Annales Univ. Sci. Budapest., Sect. Comp. 29 (2008) 245-261

UNIFORM WEIGHTED CONVERGENCE OF GRÜNWALD INTERPOLATION PROCESS ON THE ROOTS OF JACOBI POLYNOMIALS

L. Szili (Budapest, Hungary)[1]

Dedicated to Professor Imre Kátai
on the occasion of his 70th birthday

Abstract. The aim of this paper is to investigate of weighted convergence the interpolation process on the root system of Jacobi polynomials introduced by G. Grünwald [2].

Mathematical Subject Classification: 41A05, 41A10

1. Introduction and notations

For an interpolatory point system

$$X_n := \{a < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < b\} \subset (a,b)$$
$$(n \in \mathbf{N} := \{1, 2, \ldots\})$$

G. Grünwald [2] investigated first the interpolation process

$$G_n\big(f, X_n; x\big) := \sum_{k=1}^{n} f(x_{k,n})\ell_{k,n}^2\big(X_n; x\big)$$
$$\big(x \in (a,b),\ n \in \mathbf{N}\big),$$

[1]The research was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T047132.

where

$$\ell_{k,n}(x) := \ell_{k,n}\big(X_n, x\big) \qquad (k = 1, 2, \ldots, n,\ n \in \mathbf{N})$$

is the kth the fundamental polynomial of Lagrange interpolation with respect to X_n. He proved the following result:

If X_n $(n \in \mathbf{N})$ is a strongly normal point system and f is a continuous function on $[-1,1]$ ($f \in C[-1,1]$ shortly), then $G_n\big(f, X_n\big)$ tends to f for every point $x \in (-1,1)$ and the convergence is uniform on every interval $[-1+\varepsilon, 1-\varepsilon]$ $(0 < \varepsilon < 1)$, moreover there is no convergence – in general – at the points ± 1.

In this paper we shall consider the case, when $X_n's$ are the root system of the Jacobi polynomials.

Let $w_{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta$ $(x \in (-1,1),\ \alpha, \beta > -1)$ be a Jacobi weight. On the root system

$$\big\{y_{k,n} := y_{k,n}^{(\alpha,\beta)} \mid k = 1, \ldots, n\big\} \subset (-1,1) \qquad (n \in \mathbf{N})$$

of the orthonormal Jacobi polynomials

$$p_n(x) := p_n^{(\alpha,\beta)}(x) \qquad \big(x \in [-1,1],\ \alpha, \beta > -1,\ n \in \mathbf{N}_0 := \{0, 1, 2, \ldots\}\big)$$

we shall consider the Grünwald process:

$$G_n^{(\alpha,\beta)}(f; x) := \sum_{k=1}^{n} f(y_{k,n}) \ell_{k,n}^2(w_{\alpha,\beta}; x) \qquad (x \in [-1,1],\ n \in \mathbf{N}). \tag{1.1}$$

It is well known that the roots of polynomial $p_n^{(\alpha,\beta)}$ form a strongly normal point system in $[-1,1]$, if $-1 < \alpha, \beta < 0$. Grünwald's result for other Jacobi parameters were extended by J. Balázs [1] (for $\alpha = \beta \geq 0$) and I. Joó [4], [5] (for $\alpha, \beta > -1$). They also gave an order of convergence.

I. Joó [4] observed that *on the whole interval* $[-1,1]$ uniform convergence may be attained, if one takes some weighted convergence of the above process. He proved the following result:

Theorem A. (cf. [4, Theorem]) *Let $f(x)$ be a continuous function on $[-1,1]$ then for every $x \in [-1,1]$ and $n \in \mathbf{N}$ we get*

$$(1-x)^{\alpha+\frac{3}{2}} \cdot (1+x)^{\beta+\frac{3}{2}} \big|f(x) - G^{(\alpha,\beta)}\big(f, x\big)\big| \leq \\ \leq C \left\{ \omega\left(f; \frac{\log n}{n}\right) + \|f\|_\infty n^a \log n \right\},$$

where

$$a := \begin{cases} -1, & \text{if} \quad q := \min(\alpha, \beta) \geq -\frac{1}{2}, \\ \\ -2(q+1), & \text{if} \quad q < -\frac{1}{2}, \end{cases}$$

moreover the constant $C > 0$ depends only on α and β.

Here and in what follows, $\|\cdot\|_\infty$ is the supremum norm on $C[-1,1]$; moreover $\omega(f,\delta)$ denotes the modulus of continuity of $f \in C[-1,1]$.

Our aim is to extend and to sharpen Joó's result. We shall give conditions for (α,β) and (γ,δ) for which

$$\lim_{n\to+\infty} \left\| \left(f - G_n^{\alpha,\beta}\right) w_{\gamma,\delta} \right\|_\infty = 0$$

for all $f \in C[-1,1]$.

