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SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION OF n-TH ORDER WITH FOUR SINGULAR POINTS

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Abstract. The solution of the recurrent equation of 2nd order is found in explicit form. This allows to represent the Frobenius solution (series) of corresponding linear differential equation through hypergeometric function and a new special function. The structure of coefficients of the Frobenius series is obtained in explicit form through the parameters of the differential equation, too.

1. Introduction

One possibility of solving linear differential equations with polynomial coefficients is the Frobenius method [1]. Its substance is to look for the solution of differential equation in the neighbourhood of singular point $t = t_0$ in the form of power series

$$\sum_{m=0}^{\infty} e_m (t - t_0)^{\rho+m},$$

where the coefficients e_m satisfy a certain recurrent equation.

As noted in [2, p. 551], the Frobenius method gives completely satisfactory results for practical purposes. From our point of view the effectiveness of Frobenius series would be much higher, if it were possible to express it in terms of known special functions. But it requires to find the coefficients e_m, i.e. to solve the corresponding recurrent equation in explicit form. In the paper we propose a rather simple solution of recurrent equation of 2nd order.

2. Problem statement

Let

$$\sum_{i=0}^{n}\left(A_i t^2 + B_i t + C_i\right) t^i u^{(i)} = 0, \tag{1}$$

where A_i, B_i, C_i are real or complex numbers, $C_n \neq 0$.

We look for the solution of (1) in the neighbourhood of point $t = 0$ in the form

$$u(t) = e_0 t^{\rho} + e_1 t^{\rho+1} + \ldots \tag{2}$$

Let us denote for $m = 0, 1, \ldots$

$$\begin{aligned}
\gamma_m &= C_0+(\rho+m)C_1+\ldots+(\rho+m)(\rho+m-1)\ldots(\rho+m-n+1)C_n,\\
\beta_m &= B_0+(\rho+m)B_1+\ldots+(\rho+m)(\rho+m-1)\ldots(\rho+m-n+1)B_n,\\
\alpha_m &= A_0+(\rho+m)A_1+\ldots+(\rho+m)(\rho+m-1)\ldots(\rho+m-n+1)A_n.
\end{aligned} \tag{3}$$

Substituting (2) into (1) and equating the coefficients with zero at the powers t, we get the trinomial recurrence equation

$$e_1 = -\frac{\beta_0}{\gamma_1} e_0, \quad e_{m+1} = -\frac{\beta_m}{\gamma_{m+1}} e_m - \frac{\alpha_{m-1}}{\gamma_{m+1}} e_{m-1}, \quad m = 1, 2, \ldots \tag{4}$$

Here it is supposed that $\gamma_m \neq 0$, $m = 1, 2, \ldots$ and from the equation

$$\gamma_0 = C_0 + \rho C_1 + \ldots + \rho(\rho-1)\ldots(\rho-n+1)C_n = 0 \tag{5}$$

the values of parameter ρ are found. Hereinafter all the reasonings are carried out for the known parameter ρ.

3. Solution of the recurrent equation

Now the obvious solution of equation (4) is obtained having put $e_0 = 1$ and temporarily $\gamma_m = -1$, $m = 1, 2, \ldots$. In this case the equation (4) has the simplified form

$$e'_1 = \beta_0, \quad e'_{m+1} = \beta_m e'_m + \alpha_{m-1} e'_{m-1}, \quad m = 1, 2, \ldots \tag{6}$$

It is easy to see that in order to find the coefficients e'_{m+1} the set $I_{m+1} = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1}; \beta_0, \beta_1, \ldots, \beta_m)$ is used.

β_m is an element of rank 1, α_m an element of rank 2, $m = 0, 1, \ldots,$ respectively. The product of the maximal possible number of elements from I_{m+1} is called a path, these elements should have different indices forming an ascending sequence. The order of path (ord Δ) will be the sum of ranks of all elements from the path.

Theorem 1. *The arrangement of elements in the path complies with the following hierarchy:*

1. Any path starts with an element with zero index and ends either with element α_{m-1} or with element β_m.

2. The element β_i of rank 1 is multiplied by the next element whose index is one greater than that of β_i. This element may have any rank, namely $\ldots\beta_i\beta_{i+1}\ldots$ or $\ldots\beta_i\alpha_{i+1}$. The element α_i of rank 2 is multiplied by the next element whose rank is two units higher than the index of α_i, this element may be of any rank, namely $\ldots\alpha_i\beta_{i+2}\ldots$ or $\ldots\alpha_i\alpha_{i+2}\ldots$.

