OBITUARY ON

Aleksandr Andrejevich Samarskij

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by G. Stoyan

This famous mathematician was born at 19th February 1919 in Novo-Ivanovskoye in Ukraine, in 1936 began studying physics at Moscow State University (MSU), in 1941 he went as voluntary soldier into the Second World War and became severely wounded yet in the same year on the Moscow front. In 1943 he could leave the sanatorium and in 1944 continued his studies at MSU.

There, in 1948, he defended his dissertation (investigating the influence of perturbations of the form of the domain on the eigenvalues of the Laplace operator) obtaining the first doctoral degree. Since 1953 he led a department at the Institute of Applied Mathematics of the soviet academy, and, after earning the second doctoral degree in 1957, became professor at MSU in 1958 and academician in 1976.

Samarskij was very active in soviet and Russian scientific life, not only as member of numerous academic and state committees, but also by founding two departments (one at MSU, one at the Moscow Physico-Technical Institute) and, in 1990, his own Institute of Mathematical Modelling, this in the frames of the Russian Academy of Sciences. He stayed its director until 1998, and died this year, at 11th February 2008 when he was nearly 89 years old.

The name of Samarskij is well-known for his achievements in numerical mathematics and in computer-aided mathematical modelling. Some of the important monographies and textbooks written or coauthored by A.A. Samarskij, see the not exhaustive list [1]-[15] at the end, were translated for example to English, German, French and Chinese. Moreover, he authored or coauthored about 450 scientific papers. These sometimes were devoted to the development of numerical methods in general, but his speciality was the numerical solution of partial differential equations and mathematical physics. His name is connected, among others, with a practical stability theory for difference schemes, with an approximation of the convection-diffusion equation, and with the locally one-dimensional approximation of multidimensional parabolic and hyperbolic equations.

We give a short account of some of the listed results of Samarskij.

Let $\Omega \in \mathbb{R}^d$ be a bounded domain and $Q_T := \Omega \times (0,T)$ where $T > 0$. Then a parabolic equation is of the form

$$\begin{aligned} \frac{\partial u}{\partial t} =& \mathcal{L}u + f(x,t), \quad (x,t) = (x_1, \dots, x_d, t) \in Q_T, \\ \mathcal{L}u :=& \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} a_i(x,t)\frac{\partial u}{\partial x_i} + a_0(x,t)u, \end{aligned}$$

if the operator is elliptic:

$$0 \le c_0 \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \le c_1 \sum_{i=1}^{d} \xi_i^2 \text{ for all } (x,t) \in Q_T.$$

This equation is to be solved along with initial values at $t = 0$ and boundary values at $S_T := \Gamma \times [0,T]$ where $\Gamma := \partial\Omega$, say homogenous Dirichlet data at S_T:

$$t = 0,\ x \in \Omega : \quad u = u_0(x); \qquad (x,t) \in S_T : \quad u = 0.$$

For the numerical solution of this problem we consider the two-layer difference scheme with solution $y^j,\ j = 0, 1, \dots, m$ in a real Hilbert space H with scalar product $(\cdot,\cdot)$, where $m := T/\tau$, for a given time step $\tau > 0$ and right-hand sides $\varphi^j \in H$:

$$By_t + Ay^j = \varphi^j, \quad j = 0, 1, \dots, m-1, \quad y^0 \text{ is given.}$$

Here, y^j approximates $u(x, j\tau)$ for given points x in Ω, $y_t := (y^{j+1} - y^j)/\tau$ is an approximation of the time derivative $\frac{\partial u}{\partial t}$, A approximates the operator $-\mathcal{L}$. Then, to approximate the parabolic equation it seems sufficient to choose $B = I$ but it turns out that this may lead to severe stability problems necessitating very small time steps.

A better way is to add a "regularisator" R to I giving $B = I + \tau R$. The choice of R needs experience especially in the multidimensional case where it may be possible to take R such that B becomes factorized into one-dimensional operators, see e.g. [16] or [1] and later books of Samarskij. One (non-factorized but customary) choice is $R = A$.

Now, let us assume that the operators do not depend on t and satisfy

$$A = A^* > 0, \quad B = B^* > 0,$$

that is they are selfadjoint and positive definite: $(Av, v) = (v, Av) > 0$ for all $0 \neq v \in H$, etc.