2. Result

Let us introduce some notations:

$$a^+ := \begin{cases} a, & \text{if } a \ge 0, \\ 0, & \text{if } a < 0. \end{cases}$$

$c, C, c_1, C_2, \dots$ denote positive constants not necessarily the same at each occurrence. If (a_n) and (b_n) are positive real sequences, then $a_n \sim b_n$ means that there are constants $c_1, c_2 > 0$ independent of n such that $c_1 \le a_n/b_n < c_2$ for every $n \in \mathbf{N}$.

If $\alpha, \beta > -1$, $\gamma, \delta > 0$ and $n \in \mathbf{N}$, then let

$$\frac{1}{N_n^{(1)}} := \frac{1}{N_n^{(1)}(\alpha,\beta,\gamma,\delta)} := \tag{2.1}$$

$$:= \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \ge \alpha^+ + \frac{1}{2} \text{ and } \delta \ge \beta^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \text{ and } \delta - \beta^+ \ge \gamma - \alpha^+, \\ \dfrac{1}{n^{2(\gamma-\beta^+)}}, & \text{if } \beta^+ < \delta < \beta^+ + \frac{1}{2} \text{ and } \gamma - \alpha^+ \ge \delta - \beta^+; \end{cases}$$

$$\frac{1}{N_n^{(2)}} := \frac{1}{N_n^{(2)}(\alpha,\beta,\gamma,\delta)} := \tag{2.2}$$

$$:= \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \ge \alpha^+ + \frac{1}{2} \text{ and } \delta \ge \beta^+ + \frac{1}{2}, \\ \dfrac{\log n}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \text{ and } \delta - \beta^+ \ge \gamma - \alpha^+, \\ \dfrac{\log n}{n^{2(\gamma-\beta^+)}}, & \text{if } \beta^+ < \delta < \beta^+ + \frac{1}{2} \text{ and } \gamma - \alpha^+ \ge \delta - \beta^+. \end{cases}$$

It is clear that in each of above cases we get

$$\lim_{n\to+\infty} \frac{1}{N_n^{(1)}} = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{1}{N_n^{(2)}} = 0.$$

Theorem. *Suppose that* $\alpha, \beta > -1$ *and* $\gamma > \alpha^+$, $\delta > \beta^+$. *Then*

$$\lim_{n\to+\infty} \left\| \left(f - G_n^{(\alpha,\beta)}(f,\cdot)\right) w_{\gamma,\delta} \right\|_\infty = 0 \tag{2.3}$$

holds for every function $f \in C[-1,1]$.

For the order of approximation we have

$$\left\| \left(f - G_n^{(\alpha,\beta)}(f,\cdot)\right) w_{\gamma,\delta} \right\|_\infty \le C \left\{ \omega\left(f; \frac{1}{N_n^{(1)}}\right) + \frac{\|f\|_\infty}{N_n^{(2)}} \right\}, \tag{2.4}$$

where $N_n^{(1)}$ *and* $N_n^{(2)}$ *are defined by* (2.1) *and* (2.2), *further the constant* $C > 0$ *depends only on* α, β, γ *and* δ.

3. Proof of the Theorem

3.1. First we mention some basic relations with respect to the Jacobi polynomials which will be used later.

If $\alpha, \beta > -1$, $x = \cos\vartheta$ and $y_{k,n} =: \cos\vartheta_{k,n}$ $(k = 1, 2, \ldots, n,\ n \in \mathbf{N})$ are the roots of p_n then with $y_{0,n} := 1$, $y_{n+1,n} := -1$, $\vartheta_{0,n} := 0$, $\vartheta_{n+1,n} := \pi$ we have

$$\vartheta_{k+1,n} - \vartheta_{k,n} \sim \frac{1}{n}, \quad \vartheta_{k,n} \sim \frac{k}{n} \quad (k = 0, 1, \ldots, n,\ n \in \mathbf{N}) \tag{3.1}$$

(see [10, (8.9.2)]). Moreover

$$\begin{cases} \left|1 - y_{k,n}\right| \sim \left(\dfrac{k}{n}\right)^2, & \text{if } y_{k,n} \in [0,1], \\ \left|1 + y_{k,n}\right| \sim \left(\dfrac{K}{n}\right)^2, & \text{if } y_{k,n} \in [-1,0],\ K := n+1-k \end{cases} \tag{3.2}$$

(see [8, (9.9)]).