3. The order of any path Δ from the set I_{m+1} is equal to $m+1$, ord $\Delta = m+1$.

4. The coefficient e'_{m+1} is equal to the sum of all the paths from the set I_{m+1}.

Proof. Let us prove properties 1) - 4) by induction.

$m = 0$, $e'_1 = \beta_0$, $I_1 = (\beta_0)$, the paths - β_0; *ord* $\beta_0 = 1$;

$m = 1$, $e'_2 = \beta_0\beta_1 + \alpha_0$, $I_2 = (\alpha_0; \beta_0, \beta_1)$, the paths - $\beta_0\beta_1$, α_0, *ord* $\beta_0\beta_1 =$ *ord* $\alpha_0 = 2$;

$m = 2$, $e'_3 = \beta_2 e'_2 + \alpha_1 e'_1 = \beta_0\beta_1\beta_2 + \alpha_0\beta_2 + \beta_0\alpha_1$, $I_3 = (\alpha_0, \alpha_1; \beta_0, \beta_1, \beta_2)$, the paths - $\beta_0\beta_1\beta_2$, $\alpha_0\beta_2$, $\beta_0\alpha_1$, the order of each path is equal to 3.

Let the properties 1) - 4) be true for $m = k$, i.e. for the paths made up of the elements of the set $I_k = (\alpha_0, \ldots, \alpha_{k-2};\ \beta_0, \ldots, \beta_{k-1})$. Consequently, they are valid for any path made up of the elements of $I_{k-1} = (\alpha_0, \ldots, \alpha_{k-3};\ \beta_0, \ldots, \beta_{k-2}) \subset I_k$. We show that these properties are valid for $m = k+1$. The maximum possible filling of path and the hierarchy of elements in it mean that all the paths made up of the elements from I_k are finished by α_{k-2} or β_{k-1}, and that from the elements of I_{k-1} by α_{k-3} or β_{k-2}.

Since e'_k is the sum of all the paths of the order k, from (6) at $m = k$ each path made up of the element of set I_k is multiplied by β_k. Therefore, element β_k is added to the set I_k, so the order of new paths is increased by one and becomes equal to $k+1$; all obtained paths are finished by the products $\beta_{k-1}\beta_k$ or $\alpha_{k-2}\beta_k$, this does not destroy the hierarchy of elements in the path.

Further, as e'_{k-1} is the sum of all the paths of order $k-1$ made up of the elements of I_{k-1}, from (6) at $m=k$ each such path is multiplied by α_{k-1}. Therefore, the element α_{k-1} is added to the set I_{k-1} which completes the formation of the set $I_{k+1}=(\alpha_0,\dots,\alpha_{k-1};\beta_0,\dots,\beta_k)$. The order of obtained paths is increased by two units and becomes equal to $k+1$, they all are finished by the products $\beta_{k-2}\alpha_{k-1}$ or $\alpha_{k-3}\alpha_{k-1}$, this does not destroy the hierarchy of elements in the path.

From (6) it is clear that the multiplication by β_k and α_{k-1} does not affect the initial element of paths, it preserves the index zero.

Thus e'_{m+1} is the sum of all possible paths of order $m+1$ made up of the elements of the set $I_{m+1}=(\alpha_0,\alpha_1,\dots,\alpha_{m-1};\ \beta_0,\beta_1,\dots,\beta_m)$ with the specified hierarchy of elements in the paths.

The solution e'_m of equation (6) is defined by the formula

$$e'_m=\sum_{s=0}^{[m/2]},\ \sum_{i_1=0}^{m-2s}\ \sum_{i_2=i_1+2}^{m-2s+2}\dots\sum_{i_s=i_{s-1}+2}^{m-2}\Delta'(i_1,i_2,\dots,i_s), \tag{7}$$

where

$$\Delta'(i_1,i_2,\dots,i_s)=\beta_0\dots\beta_{i_1-1}\alpha_{i_1}\beta_{i_1+2}\dots\beta_{i_s-1}\alpha_{i_s}\beta_{i_s+2}\dots\beta_{m-1}, \tag{8}$$

and the first term in the sum (7) (i.e. at $s=0$) is equal to $\beta_0\beta_1\dots\beta_{m-1}$; if $s=[m/2]$, then (7) finishes at $m=2k$ with $\alpha_0\alpha_2\dots\alpha_{2k-2}$, at $m=2k+1$ with $\alpha_0\alpha_2\dots\alpha_{2k-2}\beta_{2k}$ and $\beta_0\alpha_1\alpha_3\dots\alpha_{2k-1}$, respectively.