Then, let

$$\|y\|_A = (Ay, y)^{1/2} = \|A^{1/2}y\|$$

denote the energetical norm to az A.

Theorem 1. (see [17] or [2]) *If $\varphi \equiv 0$ then*

$$(Bv, v) \geq \frac{\tau}{2}(Av, v) \text{ for all } v \in H, \text{ or shortly: } B \geq \frac{\tau}{2}A,$$

is necessary and sufficient for the stability of the two-layer scheme with respect to initial values in the norm $\|y\|_A$, i.e. for the estimate

$$\|y^j\|_A \leq \|y^0\|_A, \quad j = 1, \ldots, m.$$

It is well-known that stability with respect to initial values means also stability with respect to right-hand sides, but possibly in a non-convenient norm. The following theorem improves on this by showing an estimate in a convenient norm.

Theorem 2. (see [17] or [2]) *If $\varphi \not\equiv 0$ and $y^0 = 0$, then*

$$B - \frac{\tau}{2}A \geq \varepsilon I, \quad \varepsilon = \text{const} > 0,$$

is sufficient for the stability of the two-layer scheme with respect to right-hand sides in the sense of the following estimate

$$\|y^j\|_A \leq \left(\frac{1}{2\varepsilon} \sum_{k=0}^{j-1} \tau \|\varphi^k\|^2 \right)^{1/2}.$$

The conditions of these theorems have the advantage to be simpler verified also in the case of variable coefficients of the operator $\mathcal{L}$. They are true for $a_0 \leq 0$, $a_i \equiv 0$, $i = 1, \ldots, d$, and for $B = I + \sigma\tau A$ if $\sigma \geq \frac{1}{2} - \frac{1}{\tau\|A\|}$. The stability theories in [18] or [19] are in principle more general - but suppose the solution of auxiliary equations and further estimates.

Among the above conditions, that on B is not necessary and can be replaced by $(Bv, v) > 0$ for all $v \neq 0$ in the (real) Hilbert space H (i.e. B may be nonsymmetric) - what allows handling of the case when all a_i, $i = 1, \ldots, d$, are constant.

For three-layer difference schemes (which are of interest for parabolic and hyperbolic equations), there are similar results like Theorems 1 and 2, by Gulin

[20], a former student of Samarskij, and these have been further generalized in [2].

The above results in their application to the parabolic equation suggest that the case $a_i \neq 0$ for some i may generate difficulties for the stability and numerical solution (even when these parts of $\mathcal{L}$ from a theoretical point of view are not essential). This is indeed the case. In 1965, Samarskij [21] proposed an approximation for the one-dimensional parabolic equation ($d = 1$) which guarantees that the "discrete" maximum principle (or "monotonicity", see e.g. [1], etc.) can be applied unconditionally (and then stability also follows). In short, Samarskij's approximation for the case of a constant coefficient ordinary boundary value problem

$$au'' + bu' + f(x) = 0, \;\; 0 < x < 1, \;\; u(0) = u(1) = 0, \;\; a > 0,$$

looks as follows (where $y_i \approx u(x_i)$ and $x_i = ih, \;\; h = 1/N$ being the steplength of the grid):

$$a\rho(q)\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + b\frac{y_{i+1} - y_{i-1}}{2h} + f(x_i) = 0,$$

$$1 \leq i \leq N-1, \;\; y_0 = y_N = 0,$$

where, instead of $\rho(q) \equiv 1$ which does not assure unconditional monotonicity, Samarskij takes

$$\rho(q) = 1 + \frac{q^2}{1+|q|}, \quad q := \frac{bh}{2a}.$$

That is, for small h (and q) we have $\rho(q) = 1 + O(h^2)$, whereas for large $|q|$ there holds $\rho(q) \leq 1 + |q|$, both properties being necessary for second order approximation in h and for unconditional monotonicity.

Later, other approximations have been proposed which guarantee even convergence uniformly in a - but this does not carry over to higher dimensions. The approximation of Samarskij is nowadays modified for "upwinding" finite element approximations of multidimensional Navier-Stokes equations.