If $\alpha, \beta > -1$ and $y_{j,n}$ $(1 \le j \le n)$ denotes (one of) the closest root(s) to x (shortly $x \approx y_{j,n}$, $j = j(n)$) then

$$\left|p_n(x)\right| \sim \left|p_n'(y_{j,n})\right| \cdot \left|x - y_{j,n}\right| \quad (x \approx y_{j,n} \in [-1,1]) \tag{3.3}$$

(see [6, (3.6)]),

$$\left|p_n'(y_{k,n})\right| \sim \frac{n}{\sqrt{1 - y_{k,n}^2}} \cdot \frac{1}{\left(w_{\alpha,\beta}(y_{k,n})\sqrt{1 - y_{k,n}^2}\right)^{1/2}}$$

$$(k = 1, 2, \dots, n,\ n \in \mathbf{N})$$

(see [12, (3.3)]), therefore

$$|p_n'(y_{k,n})| \sim \begin{cases} \dfrac{n^{\alpha+5/2}}{k^{\alpha+3/2}}, & \text{if } y_{k,n} \in [0,1], \\ \dfrac{n^{\beta+5/2}}{K^{\beta+3/2}}, & \text{if } y_{k,n} \in [-1,0]. \end{cases} \tag{3.4}$$

From a result of I. Joó [5, Theorem III] it follows that

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \ell_{k,n}^2(w_{\alpha,\beta};x) = \begin{cases} -\dfrac{1}{\alpha}, & \text{if } x = 1,\ -1 < \alpha < 0,\ \beta > -1, \\ +\infty, & \text{if } x = 1,\ \alpha \ge 0,\ \beta > -1, \\ 1, & \text{if } -1 < x < 1,\ \alpha, \beta > -1, \\ -\dfrac{1}{\beta}, & \text{if } x = -1,\ -1 < \beta < 0,\ \alpha > -1, \\ +\infty, & \text{if } x = -1,\ \beta \ge 0,\ \alpha > -1. \end{cases} \tag{3.5}$$

The convergence is uniform on every interval $[-1+\varepsilon, 1-\varepsilon]$ $(0 < \varepsilon < 1)$.

3.2. For the proof of the Theorem we start from the following estimates:

$$\begin{aligned} &(1-x)^\gamma(1+x)^\delta \left|f(x) - G_n^{(\alpha,\beta)}(f;x)\right| = \\ &= (1-x)^\gamma(1+x)^\delta \left| f(x) - \sum_{k=1}^{n} f(y_{k,n})\ell_{k,n}^2(x) \right| = \\ &= (1-x)^\gamma(1+x)^\delta \left| \sum_{k=1}^{n} \left(f(x) - f(y_{k,n})\right)\ell_{k,n}^2(x) + f(x)\left(1 - \sum_{k=1}^{n} \ell_{k,n}^2(x)\right) \right| \le \\ &\le (1-x)^\gamma(1+x)^\delta \left\{ \sum_{k=1}^{n} |f(x) - f(y_{k,n})|\,\ell_{k,n}^2(x) + |f(x)| \cdot \left| 1 - \sum_{k=1}^{n} \ell_{k,n}^2(x) \right| \right\}. \end{aligned}$$

For arbitrary $x, y \in [-1,1]$ and $\delta > 0$ we denote by $\lambda := \lambda(x,y,\delta)$ the integer $\left[\frac{|x-y|}{\delta}\right]$. It is well known that

$$|f(x) - f(y)| \le \omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta).$$

Therefore

$$|f(x) - f(y_{k,n})| \le \left(\frac{|x - y_{k,n}|}{\delta} + 1 \right) \omega\big(f;\delta\big).$$

Now let $\delta := \dfrac{1}{N_n^{(1)}}$. Then for every $x \in [-1,1]$ and $n \in \mathbf{N}$ we get

$$(1-x)^\gamma(1+x)^\delta \sum_{k=1}^{n} \big|f(x) - f(y_{k,n})\big|\ell_{k,n}^2(x) \le$$

$$\leq \omega\left(f; \frac{1}{N_n^{(1)}}\right) \cdot \left\{ N_n^{(1)} (1-x)^\gamma (1+x)^\delta \sum_{k=1}^n |x - y_{k,n}| \ell_{k,n}^2(x) + \right.$$

$$\left. + (1-x)^\gamma (1+x)^\delta \sum_{k=1}^n \ell_{k,n}^2(x) \right\} \leq$$

$$\leq 2 \left(N_n^{(1)} (1-x)^\gamma (1+x)^\delta \sum_{k=1}^n |x - y_{k,n}| \ell_{k,n}^2(x) \right) \cdot \omega\left(f; \frac{1}{N_n^{(1)}}\right).$$

Here we used that $\lim\limits_{n\to+\infty} N_n^{(1)} = +\infty$ and (3.5).

