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REGULAR INTEGERS MODULO n

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. Let $n = p_1^{\nu_1} \cdots p_r^{\nu_r} > 1$ be an integer. An integer a is called regular (mod n) if there is an integer x such that $a^2x \equiv a \pmod{n}$. Let $\varrho(n)$ denote the number of regular integers a (mod n) such that $1 \leq a \leq n$. Here $\varrho(n) = (\phi(p_1^{\nu_1}) + 1) \cdots (\phi(p_r^{\nu_r}) + 1)$, where $\phi(n)$ is the Euler function. In this paper we first summarize some basic properties of regular integers (mod n). Then in order to compare the rates of growth of the functions $\varrho(n)$ and $\phi(n)$ we investigate the average orders and the extremal orders of the functions $\varrho(n)/\phi(n)$, $\phi(n)/\varrho(n)$ and $1/\varrho(n)$.

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1. Introduction

Let $n > 1$ be an integer. Consider the integers a for which there exists an integer x such that $a^2x \equiv a \pmod{n}$. In the background of this property is that an element a of a ring R is said to be regular (following J. von Neumann) if there is an $x \in R$ such that $a = axa$. In case of the ring $\mathbb{Z}_n$ this is exactly the condition above.

Properties of these integers were investigated by J. Morgado [7], [8], who called them regular (mod n). In a recent paper O. Alkam and E.A. Osba [1] using ring theoretic considerations rediscovered some of the statements proved elementarily by J. Morgado. It was observed in [7], [8] that $a > 1$ is regular (mod n) if and only if the gcd (a, n) is a unitary divisor of n. We recall that d is said to be a unitary divisor of n if $d \mid n$ and gcd $(d, n/d) = 1$, notation $d \parallel n$.

These integers occur in the literature also in an other context. It is said that an integer a possesses a weak order (mod n) if there exists an integer $k \geq 1$ such that $a^{k+1} \equiv a \pmod{n}$. Then the weak order of a is the smallest k with this property, see [4], [2]. It turns out that a is regular (mod n) if and only if a possesses a weak order (mod n).

Let $\text{Reg}_n = \{a : 1 \leq a \leq n,\ a \text{ is regular} \pmod{n}\}$ and let $\varrho(n) = \#\text{Reg}_n$ denote the number of regular integers a (mod n) such that $1 \leq a \leq n$. This function is multiplicative and $\varrho(p^\nu) = \phi(p^\nu) + 1 = p^\nu - p^{\nu-1} + 1$ for every prime power p^ν ($\nu \geq 1$), where ϕ is the Euler function. Consequently, $\varrho(n) = \sum_{d||n} \phi(d)$ for every $n \geq 1$, also $\phi(n) < \varrho(n) \leq n$ for every $n > 1$, and $\varrho(n) = n$ if and only if n is squarefree, see [7], [4], [1].

Let us compare the functions $\varrho(n)$ and $\phi(n)$. The first few values of $\varrho(n)$ and $\phi(n)$ are given by the next tables ($\varrho(n)$ is sequence $A055653$ in Sloane's On-Line Encyclopedia of Integer Sequences [10]). Note that $\varrho(n)$ is even iff $n \equiv 2 \pmod{4}$, and $\sqrt{n} \leq \varrho(n) \leq n$ for every $n \geq 1$, see [1].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\varrho(n)$	1	2	3	3	5	6	7	5	7	10	11	9	13	14	15
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\varrho(n)$	9	17	14	19	15	21	22	23	15	21	26	19	21	29	30
$\phi(n)$	8	16	6	18	8	12	10	22	8	20	12	18	12	28	8

Figure 1 is a plot of the function $\varrho(n)$ for $1 \leq n \leq 10\,000$.