Now we shall cancel the restriction $\gamma_m=-1, m=1,2,\dots$. From (4) it is easy to see that elements β_m and α_{m-1} always go in pair with $-\gamma_{m+1}$, i.e. they are $-\beta_m/\gamma_{m+1}$ and $-\alpha_{m-1}/\gamma_{m+1}$. It means that considering a path of order $m+1$ one can construct a fraction: in the numerator there will be the path, in the denominator the product of corresponding elements γ_i, respectively. The quantity of multipliers in the denominator is the same as the number of element in the numerator. The indices of multipliers in the denominator are established as follows: if in the numerator appears element β_i, then in the denominator there will necessarily be the multiplier γ_{i+1}, if in the numerator there is the element α_i, in the denominator there will necessarily be the element γ_{i+2}.

Finally, let us establish the sign of fractions. For the path $\beta_0\beta_1\dots\beta_{m-1}$ the sign is $(-1)^m$. Further, each element α_i of rank 2 eliminates two elements of rank 1 from the path, namely $\beta_i\beta_{i+1}$, since the hierarchy of elements in this path is $\beta_{i-1}\alpha_i\beta_{i+2}$. Hence, each element α_i of rank 2 reduces the quantity of multipliers in the path by one, so the sign of fraction with path containing s elements of rank 2, is $(-1)^{m-s}$.

Thus, the solution of equation (4) is finally given by

$$e_m = \sum_{s=0}^{[m/2]} \sum_{i_1=0}^{m-2s} \sum_{i_2=i_1+2}^{m-2s+2} \cdots \sum_{i_s=i_{s-1}+2}^{m-2} (-1)^{m-s} \Delta(i_1, i_2, \dots, i_s), \tag{9}$$

where

$$\Delta(i_1, i_2, \dots, i_s) = \Delta'(i_1, i_2, \dots, i_s)/[\gamma_1 \dots \gamma_{i_1} \gamma_{i_1+2} \dots \gamma_{i_s} \gamma_{i_s+2} \gamma_{i_s+3} \dots \gamma_m]. \tag{10}$$

4. Classification of paths

Let us present the series (2) in a more usable form. For this purpose we carry out the classification of paths which is equivalent to the regrouping of terms in the power series (2).

In the basic path the last two elements are of different ranks, i.e. it is finished by either $\beta_k \alpha_{k+1}$ or $\alpha_k \beta_{k+2}$. In the non-basic path at least the two last elements have identical ranks. We note if the non-basic path ends with elements of rank 2, then the indices of these elements have identical parity. This follows from the hierarchy of elements in the path. $\beta_0, \alpha_0, \beta_0 \alpha_1$ are the trivial basic paths.

Let us consider more closely the structure of paths of order m. All of them are divided into basic and non-basic ones. Basic paths end in pairs either $\beta_{m-3} \alpha_{m-2}$ or $\alpha_{m-3} \beta_{m-1}$. We act with the non-basic path as follows: we discard elements of the same rank while there appears either the first pair of elements of different ranks, or one of trivial paths, i.e. we exclude elements till the appearance of basic path. It is obvious that such basic path has order less than m. Inversely, having a basic path with order less than m and multiplying it (taking into account the hierarchy) by elements of the same rank as the one of last element in the basic path we get a non-basic path of order m, generated by this basic path.

The basic path and the generated by it non-basic ones belong to the same path class.

We unite the different basic paths of order m ending with the same pair of elements of different ranks into a sheaf of basic paths. Thus, the paths of order m contain two sheaves of basic paths. The sheaf of basic paths contains not less than one basic path. Each basic path from the sheaf generates its own class of paths. These classes do not overlap as they are generated by different basic paths from the sheaf. Thus, each path of order m is either a basic path of order m which generates a class of paths of order not smaller than m, or a non-basic path which belongs to a class of paths generated by some basic path of order smaller than m. Hence, each path of order m belongs to one and only one class.

As example we mention all the paths of orders 4 and 5. Paths of order 4 are $\beta_0\beta_1\beta_2\beta_3$, $\alpha_0\beta_2\beta_3$, $\beta_0\alpha_1\beta_3$, $\beta_0\beta_1\alpha_2$, $\alpha_0\alpha_2$. Here the basic paths are $\beta_0\alpha_1\beta_3$, $\beta_0\beta_1\alpha_2$; non-basic paths are all the rest: the first path refers to a class generated by the trivial basic path β_0, the second path is related to a class generated by the basic path $\alpha_0\beta_2$, the last path refers to a class generated by the trivial basic path α_0.