From the above relations we obtain that

$$(1-x)^\gamma (1+x)^\delta \left| f(x) - G_n^{(\alpha,\beta)}(f;x) \right| \leq \tag{3.6}$$

$$\leq C \left(N_n^{(1)} (1-x)^\gamma (1+x)^\delta \sum_{k=1}^n |x - y_{k,n}| \ell_{k,n}^2(x) \right) \omega\left(f; \frac{1}{N_n^{(1)}}\right) +$$

$$+ |f(x)| (1-x)^\gamma (1+x)^\delta \left| 1 - \sum_{k=1}^n \ell_{k,n}^2(x) \right|$$

$$\big(x \in [-1,1], \quad n \in \mathbf{N}, \ \alpha, \beta > -1, \ \gamma, \delta > 0\big).$$

We shall prove estimations for the above two terms separately in the next two subsections.

3.3. Lemma 1. *Let $\alpha, \beta > -1$. Then for every $x \in [-1,1]$ and $n \in \mathbf{N}$ we have*

$$(1-x)^\gamma (1+x)^\delta \left| 1 - \sum_{k=1}^n \ell_{k,n}^2(x) \right| \leq C \, \frac{1}{N_n^{(2)}}, \tag{3.7}$$

where $N_n^{(2)}$ is given by (2.2) and C is a positive constant depending only on parameters α, β, γ and δ.

Proof. It is well known that for the fundamental polynomials of the first kind of Hermite interpolation satisfy the following identity:

$$\sum_{k=1}^n h_{k,n}(x) = \sum_{k=1}^n \left(1 - \frac{p_n''(y_{k,n})}{p_n'(y_{k,n})} (x - y_{k,n}) \right) \ell_{k,n}^2(x) =$$

$$= \sum_{k=1}^n \left(1 - \frac{(\alpha+\beta+2) y_{k,n} + \alpha - \beta}{1 - y_{k,n}^2} (x - y_{k,n}) \right) \ell_{k,n}^2(x) = 1$$

$$\big(x \in [-1,1], \quad n \in \mathbf{N}\big).$$

From it we get

$$\left|1-\sum_{k=1}^{n}\ell_{k,n}^2(x)\right| \le C\sum_{k=1}^{n}\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\ell_{k,n}^2(x)$$
$$\bigl(x\in[-1,1],\quad n\in\mathbf{N}\bigr),$$

therefore it is enough to estimate the expression

$$F_n(x) := (1-x)^{\gamma}(1+x)^{\delta}\sum_{k=1}^{n}\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\ell_{k,n}^2(x) =$$

(3.8)
$$= (1-x)^{\gamma}(1+x)^{\delta}\sum_{y_{k,n}\in[0,1)}\ldots + (1-x)^{\gamma}(1+x)^{\delta}\sum_{y_{k,n}\in(-1,0)}\ldots =:$$
$$=: A_n^{(1)}(x) + A_n^{(2)}(x)$$
$$\bigl(x\in[-1,1],\ n\in\mathbf{N},\ \alpha,\beta>-1,\ \gamma,\delta>0\bigr).$$

Case 1. Assume that $x\in[0,1]$.

Case 1a. Consider first the sum $A_n^{(1)}(x)$. Let $y_{j,n}\approx x\in[0,1]$ and $y_{k,n}\in$ $\in[0,1)$. From (3.2) it follows that there exists $c\in(0,1)$ independent of j,k,n such that $1\le j,k\le[cn]$, thus

$$A_n^{(1)}(x) \le (1-x)^{\gamma}(1+x)^{\delta}\sum_{k=1}^{[cn]}\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\ell_{k,n}^2(x)$$
$$\bigl(x\in[0,1],\ n\in\mathbf{N}\bigr).$$

Using formulas in Section 3.1 we have uniformly for the above indices j,k and $n\in\mathbf{N}$

$$(1-x)^{\gamma}(1+x)^{\delta}\sim(1-x)^{\gamma}\sim(1-y_{j,n})^{\gamma}\sim\left(\frac{j}{n}\right)^{2\gamma},$$
$$1-y_{k,n}^2\sim\left(\frac{k}{n}\right)^2,$$
$$\ell_{k,n}^2(x)=\left[\frac{p_n(x)}{p_n'(y_{k,n})(x-y_{k,n})}\right]^2\sim\left[\frac{p_n'(y_{j,n})(x-y_{j,n})}{p_n'(y_{k,n})(x-y_{k,n})}\right]^2\sim$$
$$\sim\left(\frac{k}{j}\right)^{2\alpha+3}\cdot\frac{|x-y_{j,n}|^2}{|x-y_{k,n}|^2}.$$

If $k\ne j$ $(j,k=1,2,\ldots,n)$ then

(3.9)
$$|x-y_{k,n}|\sim|y_{j,n}-y_{k,n}|\sim\frac{|j^2-k^2|}{n^2},$$
$$|x-y_{j,n}|\le c\frac{j}{n^2}$$