For the Euler ϕ-function

$$\lim_{x\to\infty} \frac{1}{x^2} \sum_{n\leq x} \phi(n) = \frac{3}{\pi^2} \approx 0.3039.$$

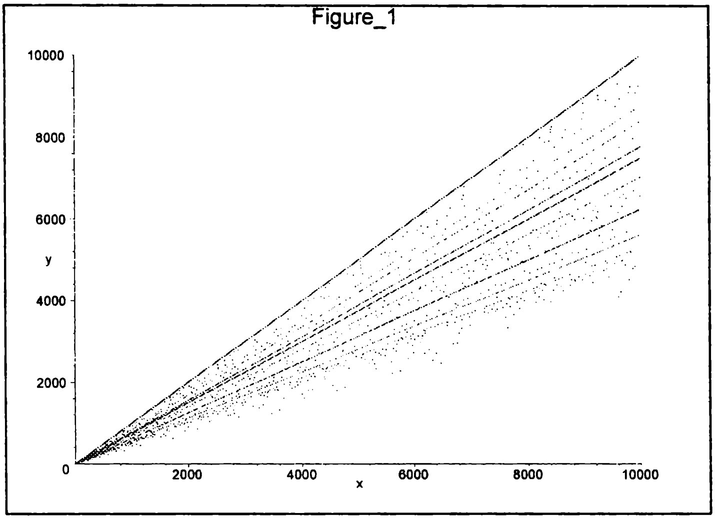
The average order of the function $\varrho(n)$ was considered in [4], [2]. One has

$$\lim_{x\to\infty} \frac{1}{x^2} \sum_{n\leq x} \varrho(n) = \frac{1}{2} A \approx 0.4407,$$

where

$$A = \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = \zeta(2) \prod_p \left(1 - \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4}\right) \approx 0.8815$$

is the so called quadratic class-number constant. For its numerical evaluation see [9].



More exactly,

$$\sum_{n \le x} \varrho(n) = \frac{1}{2} A x^2 + R(x),$$

where $R(x) = O(x \log^3 x)$, given in [4] using elementary arguments. This was improved into $R(x) = O(x \log^2 x)$ in [12], and into $R(x) = O(x \log x)$ in [3], using analytic methods. Also, $R(x) = \Omega_{\pm}(x\sqrt{\log\log x})$, see [3].

In this paper we first summarize some basic properties of regular integers (mod n). We give also their direct proofs, because the proofs of [7], [8] are lengthy and those of [1] are ring theoretical.

Then in order to compare the rates of growth of the functions $\varrho(n)$ and $\phi(n)$ we investigate the average orders and the extremal orders of the functions $\varrho(n)/\phi(n)$, $\phi(n)/\varrho(n)$ and $1/\varrho(n)$. The study of the minimal order of $\varrho(n)$ was initiated in [1].

2. Characterization of regular integers (mod n)

The integer $a = 0$ and those coprime to n are regular (mod n) for each $n > 1$. If $a \equiv b$ (mod n), then a and b are regular (mod n) simultaneously. If a and b are regular (mod n), then ab is also regular (mod n).

In what follows let $n > 1$ be of the canonical form $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$.

Theorem 1. *For an integer $a \geq 1$ the following assertions are equivalent:*

i) a is regular (mod n),

ii) for every $i \in \{1, ..., r\}$ either $p_i \nmid a$ or $p_i^{\nu_i} \mid a$,

iii) $(a, n) = (a^2, n)$,

iv) $(a, n) \mid\mid n$,

v) $a^{\phi(n)+1} \equiv a$ (mod n),

vi) there exists an integer $k \geq 1$ such that $a^{k+1} \equiv a$ (mod n).

Proof. i) $\Rightarrow$ ii). If $a^2x \equiv a$ (mod n) for an integer x, then $a(ax-1) \equiv 0$ (mod $p_i^{\nu_i}$) for every i. We have two cases: $p_i \nmid a$ and $p_i \mid a$. In the second case, since $(a, ax-1) = 1$, we obtain that $a \equiv 0$ (mod $p_i^{\nu_i}$).

ii) $\Rightarrow$ i). If $p_i^{\nu_i} \mid a$, then $a^2x \equiv a$ (mod $p_i^{\nu_i}$) for any x. If $p_i \nmid a$, then the linear congruence $ax \equiv 1$ (mod $p_i^{\nu_i}$) has solutions in x and obtain also $a^2x \equiv a$ (mod $p_i^{\nu_i}$).