Basic paths of order 5 are $\beta_0\beta_1\beta_2\beta_3\beta_4$, $\alpha_0\beta_2\beta_3\beta_4$, $\beta_0\alpha_1\beta_3\beta_4$, $\beta_0\beta_1\alpha_2\beta_4$, $\beta_0\beta_1\beta_2\alpha_3$, $\alpha_0\alpha_2\beta_4$, $\alpha_0\beta_2\alpha_3$, $\beta_0\alpha_1\alpha_3$.

Basic paths are $\beta_0\beta_1\alpha_2\beta_4$, $\alpha_0\alpha_2\beta_4$, $\beta_0\beta_1\beta_2\alpha_3$, $\alpha_0\beta_2\alpha_3$, here the first two and the last two ones form sheaves.

All the other paths are non-basic: the first path relates to a class generated by the trivial basic path β_0, the second path is related to a class generated by the basic path $\alpha_0\beta_2$, the third path is referred to a class generated by the basic path $\beta_0\alpha_1\beta_3$, the last path refers to a class generated by the trivial basic path $\beta_0\alpha_1$.

Multiplying the basic path unlimitedly by elements of the same rank as the last element of this path, we obtain the expanded class of paths generated by the given basic path.

5. Solution formula of equation (1)

Now we shall rearrange the terms of power series (2) concerning the expanded classes of paths. Let us assume that $e_0 = 1$.

The expanded class of paths generated by the trivial basic path β_0, i.e. β_0, $\beta_0\beta_1$, …, $\beta_0\beta_1\dots\beta_k$, …, defines the function

$$F_0(t) = 1 - \frac{\beta_0}{\gamma_1}t + \frac{\beta_0\beta_1}{\gamma_1\gamma_2}t^2 - \dots + (-1)^k\frac{\beta_0\dots\beta_{k-1}}{\gamma_1\dots\gamma_k}t^k + \dots \tag{11}$$

The expanded class of paths generated by the trivial basic path α_0, i.e. $\alpha_0,\ \alpha_0\alpha_2, \dots,\ \alpha_0\alpha_2\dots\alpha_{2k}, \dots,$ defines the function

$$F_1(t) = -\frac{\alpha_0}{\gamma_2}t^2 + \frac{\alpha_0\alpha_2}{\gamma_2\gamma_4}t^4 - \dots + (-1)^k\frac{\alpha_0\alpha_2\dots\alpha_{2k-2}}{\gamma_2\gamma_4\dots\gamma_{2k}}t^{2k} + \dots \tag{12}$$

The expanded class of paths generated by the trivial basic path $\beta_0\alpha_1$, i.e. $\beta_0\alpha_1,\ \beta_0\alpha_1\alpha_3,\dots,\ \beta_0\alpha_1\alpha_3\dots\alpha_{2k+1},\dots$, defines the function

$$\frac{\beta_0 t}{\gamma_1}F_{1/2}(t) = \tag{13}$$

$$= -\frac{\beta_0 t}{\gamma_1}\left[-\frac{\alpha_1}{\gamma_3}t^2+\frac{\alpha_1\alpha_3}{\gamma_3\gamma_5}t^4-\dots+(-1)^k\frac{\alpha_1\alpha_3\dots\alpha_{2k-1}}{\gamma_3\gamma_5\dots\gamma_{2k+1}}t^{2k}+\dots\right].$$

The meaning of index 1/2 will be explained below.

Let us introduce the functions

$$F_0^{(m)}(t) = 1-\frac{\beta_0}{\gamma_1}t-\dots+(-1)^m\frac{\beta_0\dots\beta_{m-1}}{\gamma_1\dots\gamma_m}t^m, \tag{14}$$

$$F_1^{(m)}(t) = -\frac{\alpha_0}{\gamma_2}t^2+\frac{\alpha_0\alpha_2}{\gamma_2\gamma_4}t^4-\dots+(-1)^m\frac{\alpha_0\alpha_2\dots\alpha_{2m-2}}{\gamma_2\gamma_4\dots\gamma_{2m}}t^{2m}, \tag{15}$$

$$F_{1/2}^{(m)}(t) = \left[-\frac{\alpha_1}{\gamma_3}t^2+\frac{\alpha_1\alpha_3}{\gamma_3\gamma_5}t^4-\dots+(-1)^m\frac{\alpha_1\alpha_3\dots\alpha_{2m-1}}{\gamma_3\gamma_5\dots\gamma_{2m+1}}t^{2m}\right]. \tag{16}$$