(see [11, Lemma 3.1]), hence

$$\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\,\ell_{k,n}^2(x)\sim$$

$$\sim\frac{|x-y_{k,n}|}{\left(\frac{k}{n}\right)^2}\cdot\left(\frac{k}{j}\right)^{2\alpha+3}\cdot\frac{|x-y_{j,n}|^2}{|x-y_{k,n}|^2}\sim n^4\cdot\frac{k^{2\alpha+1}}{j^{2\alpha+3}}\cdot\frac{|x-y_{j,n}|^2}{|j^2-k^2|}\le$$
$$\le c\left(\frac{k}{j}\right)^{2\alpha+1}\cdot\frac{1}{|j^2-k^2|}.$$

If $k=j$, then

$$\frac{|x-y_{j,n}|}{1-y_{j,n}^2}\,\ell_{j,n}^2(x)\sim\frac{|x-y_{j,n}|}{1-y_{j,n}^2}\cdot\left[\frac{p_n'(y_{j,n})(x-y_{j,n})}{p_n'(y_{j,n})(x-y_{j,n})}\right]^2\le c\,\frac{1}{j}.$$

Using the above formulas we obtain that

$$A_n^{(1)}(x)\le C\left(\frac{j}{n}\right)^{2\gamma}\left\{\sum_{\substack{k=1\\k\ne j}}^{[cn]}\left(\frac{k}{j}\right)^{2\alpha+1}\frac{1}{|j^2-k^2|}+\frac{1}{j}\right\}\le$$

$$\le C\left(\frac{j}{n}\right)^{2\gamma}\left\{\sum_{k=1}^{j/2}\left(\frac{k}{j}\right)^{2\alpha+1}\frac{1}{j^2}+\sum_{\substack{k=j/2\\k\ne j}}^{2j}\frac{1}{j\cdot|j-k|}+\sum_{k=2j}^{[cn]}\left(\frac{k}{j}\right)^{2\alpha+1}\frac{1}{k^2}+\frac{1}{j}\right\}\le$$

$$\le C\left(\frac{j}{n}\right)^{2\gamma}\left\{\frac{1}{j}+\frac{\log(j+1)}{j}+\frac{1}{j^{2\alpha+1}}\sum_{k=2j}^{[cn]}k^{2\alpha-1}+\frac{1}{j}\right\}\le$$

$$\le C\left(\frac{j}{n}\right)^{2\gamma}\left(\frac{\log(j+1)}{j}+\frac{1}{j^{2\alpha+1}}\begin{cases}j^{2\alpha}-n^{2\alpha}, & \text{if } \alpha<0\\ \log(n/j), & \text{if } \alpha=0\\ n^{2\alpha}-j^{2\alpha}, & \text{if } \alpha>0\end{cases}\right)=$$

$$=C\left(\frac{j}{n}\right)^{2\gamma}\left(\frac{\log(j+1)}{j}+\frac{1}{j}\begin{cases}1-(n/j)^{2\alpha}, & \text{if } \alpha<0\\ \log(n/j), & \text{if } \alpha=0\\ (n/j)^{2\alpha}-1, & \text{if } \alpha>0\end{cases}\right)\le$$

$$\le C\left(\frac{j}{n}\right)^{2\gamma}\left(\frac{\log(j+1)}{j}+\frac{1}{j}\begin{cases}1, & \text{if } \alpha<0\\ \log(n/j), & \text{if } \alpha=0\\ (n/j)^{2\alpha}, & \text{if } \alpha>0\end{cases}\right)\le$$

$$\le C\frac{j^{2\gamma-1}}{n^{2\gamma}}\cdot\begin{cases}\log(j+1), & \text{if } \alpha<0,\\ \log n, & \text{if } \alpha=0,\\ \log(j+1)+(n/j)^{2\alpha}, & \text{if } \alpha>0.\end{cases}$$

Our aim is to choose the index $\gamma > 0$ (for fixed $\alpha > -1$) such that $\lim_{n\to+\infty} A_n^{(1)}(x) = 0$ uniformly in $x \in [0,1]$.

Let $\alpha < 0$. Using that for $s \geq 0$ the function $x^s \log x$ is increasing on $[1,+\infty)$ and for $t \in (-1,0)$ the function $x^t \log x$ is bounded on $[1,+\infty)$ we have

$$A_n^{(1)}(x) \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \frac{1}{2}, \\ \dfrac{1}{n^{2\gamma}}, & \text{if } 0 < \gamma < \frac{1}{2} \end{cases}$$

uniformly in $x \in [0,1]$ and $n \in \mathbf{N}$.

Let $\alpha = 0$. Then

$$A_n^{(1)}(x) \leq C \frac{j^{2\gamma-1}\log n}{n^{2\gamma}} \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \frac{1}{2}, \\ \dfrac{\log n}{n^{2\gamma}}, & \text{if } 0 < \gamma < \frac{1}{2} \end{cases}$$

uniformly in $x \in [0,1]$ and $n \in \mathbf{N}$.