ii) $\Leftrightarrow$ iii). Follows at once by the property of the gcd.

ii) $\Leftrightarrow$ iv) Follows at once by the definition of the unitary divisors (the unitary divisors of a prime power p^ν are 1 and p^ν).

ii) $\Rightarrow$ v) ([1]) If $p_i^{\nu_i} \mid a$, then $a^{\phi(n)+1} \equiv a$ (mod $p_i^{\nu_i}$). If $p_i \nmid a$, then using Euler's theorem, $a^{\phi(n)+1} \equiv a(a^{\phi(p_i^{\nu_i})})^{\phi(n)/\phi(p_i^{\nu_i})} \equiv a$ (mod $p_i^{\nu_i}$). Therefore $a^{\phi(n)+1} \equiv a$ (mod $p_i^{\nu_i}$) for every i and $a^{\phi(n)+1} \equiv a$ (mod n).

v) $\Rightarrow$ i) ([1]) If $a^{\phi(n)+1} \equiv a$ (mod n), then $a^2a^{\phi(n)-1} \equiv a$ (mod n), hence $a^2x \equiv a$ (mod n) is verified for $x = a^{\phi(n)-1}$ (which is the von Neumann inverse of a in $\mathbb{Z}_n$).

v) $\Rightarrow$ vi) Immediate by taking $k = \phi(n)$.

vi) $\Rightarrow$ i) If $a^{k+1} \equiv a$ (mod n) for an integer $k \geq 1$, then $a^2x \equiv a$ (mod n) holds for $x = a^{k-1}$, finishing the proof.

Note that the proof of i) $\Leftrightarrow$ v) given in [8] uses Dirichlet's theorem on arithmetic progressions, which is unnecessary.

Theorem 2. *The function $\varrho(n)$ is multiplicative and $\varrho(p^\nu) = p^\nu - p^{\nu-1} + 1$ for every prime power p^ν ($\nu \geq 1$). For every $n \geq 1$,*

$$\varrho(n) = \sum_{d \mid\mid n} \phi(d).$$

Proof. By Theorem 1, a is regular (mod n) iff for every $i \in \{1, ..., r\}$ either $p_i \nmid a$ or $p_i^{\nu_i} \mid a$.

Let $a \in \text{Reg}_n$. If $p_i \nmid a$ for every i, then $(a, n) = 1$, the number of these integers a is $\phi(n)$. Suppose that $p_i^{\nu_i} \mid a$ for exactly one value i and that for all $j \neq i$, $(p_j, a) = 1$. Then $a = bp_i^{\nu_i}$, where $1 \leq b \leq n/p_i^{\nu_i}$ and $(b, n/p_i^{\nu_i}) = 1$.

The number of such integers a is $\phi(n/p_i^{\nu_i})$. Now suppose that $p_i^{\nu_i} \mid a$, $p_j^{\nu_j} \mid a$, $i < j$, and for all $k \neq i, k \neq j$, $(p_i, a) = (p_j, a) = 1$. Then $a = cp_i^{\nu_i}p_j^{\nu_j}$, where $1 \leq c \leq n/(p_i^{\nu_i}p_j^{\nu_j})$ and $(c, n/(p_i^{\nu_i}p_j^{\nu_j})) = 1$. The number of such integers a is $\phi(n/(p_i^{\nu_i}p_j^{\nu_j}))$, etc. We obtain

$$\varrho(n) = \phi(n) + \sum_{1\leq i\leq r} \phi(n/p_i^{\nu_i}) + \sum_{1\leq i<j\leq r} \phi(n/p_i^{\nu_i}p_j^{\nu_j}) + ... + \phi(n/(p_1^{\nu_1}\cdots p_r^{\nu_r})).$$