Since the order of each path in e'_m is equal to m and all the basic paths for the coefficient e'_m end with the factors $\beta_{m-3}\alpha_{m-2}$ or $\alpha_{m-3}\beta_{m-1}$, all the previous multipliers in these paths form some path of order $m-3$. Thus, in e'_m there are only two sheaves of basic paths: $e'_{m-3}\beta_{m-3}\alpha_{m-2}$ and $e'_{m-3}\alpha_{m-3}\beta_{m-1}$. The hierarchy of arrangement of elements remains in each of these paths, since the last elements in all the paths of order $m-3$ in e'_{m-3} are either α_{m-5} or β_{m-4}.

Both sheaves of basic paths generate the expanded classes of paths

$$e'_{m-3}\alpha_{m-3}\beta_{m-1},\quad e'_{m-3}\alpha_{m-3}\beta_{m-1}\beta_m,\dots, \tag{17}$$

$$e'_{m-3}\beta_{m-3}\alpha_{m-2},\quad e'_{m-3}\beta_{m-3}\alpha_{m-2}\alpha_m,\dots \tag{18}$$

The function corresponding to the expanded class of paths (17) is determined by the series

$$-\frac{e_{m-3}\alpha_{m-3}}{\gamma_{m-1}}\left[-\frac{\beta_{m-1}}{\gamma_m}t^m+\frac{\beta_{m-1}\beta_m}{\gamma_m\gamma_{m+1}}t^{m+1}-\dots\right]. \tag{19}$$

Let us transform the expression in brackets:

$$[\ldots] = \frac{(-1)^{m-1}\gamma_1\ldots\gamma_{m-1}}{\beta_0\ldots\beta_{m-2}}\times$$

$$\times\left[(-1)^m\frac{\beta_0\ldots\beta_{m-1}}{\gamma_1\ldots\gamma_m}t^m+(-1)^{m+1}\frac{\beta_0\ldots\beta_m}{\gamma_1\ldots\gamma_{m+1}}t^{m+1}+\ldots\right]=$$

$$=\frac{(-1)^{m-1}\gamma_1\ldots\gamma_{m-1}}{\beta_0\ldots\beta_{m-2}}\left[F_0(t)-F_0^{(m-1)}(t)\right],$$

this way (19) results in the form

$$(-1)^m\frac{e_{m-3}\gamma_1\ldots\gamma_{m-2}\alpha_{m-3}}{\beta_0\beta_1\ldots\beta_{m-2}}\left[F_0(t)-F_0^{(m-1)}(t)\right].$$

Taking into account that the elementary basic path in our case is $\alpha_0\beta_2$, for the basic paths of order m $(m = 3, 4, \ldots)$ ending with the pair of elements $\alpha_{m-3}\beta_{m-1}$, we finally obtain the function

$$\sum_{m=3}^{\infty} R_m\left[F_0(t)-F_0^{(m-1)}(t)\right], \tag{20}$$

where

$$R_m=(-1)^m\frac{e_{m-3}\gamma_1\ldots\gamma_{m-2}\alpha_{m-3}}{\beta_0\beta_1\ldots\beta_{m-2}}. \tag{21}$$

The function corresponding to the expanded class of paths (18) is given by the series

$$-\frac{e_{m-3}\beta_{m-3}}{\gamma_{m-2}}\left[-\frac{\alpha_{m-2}}{\gamma_m}t^m+\frac{\alpha_{m-2}\alpha_m}{\gamma_m\gamma_{m+2}}t^{m+2}-\ldots\right]. \tag{22}$$

For the simplification of this formula it is necessary to distinguish the cases: $m = 2k$ and $m = 2k + 1$.

Let $m = 2k$. Then the expression in brackets from (22) may be rewritten as

$$\frac{(-1)^{k-1}\gamma_2\ldots\gamma_{2k-2}}{\alpha_0\ldots\alpha_{2k-4}}\times$$

$$\times\left[(-1)^k\frac{\alpha_0\ldots\alpha_{2k-2}}{\gamma_2\ldots\gamma_{2k}}t^{2k}+(-1)^{k+1}\frac{\alpha_0\ldots\alpha_{2k}}{\gamma_2\ldots\gamma_{2k+2}}t^{2k+2}+\ldots\right]=$$

$$= \frac{(-1)^{k-1}\gamma_2\ldots\gamma_{2k-2}}{\alpha_0\ldots\alpha_{2k-4}}\left[F_1(t)-F_1^{(k-1)}(t)\right],$$

and, finally, (22) will have the form

$$(-1)^k\frac{e_{2k-3}\beta_{2k-3}\gamma_2\ldots\gamma_{2k-2}}{\alpha_0\alpha_2\ldots\alpha_{2k-4}}\left[F_1(t)-F_1^{(k-1)}(t)\right].$$