Let $\alpha > 0$. Then

$$A_n^{(1)}(x) \leq C \frac{j^{2(\gamma-\alpha)-1}}{n^{2(\gamma-\alpha)}} \cdot \left\{ \left(\frac{j}{n}\right)^{2\alpha} \log(j+1) + 1 \right\} \leq$$

$$\leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha)}}, & \text{if } 0 < \alpha < \gamma < \alpha + \frac{1}{2} \end{cases}$$

uniformly in $x \in [0,1]$ and $n \in \mathbf{N}$.

Summarizing the above three formulas we obtain that

$$\text{(3.10)} \qquad A_n^{(1)}(x) \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha^+ + \frac{1}{2}, \\ \dfrac{(\log n)^a}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \end{cases}$$

$$\big(x \in [0,1],\ n \in \mathbf{N},\ \alpha,\beta > -1,\ \gamma,\delta > 0\big),$$

where the constant $C > 0$ depends only on α, β, γ and δ and

$$a := \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0. \end{cases}$$

Case 1b. Now consider the sum $A_n^{(2)}(x)$ in (3.8). Let $y_{j,n} \approx x \in [0,1]$, $y_{k,n} \in (-1,0)$ and $K := n+1-k$. From (3.2) it follows that there exist $c_1, c_2 > 0$ independent of n such that $1 \leq j \leq [c_1 n]$ and $1 \leq K \leq [c_2 n]$. Using formulas in Section 3.1 and (3.9) we have uniformly for the above indices j, k and $n \in \mathbf{N}$

$$(1-x)^\gamma (1+x)^\delta \sim (1-x)^\gamma \sim (1-y_{j,n})^\gamma \sim \left(\frac{j}{n}\right)^{2\gamma},$$

$$1-y_{k,n}^2 \sim \left(\frac{K}{n}\right)^2,$$

$$\ell_{k,n}^2(x) = \left[\frac{p_n(x)}{p_n'(y_{k,n})(x-y_{k,n})}\right]^2 \sim \left[\frac{p_n'(y_{j,n})(x-y_{j,n})}{p_n'(y_{k,n})(x-y_{k,n})}\right]^2 \sim$$

$$\sim n^{2(\alpha-\beta)} \cdot \frac{K^{2\beta+3}}{j^{2\alpha+3}} \cdot \frac{|x-y_{j,n}|^2}{|x-y_{k,n}|^2};$$

$$\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\,\ell_{k,n}^2(x) \le C\, n^{2(\alpha-\beta-1)} \cdot \frac{K^{2\beta+1}}{j^{2\alpha+1}} \cdot \frac{1}{|x-y_{k,n}|}.$$

Therefore if $x \in [0,1]$ and $n \in \mathbf{N}$, then we have

$$A_n^{(2)}(x) = (1-x)^\gamma (1+x)^\delta \sum_{y_{k,n}<0} \frac{|x-y_{k,n}|}{1-y_{k,n}^2} \ell_{k,n}^2(x) \le$$

$$\le C \left(\frac{j}{n}\right)^{2\gamma} \frac{n^{2(\alpha-\beta-1)}}{j^{2\alpha+1}} \sum_{y_{k,n}<0} \frac{K^{2\beta+1}}{|x-y_{k,n}|}.$$

Now let $\frac{1}{2} \le x \le 1$, then $|x-y_{k,n}| \sim 1$, thus

$$A_n^{(2)}(x) \le$$

$$C \left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1} \frac{1}{n^{2\beta+3}} \sum_{K=1}^{[c_2 n]} K^{2\beta+1} \le C\frac{1}{n}\left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1} = C\frac{1}{j}\left(\frac{j}{n}\right)^{2(\gamma-\alpha)}.$$

If $0 \le x \le \frac{1}{2}$, then

$$j \sim n, \quad k \sim n, \quad j \ne k \quad \text{and} \quad \frac{|j^2-k^2|}{n^2} \sim \frac{|j-k|}{n},$$

so we get

$$A_n^{(2)}(x) \le \frac{C}{n^{2\beta+3}} \sum_{y_{k,n}<0} \frac{K^{2\beta+1}}{|y_{j,n}-y_{k,n}|} \le$$

$$\le \frac{C}{n^{2\beta+3}} \left\{ \sum_{y_{k,n} \le -\frac{1}{2}} K^{2\beta+1} + \sum_{-\frac{1}{2} \le y_{k,n} < 0} \frac{K^{2\beta+1}}{(|j^2-k^2|/n^2)} \right\} \le$$

$$\le \frac{C}{n^{2\beta+3}} \left\{ \sum_{K=1}^{[c_2 n]} K^{2\beta+1} + n^{2\beta+2} \sum_{-\frac{1}{2} \le y_{k,n} < 0} \frac{1}{|j-k|} \right\} \le$$

$$\leq C\left\{\frac{1}{n}+\frac{1}{n}\sum_{l=1}^{n}\frac{1}{l}\right\}\leq C\frac{\log n}{n}.$$