Let $y_i = \phi(p_i^{\nu_i})$, $1 \leq i \leq r$, and $y = y_1 \cdots y_r$. Then $\phi(n) = y$ and

$$\varrho(n) = y + \sum_{1\leq i\leq r} \frac{y}{y_i} + \sum_{1\leq i<j\leq r} \frac{y}{y_i y_j} + ... + \frac{y}{y_1\cdots y_r} =$$

$$= (y_1 + 1)\cdots(y_r + 1) = (\phi(p_1^{\nu_1}) + 1)\cdots(\phi(p_r^{\nu_r}) + 1).$$

The given representation of $\varrho(n)$ now follows at once taking into account that the unitary convolution preserves the multiplicativity of functions, see for example [5].

Another method, see [7]: Group the integers $a \in \{1, 2, ..., n\}$ according to the value (a, n). Here $(a, n) = d$ if and only if $(j, n/d) = 1$, where $a = jd$, $1 \leq j \leq n/d$, hence the number of integers a with $(a, n) = d$ is $\phi(n/d)$. According to Theorem 1, a is regular (mod n) if and only if $d = (a, n) \parallel n$, and obtain that

$$\varrho(n) = \sum_{d\parallel n} \phi(n/d) = \sum_{d\parallel n} \phi(d).$$

Now the multiplicativity of $\varrho(n)$ is a direct consequence of this representation.

Let $S(n)$ denote the sum of regular integers $a \in \mathrm{Reg}_n$. We give a simple formula for $S(n)$, not considered in the cited papers, which is analogous to

$$\sum_{1\leq a\leq n, (a,n)=1} a = n\phi(n)/2 \qquad (n > 1).$$

Theorem 3. *For every $n \geq 1$,*

$$S(n) = \frac{n(\varrho(n) + 1)}{2}.$$

Proof. Similar to the counting procedure above or by grouping the integers $a \in \{1, 2, ..., n\}$ according to the value (a, n):

$$S(n) = \sum_{a\in \mathrm{Reg}_n} a = \sum_{d\parallel n} \sum_{\substack{a\in \mathrm{Reg}_n \\ (a,n)=d}} a = \sum_{d\parallel n} d \sum_{\substack{j=1 \\ (j,n/d)=1}}^{n/d} j =$$

$$= n + \sum_{\substack{d||n \\ d<n}} d\frac{n\phi(n/d)}{2d} = n + \frac{n}{2}\sum_{\substack{d||n \\ d<n}} \phi(n/d) = \frac{n(\varrho(n)+1)}{2}.$$

3. Average orders

Theorem 4. *For the quotient $\varrho(n)/\phi(n)$ we have*

$$\sum_{n\le x} \frac{\varrho(n)}{\phi(n)} = Bx + O(\log^2 x),$$

where $B = \pi^2/6 \approx 1.6449$.

Proof. By Theorem 2, $\varrho(p^\nu)/\phi(p^\nu) = 1 + 1/\phi(p^\nu)$ for every prime power p^ν ($\nu \ge 1$). Hence, taking into account the multiplicativity, for every $n \ge 1$,

$$\frac{\varrho(n)}{\phi(n)} = \sum_{d||n} \frac{1}{\phi(d)}.$$

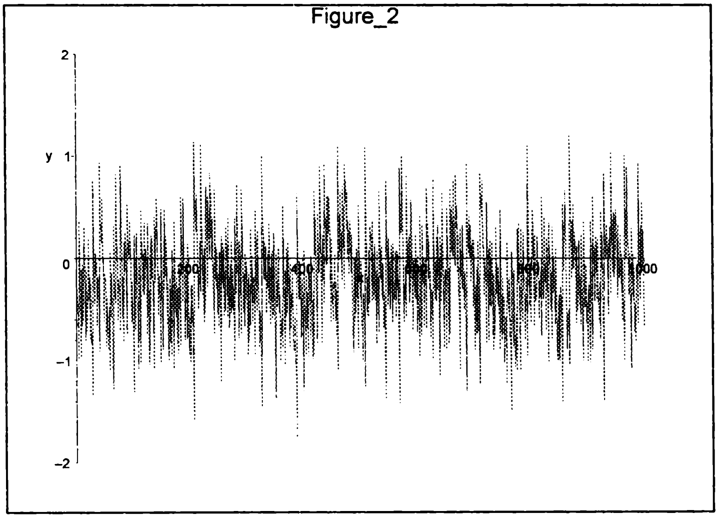
Using this representation (given also in [1]) we obtain