Taking into account that the elementary basic path in the actual case is $\beta_0\beta_1\alpha_2$, for the basic path of any order ending with the pair $\beta_{2k-3}\alpha_{2k-2}$, $k=2,3,\ldots$ we obtain the function

$$\sum_{k=2}^{\infty} P_k\left[F_1(t)-F_1^{(k-1)}(t)\right], \tag{23}$$

where

$$P_k=\frac{(-1)^k e_{2k-3}\beta_{2k-3}\gamma_2\ldots\gamma_{2k-2}}{\alpha_0\alpha_2\ldots\alpha_{2k-4}}. \tag{24}$$

Let $m=2k+1$. Then we can rewrite the brackets in (22) as follows

$$\frac{(-1)^{k-1}\gamma_3\gamma_5\ldots\gamma_{2k-1}}{\alpha_1\alpha_3\ldots\alpha_{2k-3}}\times$$

$$\times\left[(-1)^k\frac{\alpha_1\alpha_3\ldots\alpha_{2k-1}}{\gamma_3\gamma_5\ldots\gamma_{2k+1}}t^{2k+1}+(-1)^{k+1}\frac{\alpha_1\ldots\alpha_{2k+1}}{\gamma_3\ldots\gamma_{2k+3}}t^{2k+3}+\ldots\right]=$$

$$=\frac{(-1)^{k-1}\gamma_3\ldots\gamma_{2k-1}t}{\alpha_1\ldots\alpha_{2k-3}}\left[F_{1/2}(t)-F_{1/2}^{(k-1)}(t)\right].$$

It means the series (22) will be transformed to the function

$$\frac{(-1)^k e_{2k-2}\beta_{2k-2}\gamma_3\ldots\gamma_{2k-3}t}{\alpha_1\alpha_3\ldots\alpha_{2k-3}}\left[F_{1/2}(t)-F_{1/2}^{(k-1)}(t)\right].$$

Since the elementary basic path in this case is $\beta_0\beta_1\beta_2\alpha_3$, for the basic paths of any order ending with the pair of elements $\beta_{2k-2}\alpha_{2k-1}$, $k=2,3,.,$ we obtain the function

$$t\sum_{k=2}^{\infty} S_k\left[F_{1/2}(t)-F_{1/2}^{(k-1)}(t)\right], \tag{25}$$

where

$$S_k = \frac{(-1)^k e_{2k-2}\beta_{2k-2}\gamma_3\gamma_5\dots\gamma_{2k-3}}{\alpha_1\alpha_3\dots\alpha_{2k-3}}, \tag{26}$$

but $S_2 = \frac{e_2\beta_2}{\alpha_1}$.

Thus, the solution of equation (1) can be found in the form

$$u(t) = t^\rho \left\{ F_0(t) - \frac{\beta_0}{\gamma_1} t F_{1/2}(t) + F_1(t) + \sum_{k=2}^{\infty} \Big[R_{k+1}(F_0(t) - F_0^{(k)}(t)) + \right.$$

$$\left. + P_k(F_1(t) - F_1^{(k-1)}(t)) + S_k t \left(F_{1/2}(t) - F_{1/2}^{(k-1)}(t) \right) \Big] \right\}.$$

Let us rewrite functions $F_0(t)$, $F_1(t)$ and $F_{1/2}(t)$ in other forms using the numbers α_m, β_m and γ_m in product forms.

Let

$$\alpha_m = A_n(m+p_1)\dots(m+p_n), \quad \beta_m = B_n(m+q_1)\dots(m+q_n), \quad m = 0,1,\dots$$

Since $\gamma_0 = 0$, we obtain

$$C_0 = -(\rho+m)C_1 - \dots - (\rho+m)\dots(\rho+m-n+1)C_n,$$

and, hence $\gamma_m = C_n m(m+r_1)\dots(m+r_{n-1})$, $m = 1,2,\dots$.