Consequently, for $x\in[0,1]$ and $n\in\mathbf{N}$ we obtain that

(3.11)
$$A_n^{(2)}(x)\leq C\left\{\frac{1}{j}\left(\frac{j}{n}\right)^{2(\gamma-\alpha)}+\frac{\log n}{n}\right\}=C\left\{\frac{1}{n}\left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1}+\frac{\log n}{n}\right\}.$$

Let $\alpha<0$. Since

$$\frac{1}{n}\left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1}=\frac{1}{n}\cdot\frac{j^{2\gamma-1}}{n^{2\gamma-1}}\cdot\left(\frac{j}{n}\right)^{-2\alpha}\leq\begin{cases}\dfrac{1}{n}, & \text{if } \gamma\geq\frac{1}{2},\\ \dfrac{1}{n^{2\gamma}}, & \text{if } 0<\gamma<\frac{1}{2},\end{cases}$$

thus

$$A_n^{(2)}(x)\leq C\cdot\begin{cases}\dfrac{\log n}{n}, & \text{if } \gamma\geq\frac{1}{2},\\ \dfrac{1}{n^{2\gamma}}, & \text{if } 0<\gamma<\frac{1}{2}\end{cases}$$

uniformly in $x\in[0,1]$ and $n\in\mathbf{N}$.

If $\alpha\geq0$, then we have

$$A_n^{(2)}(x)\leq C\cdot\begin{cases}\dfrac{\log n}{n}, & \text{if } \gamma\geq\alpha+\frac{1}{2},\\ \dfrac{1}{n^{2(\gamma-\alpha)}}, & \text{if } 0\leq\alpha<\gamma<\alpha+\frac{1}{2}\end{cases}$$

uniformly in $x\in[0,1]$ and $n\in\mathbf{N}$.

Summarizing the above formulas we have

(3.12)
$$A_n^{(2)}(x)\leq C\begin{cases}\dfrac{\log n}{n}, & \text{if } \gamma\geq\alpha^{+}+\frac{1}{2},\\ \dfrac{1}{n^{2(\gamma-\alpha^{+})}}, & \text{if } \alpha^{+}<\gamma<\alpha^{+}+\frac{1}{2}\end{cases}$$

$$\big(x\in[0,1],\quad \alpha,\beta>-1,\quad \gamma,\delta>0,\quad n\in\mathbf{N}\big),$$

where C is independent of x and n. By (3.8), (3.10) and (3.12) we get

(3.13)
$$F_n(x)=(1-x)^{\gamma}(1+x)^{\delta}\sum_{k=1}^{n}\frac{|x-y_{k,n}|}{1-y_{k,n}^2}\ell_{k,n}^2(x)\leq$$

$$\leq C\cdot\begin{cases}\dfrac{\log n}{n}, & \text{if } \gamma\geq\alpha^{+}+\frac{1}{2},\\ \dfrac{(\log n)^{a}}{n^{2(\gamma-\alpha^{+})}}, & \text{if } \alpha^{+}<\gamma<\alpha^{+}+\frac{1}{2}\end{cases}$$

$$\big(x\in[0,1],\ n\in\mathbf{N},\alpha,\beta>-1,\ \gamma,\delta>0\big),$$

where the constant $C>0$ depends only on α,β,γ and δ.

Case 2. Let us consider (3.8) for $x\in[-1,0]$. Using the symmetry of Jacobi polynomials

$$p_n^{(\alpha,\beta)}(x)=(-1)^n p_n^{(\beta,\alpha)}(-x)$$
$$\big(x\in[-1,1],\ n\in\mathbf{N},\ \alpha,\beta>-1\big)$$

(see [10, (4.1.3)]) and (3.13) we get

$$F_n(x)\le C\cdot\begin{cases}\dfrac{\log n}{n}, & \text{if } \delta\ge\beta^++\frac12,\\[2mm] \dfrac{(\log n)^b}{n^{2(\delta-\beta^+)}}, & \text{if } \beta^+<\delta<\beta^++\frac12\end{cases}\tag{3.14}$$

$$\big(x\in[-1,0],\ n\in\mathbf{N},\ \alpha,\beta>-1,\ \gamma,\delta>0\big),$$

where the constant $C>0$ depends only on α,β,γ and δ and

$$b:=\begin{cases}1, & \text{if } \beta=0,\\ 0, & \text{if } \beta\ne 0.\end{cases}$$

Finally collecting Cases 1 and 2 (see (3.13) and (3.14)) we obtain that

$$F_n(x)\le C\begin{cases}\dfrac{\log n}{n}, & \text{if } \gamma\ge\alpha^++\frac12 \text{ and } \delta\ge\beta^++\frac12,\\[2mm] \dfrac{\log n}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+<\gamma<\alpha^++\frac12 \text{ and } \delta-\beta^+\ge\gamma-\alpha^+,\\[2mm] \dfrac{\log n}{n^{2(\gamma-\beta^+)}}, & \text{if } \beta^+<\delta<\beta^++\frac12 \text{ and } \gamma-\alpha^+\ge\delta-\beta^+\end{cases}$$

$$\big(x\in[-1,1],\ n\in\mathbf{N},\ \alpha,\beta>-1,\ \gamma,\delta>0\big),$$

where the constant $C>0$ depends only on α,β,γ and δ, which proves Lemma 1.