$$\sum_{n\le x} \frac{\varrho(n)}{\phi(n)} = \sum_{\substack{de\le x \\ (d,e)=1}} \frac{1}{\phi(d)} = \sum_{d\le x} \frac{1}{\phi(d)} \sum_{\substack{e\le x/d \\ (e,d)=1}} 1 =$$

$$= \sum_{d\le x} \frac{1}{\phi(d)} \left(\frac{\phi(d)x}{d^2} + O(2^{\omega(d)}) \right) = x\sum_{d\le x} \frac{1}{d^2} + O\left(\sum_{d\le x} \frac{2^{\omega(d)}}{\phi(d)} \right),$$

where $\omega(d)$ denotes, as usual, the number of distinct prime factors of d. Furthermore, let $\tau(n)$ and $\sigma(n)$ denote the number and the sum of divisors of n, respectively. Using that $\phi(n)\sigma(n) \gg n^2$, we have $2^{\omega(d)}/\phi(d) \ll \tau(d)\sigma(d)/d^2$. Here $\sum_{d\le x} \tau(d)\sigma(d) \ll x^2 \log x$, according to a result of Ramanujan, and obtain by partial summation that the error term is $O(\log^2 x)$.

Figure 2 is a plot of the error term $\sum_{n\le x} \varrho(n)/\phi(n) - Bx$ for $1 \le x \le 1000$.



Consider now the quotient $f(n) = \phi(n)/\varrho(n)$, where $f(n) \leq 1$. According to a well-known result of H. Delange, $f(n)$ has a mean value given by

$$C = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\nu=1}^{\infty} \frac{f(p^\nu)}{p^\nu}\right) =$$

$$= \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \left(1 - \frac{1}{p}\right)\sum_{\nu=1}^{\infty} \frac{1}{p^\nu - p^{\nu-1} + 1}\right).$$

Here $C \approx 0.6875$, which can be obtained using that for every $k \geq 1$,

$$C = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \left(1 - \frac{1}{p}\right)\sum_{\nu=1}^{k} \frac{1}{p^\nu - p^{\nu-1} + 1} + \frac{1}{p^k r_p}\right),$$

where $p - 1 < r_p < p$ for each prime p.

We prove an asymptotic formula.

Theorem 5.

$$\sum_{n \leq x} \frac{\phi(n)}{\varrho(n)} = Cx + O((\log x)^{5/3}(\log\log x)^{4/3}).$$

Proof. For $f(n) = \phi(n)/\varrho(n)$ let

$$f(n) = \sum_{d|n} \frac{\phi(d)}{d}\, v(n/d),$$

that is, in terms of the Dirichlet convolution, $f = \phi/E * v$, $f = \mu/E * I * v$, $v = f * I/E * \mu$, where $\mu(n)$ is the Möbius function, $E(n) = n$, $I(n) = 1$ $(n \geq 1)$.

The function $v(n)$ is multiplicative, for every prime power p^ν $(\nu \geq 1)$,

$$v(p^\nu) = f(p^\nu) - \left(1 - \frac{1}{p}\right)\left(f(p^{\nu-1}) + \frac{1}{p}f(p^{\nu-2}) + \ldots + \frac{1}{p^{\nu-2}}f(p) + \frac{1}{p^{\nu-1}}\right).$$

and $v(p) = 0$, $|v(p^2)| \leq 1/p$ for every prime p. Also,

$$f(p^\nu) = \frac{p^\nu - p^{\nu-1}}{p^\nu - p^{\nu-1} + 1} = \frac{1 - 1/p}{1 - (1/p - 1/p^\nu)} =$$

