, b_{mn}, c_{mn}) = \\ &= \frac{1}{U_{mn22}}(V_{mn11} - U_{mn11}a_{mn} - U_{mn21}b_{mn} - U_{mn12}c_{mn}). \end{aligned} \tag{59}$$

In the third step from (41)

$$b_{mn} = b_{mn}(a_{m1}, \ldots, a_{mn}) \tag{60}$$

and from (42)

$$c_{mn} = c_{mn}(a_{1n}, \ldots, a_{mn}) \tag{61}$$

and then

$$d_{mn} = d_{mn}(a_{1n}, \ldots, a_{mn}, \ldots, a_{m1}). \tag{62}$$

In the fourth step from (48)

$$b_{mj} = b_{mj}(a_{m1}, \ldots, a_{mn}), \quad j = 1, \ldots, n-1, \tag{63}$$

and from (49)

$$c_{in} = c_{in}(a_{1n}, \ldots, a_{mn}), \quad i = 1, \ldots, m-1. \tag{64}$$

3. Compute the coefficient a_{ij}.

From (53)

$$a_{i+1j+1} = a_{i+1j+1}(a_{11}, \ldots, a_{i1}, a_{12}, \ldots, a_{ij}) \tag{65}$$

for $i = 1, \ldots, m-1;\ j = 1, \ldots, n-1$.

From (45)

$$a_{i+1n} = a_{i+1n}(a_{11}), \quad i = 1, \ldots, m-1. \tag{66}$$

From (44)

$$a_{mj+1} = a_{mj+1}(a_{11}), \quad j = 1, \ldots, n-1. \tag{67}$$

Finally, from (42) we compute a_{11}.

4. When the stop condition is not satisfied calculate the weights p_{ijk} with the use of the spline function $g(x,y)$ and repeat from step 2.

Remarks on step 4:

- One of the possible stop conditions is the following: compute the summed square of the difference of the samples and the approximating spline function (at the k-th iteration step it is denoted by S_k). At startup set $\varepsilon > 0$ and compare ε with the value of $\frac{|S_{k-1} - S_k|}{S_{k-1}}$.
- The larger the difference between f_{ijk} and the value of the spline function in the point (x_{ijk}, y_{ijk}) the smaller the value of p_{ijk} is advised to choose, e.g.:

$$p_{ijk} = \begin{cases} \dfrac{1}{|f_{ijk} - g(x_{ijk}, y_{ijk})|} & \text{if} \quad |f_{ijk} - g(x_{ijk}, y_{ijk})| > \varepsilon', \\ \dfrac{1}{\varepsilon'} & \text{if} \quad |f_{ijk} - g(x_{ijk}, y_{ijk})| \le \varepsilon', \end{cases} \tag{68}$$

where the number $\varepsilon' > 0$ is small enough.

5. Examples

We check the applicability of the solution with the so-called stochastic simulation method: a known solution is polluted with a random variable of normal or Cauchy distribution. The method is then applied to the polluted sample.

In the example we have modified the values of the function $f(x, y)$ with a random variable of Cauchy distribution. The rectangular domain $T = [0, 1] \times [0, 1]$ is divided into $m = n = 8$ equal parts. The iteration is done with the help of the Maple V Rel. 5 computer algebra system according to the algorithm presented in the previous section. In the second example we use data of Nagy et al. [2].

6. Conclusion

We have modeled a function approximating a large number of samples with subdomain bilinear spline polynomials. The solution results a continuous spline function which gives the minimum of the variational problem according to the method of least squares and approximates the data in a similar manner as robust estimators.

The novelty of the method is that it makes global equalization with subdomain approximation on a large number of samples. Contrary to the current spline approximation methods, where a polynomial is defined between every point, in this treatment the points are grouped together. After that we search for surfaces which approximate these groups well according to specific conditions and, moreover, give a good approximation of the data points globally.

In the model calculations fast convergence is apparent and large oscillations do not appear.

References

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