Somewhat controversial is the relation of Samarskij to finite elements and the multigrid algorithms, both well-known components of successful theoretical approaches and effective programs for the solution of boundary value problems. When deriving difference schemes, he and his school used several means, not just replacing derivatives in the equation by difference quotients. So, in joint papers with Tychonov like [22], the general question was considered which functionals of the coefficients of the differential equation

$$(k(x)u')' - q(x)u + f(x) = 0, \;\; 0 < x < 1,$$

$$u(0) = u(1) = 0, \;\; k(x) \geq M_1, \;\; q \geq 0$$

lead to convergent discrete solutions of a general three-point difference scheme (which includes the finite element method with so-called "hat functions"), and best such functionals were obtained. When k and q are allowed to possess jumps, the optimal functionals are not referring to the finite element method.

To construct difference schemes for variable coefficient partial differential equations in general domains, the finite volume technique ("balance equation method") was used in Samarskij's school. But finite element approximations were termed "variational-difference schemes" and considered a complicated nonstandard way to obtain a difference scheme.

Similarly, the multigrid algorithm and the basic papers by Fedorenko [23], [24] and the apparently final result of Bakhvalov [25] were known at the end of the 60's, but the opinion was that this method is - though optimal - but too complicated ("and used only by Fedorenko").

In other words, it was not recognized that there was a lot of possible and necessary scientific work yet, like selection of diverse interpolation and restriction operators, of appropriate smoothing iterations and optimal parameters, other convergence proofs etc., a work that then remained to A. Brandt [26], [27] and W. Hackbusch [28], [29] and many more in the Western world, where soon multigrid became a whole "industry".

In the Samarskij school, all the basic knowledge was present to do this themself (see the book [5]), but even at the end of the 80's when Shaidurov prepared the first Russian multigrid book [30] (Bakhvalov being the lector), he experienced difficulties to bring it through.

The above scientific details are only one side of the activities of Samarskij. For him, a research was not finished when a convergence proof was found and published in a good journal. The methods considered had to be progammed, their real accuracy tested; computing time and memory needed were important issues.

Even more, a problem was tackled since it was of importance for society, outside of mathematics, and the outcome of research had to be evaluated and applied there. He and his collaborators were working on industrial projects and able to discuss the mathematical models used, let this be in elasticity, gas dynamics, or transfer of mass and heat.

What became known to a broader auditory only after the soviet system change is that from 1948 until 1953 Samarskij was involved into the scientific part first of the Soviet atombomb project and later of the H-bomb project where his task was among others to estimate numerically the intensity of explosion.

Later he devoted much time to the magneto-hydrodynamic equations, the modelling of lasers and of the thermonuclear synthesis.

Finally, in his joint book with Mikhailov [15] there appear financial and market problems.

Samarskij founded and led a huge scientific school: more than 100 of his students reached the first doctoral degree ("candidate of sciences"), more than 40 the second doctoral degree ("doctor of sciences"), and to him came not only students from Soviet Union resp. Russia, but also from countries of the socialist community like Eastern Germany, Hungary, Bulgaria, and, somewhat earlier also from China.

At the Department of Numerical Analysis of Eötvös University Budapest, several of his former students were or are working. Samarskij organized 3 joint summer schools with the forerunner of this department and the former Computing Center of the university and himself held lectures, too. These, earlier mostly were connected to the iterative solution of multidimensional difference schemes, but in the 80's often showed a developed scheme of the way of mathematical modelling remembering of interconnected biochemical reaction cycles.

To held a lecture in his seminar at MSU was a honour, and long queues formed of people waiting for this possibility. For an average student, he had about ten minutes per week and therefore a system of "microchiefs" was customary, that is according to the topic on which a student was working, one of Samarskij's already doctorated collaborators was appointed as tutor.

He himself devoted much time and influence to support his students in their later work and position, was interested in their personal life and their further scientific achievements.

References

[1] **Самарский А.А.,** *Введение в теорию разностных схем,* Наука, Москва, 1971. (Samarskij A.A., *Introduction into the theory of difference schemes,* Nauka, Moscow, 1971.)

[2] **Самарский А.А. и Гулин А.В.,** *Устойчивость разностных схем,* Наука, Москва, 1973. (Samarskij A.A. and Gulin A.V., *Stability of difference schemes,* Nauka, Moscow, 1973.)