$$= \left(1 - \frac{1}{p}\right)\left(1 + \left(\frac{1}{p} - \frac{1}{p^\nu}\right) + \left(\frac{1}{p} - \frac{1}{p^\nu}\right)^2 + \ldots\right) =$$

$$= \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^\nu} - \frac{1}{p^\nu} + O\left(\frac{1}{p^{\nu+1}}\right)\right),$$

and obtain that for every fixed $\nu \geq 3$,

$$f(p^\nu) = 1 - \frac{1}{p^\nu} + O\left(\frac{1}{p^{\nu+1}}\right),$$

consequently,

$$v(p^\nu) = 1 - \frac{1}{p^\nu} + O\left(\frac{1}{p^{\nu+1}}\right) - \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{\nu-1}} - \frac{\nu-1}{p^{\nu-1}} + O\left(\frac{1}{p^\nu}\right)\right)$$

$$(*) \qquad v(p^\nu) = \frac{\nu-1}{p^{\nu-1}} + O\left(\frac{1}{p^\nu}\right).$$

It follows that there exists x_0 such that for every prime $p > x_0$ and for every $\nu \geq 3$,

$$(**) \qquad |v(p^\nu)| \leq \frac{1}{p^{3\nu/5}}.$$

Now we show that

$$\sum_{n \leq x} v(n) = O(\log x), \quad \sum_{n > x} \frac{v(n)}{n} = O\left(\frac{\log x}{x}\right).$$

We deduce the first estimate, the second one will follow by partial summation. Let $\mathcal{M}_1 = \{n : p \mid n \Rightarrow p \leq x_0\}$, $\mathcal{M}_2 = \{n : p \mid n \Rightarrow p^3 \mid n, p > x_0\}$, $\mathcal{M}_3 = \{n : p \mid n \Rightarrow p^2 \mid n, p^3 \nmid n, p > x_0\}$. If $v(n) \neq 0$, then n can be written uniquely as $n = n_1 n_2 n_3$, where $n_1 \in \mathcal{M}_1$, $n_2 \in \mathcal{M}_2$, $n_3 \in \mathcal{M}_3$. We have the following estimates.

If $n_3 \in \mathcal{M}_3$, then $n_3 = m^2$ with $|\mu(m)| = 1$. Using $|v(p^2)| \le 1/p$ we have $|v(n_3)| \le 1/m$, and

$$\sum_{\substack{n_3 \le x \\ n_3 \in \mathcal{M}_3}} v(n_3) \ll \sum_{m \le \sqrt{x}} \frac{|\mu(m)|}{m} \ll \log x.$$

By (**), for x_0 sufficiently large,

$$\sum_{\substack{n_2 \le x \\ n_2 \in \mathcal{M}_2}} v(n_2) \ll \prod_{p > x_0} (1 + |v(p^3)| + |v(p^4)| + \ldots) \ll$$

$$\ll \prod_{p > x_0} \left(1 + \frac{1}{p^{9/5}} + \frac{1}{p^{12/5}} + \ldots\right) \ll \prod_{p > x_0} \left(1 + \frac{2}{p^{9/5}}\right) < \infty.$$

Using (*) we also have

$$\sum_{\substack{n_1 \le x \\ n_1 \in \mathcal{M}_1}} v(n_1) \ll \prod_{p \le x_0} \left(1 + \frac{1}{p} + |v(p^3)| + |v(p^4)| + \ldots\right) < \infty.$$

Hence

$$\sum_{n \le x} v(n) = \sum_{n_1 n_2 n_3 \le x} |v(n_1)|\,|v(n_2)|\,|v(n_3)| =$$

$$= \sum_{n_1 n_2 \le x} |v(n_1)|\,|v(n_2)| \sum_{n_3 \le x/n_1 n_2} |v(n_3)| \ll \log x.$$