Then

$$F_0(t) = \sum_{k=0}^{\infty} \frac{(q_1)_k \dots (q_n)_k}{(1)_k (r_1)_k \dots (r_{n-1})_k} \frac{(-1)^k B_n^k}{C_n^k} t^k =$$

$$= F\left(q_1, \dots, q_n; 1, r_1, \dots, r_{n-1}; \frac{-B_n t}{C_n} \right),$$

where $F_0(t)$ is the generalized hypergeometric function ${}_nF_{n-1}$ and

$$(a)_k = a(a+1)\dots(a+k-1); \quad (1)_k = k!$$

$$F_1(t) = \sum_{k=1}^{\infty} \frac{\left(\frac{p_1}{2}\right)_k \dots \left(\frac{p_n}{2}\right)_k}{(1)_k \left(1+\frac{r_1}{2}\right)_k \dots \left(1+\frac{r_{n-1}}{2}\right)_k} (-1)^k \frac{A_n^k}{C_n^k} t^{2k} =$$

$$= F\left(\frac{p_1}{2}, \dots, \frac{p_n}{2}; 1, 1+\frac{r_1}{2}, \dots, 1+\frac{r_{n-1}}{2}; -A_n t^2 / C_n \right) - 1.$$

Let us introduce the function

$$F_{1/2}(a_1,\ldots,a_n;1,b_1,\ldots,b_{n-1};t)=$$

$$=F\left(a_1+\frac{1}{2},\ldots,a_n+\frac{1}{2};1+\frac{1}{2},b_1+\frac{1}{2},\ldots,b_{n-1}+\frac{1}{2};t\right)=$$

$$=\sum_{k=0}^{\infty}\frac{\left(a_1+\frac{1}{2}\right)_k\ldots\left(a_n+\frac{1}{2}\right)_k}{\left(1+\frac{1}{2}\right)_k\left(b_1+\frac{1}{2}\right)_k\ldots\left(b_{n-1}+\frac{1}{2}\right)_k}t^k.$$

We call it the generalized hypergeometric function of half-order. Then

$$F_{1/2}(t)=F_{1/2}\left(\frac{p_1}{2},\ldots\frac{p_n}{2};1,1+\frac{r_1}{2},\ldots,1+\frac{r_{n-1}}{2};-\frac{A_n}{C_n t^2}\right)-1.$$

In particular, at $n=2\quad \gamma_m=mC_2(m+\sigma-1)$, where $\sigma=2\rho+C_1/C_2$. $F_0(t)=F\left(q_1,q_2;1,\sigma;-\frac{B_2}{C_2}t\right)$ is a hypergeometric function.

$$F_1(t)=F\left(\frac{p_1}{2},\frac{p_2}{2};1,\frac{\sigma+1}{2};-\frac{A_2}{C_2}t^2\right)-1,$$

$$F_{1/2}(t)=F_{1/2}\left(\frac{p_1}{2},\frac{p_2}{2};1,\frac{\sigma+1}{2};-\frac{A_2}{C_2}t^2\right)-1=$$

$$=\sum_{k=1}^{\infty}(-1)^k\frac{A_2^k}{C_2^k}\frac{(1+p_1)(3+p_1)\ldots(2k-1+p_1)(1+p_2)(3+p_2)\ldots(2k-1+p_2)}{3\cdot5\cdot\ldots\cdot(2k+1)(\sigma+2)(\sigma+4)\ldots(\sigma+2k)}t^{2k}.$$

If $B_2=0$, then the confluent hypergeometric function is $F_0(t)$, if $A_2=0$, then the confluent hypergeometric function is $F_1(t)$.

At $n=2$ and $A_2=1$, $A_1=a+b+1$, $A_0=ab$; $B_2=-(\tau+1)$, $B_1=d-a-b-1-(c+d)\tau$, $B_0=-\lambda$, $C_2=\tau$, $C_1=c\tau$, $C_0=0$ the equation (1) transfers to the Heun differential equation [3, p.129].

6. Structure of coefficient e_m

We write the formula (10) in the form

$$\Delta(i_1,i_2,\ldots,i_s)=\Delta'(i_1,i_2,\ldots,i_s)\gamma_{i_1+1}\ldots\gamma_{i_s+1}/(\gamma_1\gamma_2\ldots\gamma_m). \tag{27}$$

We shall define the product $\beta_0\beta_1\dots\beta_{m-1}$ taking into account the formulas (3) (we assume that $\rho\neq 0$, since the case $\rho=0$ is resulted in simpler formulas without complications).