[3] **Самарский А.А. и Андреев В.В.,** *Разностные методы для эллиптических уравнений,* Наука, Москва, 1976. (Samarskij A.A. and Andrejev V.B., *Difference schemes for elliptic equations,* Nauka, Moscow, 1976.)

[4] **Самарский А.А.,** *Теория разностных схем,* Наука, Москва, 1977. (Samarskij A.A., *The theory of difference schemes,* Nauka, Moscow, 1977; German translation: Samarskij A.A., *Theorie der Differenzenverfahren,* Leipzig, Teubner 1984; 3rd ed., Nauka, Moscow, 1989; English translation: Marcel Dekker, New York, 2001.)

[5] **Самарский А.А. и Николаев Е.С.,** *Методы решения сеточных уравнений,* Наука, Москва, 1978. (Samarskij A.A. and Nikolajev J.S., *Solution methods for discretized equations,* Nauka, Moscow, 1978.)

[6] **Самарский А.А. и Попов Ю.П.,** *Разностные методы решения задач газовой динамики,* Наука, Москва, 1980, 3rd. Russ. ed. 1992. (Samarskij A.A. and Popov Ju.P., *Difference methods for the solution of problems of gas dynamics,* Nauka, Moscow, 1980.)

[7] **Самарский А.А., Лазаров Р.Д. и Макаров В.Л.,** *Разностные схемы для дифференциальных уравнений с обобщенными решениями,* Наука, Москва, 1987. (Samarskij A.A., Lazarov R.D. and Makarov V.L., *Difference schemes for differential equations with generalized solutions,* Nauka, Moscow, 1987.)

[8] **Самарский А.А.,** *Ведение в вычислительные методы, 2. изд.,* Наука, Москва, 1987. (Samarskij A.A., *Numerical methods. An introduction. 2nd corr. ed.,* Nauka, Moscow, 1987.)

[9] **Самарский А.А. и Гулин А.В.,** *Вычислительные методы,* Наука, Москва, 1989. (Samarskij A.A., Gulin A.V., *Numerical methods,* Nauka, Moscow, 1989.)

[10] **Samarskii A.A. and Vabishchevitch P.N.,** *Computational heat transfer,* Wiley, Chichester, 1995.

[11] **Samarskii A.A., Galaktionov V.A., Kurdyumov S.P. and Mikhailov A.P.,** *Blow-up in quasilinear parabolic equations,* de Gruyter, Berlin and Hawthorne, 1995.

[12] **Тихонов А.Н. и Самарский А.А.,** *Уравнения математической физики, 6. изд.,* Наука, Москва, 1999. (Tychonov A.N. and Samarskij A.A., *The equations of mathematical physics, 6th ed.,* Nauka, Moscow, 1999.)

[13] **Самарский А.А. и Вабищевич П.Н.,** *Аддитивные схемы для проблем математическои физики,* Наука, Москва, 1999. (Samarskii A.A. and Vabishchevich P.N., *Additive schemes for problems of mathematical physics,* Nauka, Moscow, 1999.)

[14] **Samarskii A.A., Matus P.P. and Vabishchevich P.N.,** *Difference schemes with operator factors,* Kluwer, Dordrecht, 2002.

[15] **Самарский А.А. и Михайлов А.П.,** *Математическое моделирование. Идеи, методы, примеры, 2. изд.,* ФИЗМАТЛИТ, Москва,

2005. (Samarskij A.A. and Mikhailov A.P., *Mathematical modelling. Ideas, methods, examples, 2nd corr. ed.,* FIZMATLIT, Moscow, 2005.)

[16] **Дьяконов Е.Г.,** Разностные схемы с расщепляющимся оператором для многомерных нестационарных уравнений, *Ж. вычисл. матем. и матем. физ.,* **2** (4) (1962), 549-568. (D'yakonov E.G., Difference schemes with factorized operator for multidimensional time-dependent equations, *USSR J. Num. Mat. Mat. Phys.,* **2** (4) (1962), 549-568.)

[17] **Самарский А.А.,** Необходимые и достаточные условия устойчивости двухслойных разностных схем, *Доклады АН СССР,* **181** (4) (1968), 808-811. (Samarskij A.A., Necessary and sufficient conditions for the stability of two-layer difference schemes, *Doklady Acad. USSR,* **181** (4) (1968), 808-811.)