Now applying the following well-known result of Walfisz,

$$\sum_{n \le x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + O((\log x)^{2/3} (\log\log x)^{4/3})$$

we have

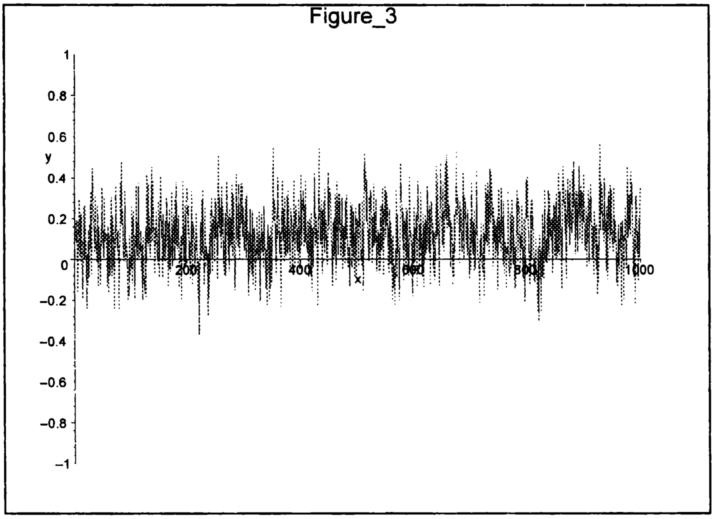
$$\sum_{n \le x} f(n) = \sum_{d \le x} v(d) \sum_{e \le x/d} \frac{\phi(e)}{e} =$$

$$= \frac{6}{\pi^2} x \sum_{d \le x} \frac{v(d)}{d} + O((\log x)^{2/3} (\log\log x)^{4/3} \sum_{d \le x} v(d)) =$$

$$= \frac{6}{\pi^2} x \sum_{d=1}^{\infty} \frac{v(d)}{d} + O((\log x)^{5/3} (\log\log x)^{4/3}),$$

ending the proof of Theorem 5.

Figure 3 is a plot of the error term $\sum_{n \le x} \phi(n)/\varrho(n) - Cx$ for $1 \le x \le 1000$.



Theorem 6.

$$\sum_{n\le x} \frac{1}{\varrho(n)} = D\log x + E + O\left(\frac{\log^9 x}{x}\right),$$

where D and E are constants,

$$D = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_p \left(1 - \frac{p(p-1)}{p^2-p+1} \sum_{\nu=1}^{\infty} \frac{1}{p^\nu(p^\nu - p^{\nu-1}+1)}\right).$$

Proof. Write

$$\frac{1}{\varrho(n)} = \sum_{\substack{de=n\\(d,e)=1}} \frac{h(d)}{\phi(e)},$$

where h is multiplicative and for every prime power p^ν ($\nu \ge 1$),

$$\frac{1}{\varrho(p^\nu)} = h(p^\nu) + \frac{1}{\phi(p^\nu)}, \quad h(p^\nu) = -\frac{1}{\phi(p^\nu)(\phi(p^\nu)+1)},$$

therefore $h(n) \ll 1/\phi^2(n)$. We need the following known result (cf. for example [6], p. 43)

$$\sum_{\substack{n\le x\\(n,k)=1}} \frac{1}{\phi(n)} = Ka(k)\left(\log x + \gamma + b(k)\right) + O\left(2^{\omega(k)}\frac{\log x}{x}\right),$$

where γ is Euler's constant,

$$K=\frac{\zeta(2)\zeta(3)}{\zeta(6)},\ a(k)=\prod_{p|k}\left(1-\frac{p}{p^2-p+1}\right)\le\frac{\phi(k)}{k},$$

$$b(k)=\sum_{p|k}\frac{\log p}{p-1}-\sum_{p\nmid k}\frac{\log p}{p^2-p+1}\ll\frac{\psi(k)\log k}{\phi(k)},\quad \text{with}\ \psi(k)=k\prod_{p|k}\left(1+\frac{1}{p}\right).$$