$$\beta_0\beta_1\dots\beta_{m-1}=$$

$$=\sum_{k=0}^{n} f_k(\rho,m)B_k^m+\sum_{\substack{j_0+j_1+\dots+j_n=m\\ 0\le j_i\le m-1}} B_0^{j_0}B_1^{j_1}\dots B_n^{j_n} f_{j_0,j_1,\dots,j_n}(\rho,m). \tag{28}$$

Here $f_0(\rho,m)\equiv 1$, and

$$f_k(\rho,m)=$$

$$=(\rho-k+1)(\rho-k+2)^2\dots(\rho-1)^{k-1}\rho^k\dots(\rho+m-k)^k$$

$$\times(\rho+m-k+1)^{k+1}(\rho+m-k+2)^{k-2}\dots(\rho+m-1)=$$

$$=\rho^{km}+\dots+(-1)^{k-1}\rho^k 2!3!\dots(k-1)!(m-k)!\dots(m-1)!$$

$f_{j_0,j_1,\dots,j_n}(\rho,m)$ is a polynomial of degree smaller than km (concerning ρ), and the ratio of each of its coefficients to the maximal coefficient among polynomials $f_k(\rho,m)$ tends to zero as $m\to\infty$.

At $\rho=0$ $f_k(\rho,m)=(m-k)!(m-k+1)!\dots(m-1)$.

In the formula (28) we call the first sum as body, and the second sum will be called an additional part of product $\beta_0\beta_1\dots\beta_{m-1}$. Note that the exponential factors of additional part are easily constructed by exhibitors of the body.

From formulas (3) we can see that the coefficients at parameters A_i, B_i, C_i are invariant concerning these parameters (for example the replacement of all the parameters B_i to the corresponding parameters A_i or C_i does not change the coefficients of these parameters). Therefore, replacing in the product $\beta_0\beta_1\dots\beta_{m-1}$ the pair $\beta_i\beta_{i+1}$ to $\alpha_i\gamma_{i+1}$, the coefficients $f_k(\rho,m)$ and $f_{j_0,j_1,\dots,j_n}(\rho,m)$ are also invariant concerning such replacement.

Hence, at such a replacement in formula (28) only the exponential part will change, and it is necessary to take into account the number of pairs $\alpha_i\gamma_{i+1}$ which can be put in the product $\beta_0\beta_1\dots\beta_{m-1}$. For example the body of product $\beta_0\dots\beta_{i_1-1}\alpha_{i_1}\gamma_{i_1+1}\beta_{i_1+2}\dots\beta_{m-1}$ is $\sum\limits_{k=0}^{n} f_k(\rho,m)B_k^{m-2}A_kC_k$ and it means that the body of the sum

$$\sum_{i_1=0}^{m-2}\beta_0\dots\beta_{i_1-1}\alpha_{i_1}\gamma_{i_1+1}\beta_{i_1+2}\dots\beta_{m-1}=\binom{m-1}{1}\sum_{k=0}^{n} f_k(\rho,m)B_k^{m-2}A_kC_k, \tag{29}$$

where $\binom{m-1}{1}=m-1$.

In the remaining part of this sum indices of parameters B_i vary from 0 to $m-3$ and the products $A_l C_p$, $l \neq p$, $l, p = 0, \ldots, n$ are added to the products of degrees of these parameters.

Taking into account (9) and (27) for the coefficients e_m we obtain

$$e_m = \frac{\sum\limits_{k=0}^{n} f_k(\rho, m) \sum\limits_{s=0}^{\left[\frac{m}{2}\right]} (-1)^{m-s} \binom{m-s}{s} B_k^{m-2s} A_k^s C_k^s + additional\ \ part}{\sum\limits_{k=0}^{n} g_k(\rho, m) C_k^m + additional\ \ part}, \tag{30}$$

where $\binom{m-s}{s}$ is a binomial coefficient and

$$g_k(\rho, m) =$$
$$= (\rho-k+2)(\rho-k+3)^2 \ldots (\rho)^{k-1}(\rho+1)^k \ldots (\rho+m-k+1)^k(\rho+m-k+2)^{k-1} \times$$
$$\times (\rho + m) = \rho^{km} + \ldots + (-1)^k \rho^{k-1} 2!3! \ldots (k-1)!(m-k+1)! \ldots (m-1)!m!$$

Since the fraction $\binom{m-s}{s} f_k(\rho, m)/g_k(\rho, m)$ concerning ρ is the ratio of polynomials of identical degrees equal to mk and concerning m it is also some polynomial, consequently the asymptotic form of e_m as $m \to \infty$ is determined by exponents whose bases are the parameters A_i, B_i, C_i.

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