[18] **Годунов С.К. и Рябенький В.С.,** Спектральные признаки устойчивости краевых задач для несамосопряженных разностных уравнений, *Успехи мат. наук,* **18** (3) (111), (1963), 3-14. (Godunov S.K. and Ryaben'kij V.S., Spectral stability conditions for boundary value problems of non-selfadjoint difference schemes, *Uspechi Mat. Nauk,* **18** (3) (111) (1963), 3-14.)

[19] **Gustafsson B., Kreiss H.-O. and Oliger J.,** *Time dependent problems and difference methods,* Wiley Interscince, New York, 1995.

[20] **Гулин А.В.,** Необходимые и достаточные условия устойчивости трехслойных разностных схем, *Ж. вычисл. матем. и мат. физ.,* **8** (4) (1968), 899-902. (Gulin A.V., Necessary and sufficient conditions for the stability of three-layer difference schemes, *USSR J. Num. Mat. Mat. Phys.,* **8** (4) (1968), 899-902.)

[21] **Самарский А.А.,** О монотонных разностных схем для эллиптических и параболических уравнений в случае несамосопряженного эллиптического оператора, *Ж. вычисл. матем. и матем. физ.,* **5** (3) (1965), 548-551. (Samarskij A.A., On monotone difference schemes for elliptic and parabolic equations in the case of a non-selfadjoint elliptic operator, *USSR J. Num. Mat. Mat. Phys.,* **5** (3) (1965), 548-551.)

[22] **Тихонов А.Н. и Самарский А.А.,** Об однородных разностных схем, *Ж. вычисл. матем. и матем. физ.,* **1** (1) (1961), 5-63. (Tychonov A.N. and Samarskij A.A., On homogeneous difference schemes, *USSR J. Num. Mat. Mat. Phys.,* **1** (1) (1961), 5-63.)

[23] **Федоренко Р.П.,** Релаксационый метод решения эллиптических разностных уравнений, *Ж. вычисл. матем. и матем. физ.,* **1** (5) (1961), 922-927. (Fedorenko R.P., A relaxation method for solving elliptic difference equations, *USSR J. Num. Mat. Mat. Phys.,* **1** (5) (1961), 922-927.)

[24] **Федоренко Р.П.,** О скорости сходимости одного итерационного процесса, *Ж. вычисл. матем. и матем. физ.,* **4** (3) (1964), 559-564. (Fedorenko R.P., On the speed of convergence of an iterative process, *USSR J. Num. Mat. Mat. Phys.,* **4** (3) (1964), 559-564.)

[25] **Бахвалов С.Н.,** О сходимости одного релаксационного метода при естественных ограничениях на эллиптический оператор, *Ж. вычисл. матем. и матем. физ.,* **6** (5) (1966), 101-135. (Bakhvalov S.N., The convergence of a relaxation method with natural constraints on the elliptic operator, *USSR J. Num. Mat. Mat. Phys.,* **6** (5) (1966), 101-135.)

[26] **Brandt A.,** Multi-level adaptive technique (MLAT) for fast numerical solution to boundary value problems, *Proc. 3. Intern. Conf. Numerical Methods in Fluid Mechanics, Paris 1972,* eds. Cabannes H. and Temam R., Lecture Notes in Physics **18**, Springer Verlag, 1973, 82-89.

[27] **Brandt A.,** Multi-level adaptive solutions to boundary value problems, *Math. Comp.,* **31** (1977), 333-390.

[28] **Hackbusch W.,** *Ein iteratives Verfahren zur schnellen Auflösung elliptischer Randwertprobleme,* Report 76-12, Institut für Angewandte Mathematik, Universität Köln 1976.

[29] **Hackbusch W.,** A fast numerical method for elliptic boundary value problems with variable coefficients, *Proc. 2. GAMM-Conference on Numerical Methods in Fluid Mechanics,* eds. Hirschel E.H. and Geller W., DFVLR, Köln 1977, 50-57.

[30] **Шайдуров В.В.,** *Многосеточные методы конечных элементов,* Наука, Москва, 1989. (Shaidurov V.V., *Multigrid methods for finite elements,* Nauka, Moscow, 1989, English transl. Kluwer, Dordrecht 1995.)