We have

$$\sum_{n\le x}\frac{1}{\varrho(n)}=\sum_{d\le x}h(d)\sum_{\substack{e\le x/d\\(e,d)=1}}\frac{1}{\phi(e)}=$$

$$=K\left((\log x+\gamma)\sum_{d\le x}h(d)a(d)+\sum_{d\le x}h(d)a(d)(b(d)-\log d)\right)+$$

$$+O\left(\frac{\log x}{x}\sum_{d\le x}d|h(d)|2^{\omega(d)}\right),$$

and we obtain the given result with the constants

$$D=K\sum_{n=1}^{\infty}h(n)a(n),\ E=K\gamma\sum_{n=1}^{\infty}h(n)a(n)+K\sum_{n=1}^{\infty}h(n)a(n)(b(n)-\log n),$$

these series being convergent taking into account the estimates above. For the error terms,

$$\sum_{n>x}|h(n)|a(n)\ll\sum_{n>x}\frac{1}{n\phi(n)}\ll\sum_{n>x}\frac{\sigma(n)}{n^3}\ll\frac{1}{x},\ \sum_{n>x}|h(n)|a(n)\log n\ll\frac{\log x}{x},$$

$$\sum_{n>x}|h(n)a(n)b(n)|\ll\sum_{n>x}\frac{\tau^3(n)\log n}{n^2}\ll\frac{\log^8 x}{x},$$

using that $\sum_{n\le x}\tau^3(n)\ll x\log^7 x$ (Ramanujan), and

$$\sum_{n\le x}n|h(n)|2^{\omega(n)}\ll\sum_{n\le x}\frac{\tau^3(n)}{n}\ll\log^8 x.$$

4. Extremal orders

Since $\varrho(n) \le n$ for every $n \ge 1$ and $\varrho(p) = p$ for every prime p, it is immediate that $\limsup_{n\to\infty} \varrho(n)/n = 1$. The minimal order of $\varrho(n)$ is also the same as that of $\phi(n)$, namely,

Theorem 7.

$$\liminf_{n\to\infty} \frac{\varrho(n)\log\log n}{n} = e^{-\gamma}.$$

Proof. We apply the following result ([11], Corollary 1): If f is a nonnegative real-valued multiplicative arithmetic function such that for each prime p

i) $\rho(p) := \sup_{\nu\ge 0} f(p^\nu) \le (1-1/p)^{-1}$, and

ii) there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ satisfying $f(p^{e_p}) \ge 1 + 1/p$,

then

$$\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^{\gamma} \prod_p \left(1 - \frac{1}{p}\right)\rho(p).$$

Take $f(n) = n/\varrho(n)$, where $f(p^\nu) = (1-1/p+1/p^\nu)^{-1} < (1-1/p)^{-1} = \rho(p)$, and for $e_p = 3$,

$$f(p^3) > 1 + \frac{p^2-1}{p^3-p^2+1} > 1 + \frac{1}{p}$$

for every prime p.

It is immediate that $\liminf_{n\to\infty} \varrho(n)/\phi(n) = 1$. The maximal order of $\varrho(n)/\phi(n)$ is given by

Theorem 8.

$$\limsup_{n\to\infty} \frac{\varrho(n)}{\phi(n)\log\log n} = e^{\gamma}.$$

Proof. Now let $f(n) = \varrho(n)/\phi(n)$ in the result given above. Here

$$f(p^\nu) = 1 + \frac{1}{p^\nu - p^{\nu-1}} \le 1 + \frac{1}{p-1} = \left(1 - \frac{1}{p}\right)^{-1} = \rho(p),$$

and for $e_p = 1$, $f(p) > 1 + 1/(p-1) > 1 + 1/p$ for every prime p.

5. The plots were produced using Maple. The function $\varrho(n)$ was generated by the following procedure:

```
rho:= proc(n) local x, i:  x:= 1:
```

```
for i from 1 to nops(ifactors(n)[ 2 ]) do
p_i:=ifactors(n)[2][i][1]:  a_i:=ifactors(n)[2][i][2];
x := x*(p_i^a_i-p_i^(a_i-1)+1):  od:  RETURN(x) end;
```

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