3.4. Lemma 2. *Let $\alpha,\beta>-1$. Then for every $x\in[-1,1]$ and $n\in\mathbf{N}$ we have*

$$H_n(x):=(1-x)^\gamma(1+x)^\delta\sum_{k=1}^{n}|x-y_{k,n}|\ell_{k,n}^2(x)\le C\frac{1}{N_n^{(1)}},\tag{3.15}$$

where $N_n^{(1)}$ is given by (2.1) and the constant $C>0$ depends only on parameters α,β,γ and δ.

Proof. The proof is similar to the proof of Lemma 1, so we only sketch it. Let

$$H_n(x) := (1-x)^\gamma(1+x)^\delta \sum_{y_{k,n}\geq 0} |x-y_{k,n}|\ell_{k,n}^2(x)+$$
$$+(1-x)^\gamma(1+x)^\delta \sum_{y_{k,n}<0} |x-y_{k,n}|\ell_{k,n}^2(x) =: B_n^{(1)}(x)+B_n^{(2)}(x).$$

Case 1. First we suppose that $x \approx y_{j,n} \in [0,1]$ and $y_{k,n} \in [0,1)$. From (3.2) it follows that there exists $c \in (0,1)$ independent of j,k,n such that $1 \leq j,k \leq [cn]$. So for $x \in [0,1]$ and $n \in \mathbf{N}$ consider the sum

$$B_n^{(1)}(x) = (1-x)^\gamma(1+x)^\delta \sum_{k=1}^{[cn]} |x-y_{k,n}|\ell_{k,n}^2(x).$$

Then

$$B_n^{(1)}(x) \leq C\left(\frac{j}{n}\right)^{2\gamma} \sum_{k=1}^{[cn]} \left(\frac{k}{j}\right)^{2\alpha+3} \frac{|x-y_{j,n}|^2}{|x-y_{k,n}|} \leq$$
$$\leq \frac{C}{n^2}\left(\frac{j}{n}\right)^{2\gamma} \left\{ \sum_{\substack{k=1\\k\neq j}}^{[cn]} \left(\frac{k}{j}\right)^{2\alpha+3} \frac{j^2}{|j^2-k^2|} + j \right\} \leq$$
$$\leq \frac{C}{n^2}\left(\frac{j}{n}\right)^{2\gamma} \left\{ \sum_{k=1}^{j/2} \left(\frac{k}{j}\right)^{2\alpha+3} \frac{j^2}{j^2} + \sum_{\substack{k=j/2\\k\neq j}}^{2j} \frac{j^2}{j\cdot|j-k|} + \sum_{k=2j}^{[cn]} \left(\frac{k}{j}\right)^{2\alpha+3} \frac{j^2}{k^2} + j \right\} \leq$$
$$\leq \frac{C}{n^2}\left(\frac{j}{n}\right)^{2\gamma} \left\{ j + j\log(j+1) + \frac{1}{j^{2\alpha+1}} \sum_{k=2j}^{[cn]} k^{2\alpha+1} + j \right\} \leq$$
$$\leq \frac{C}{n^2}\left(\frac{j}{n}\right)^{2\gamma} \left\{ j\log(j+1) + \frac{n^{2\alpha+2}-j^{2\alpha+2}}{j^{2\alpha+1}} \right\} \leq$$
$$\leq \frac{C}{n^2}\left(\frac{j}{n}\right)^{2\gamma} \left\{ j\log(j+1) + j\left(\frac{n}{j}\right)^{2\alpha+2} \right\} = C\left(\frac{j}{n}\right)^{2\gamma} \left\{ \frac{j\log(j+1)}{n^2} + \frac{1}{j}\left(\frac{n}{j}\right)^{2\alpha} \right\}.$$

If $\alpha < 0$, then by

$$\left(\frac{j}{n}\right)^{2\gamma} \frac{1}{j}\left(\frac{n}{j}\right)^{2\alpha} \leq \frac{j^{2\gamma-1}}{n^{2\gamma}}$$

we get

$$B_n^{(1)}(x) \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \frac{1}{2}, \\ \dfrac{1}{n^{2\gamma}}, & \text{if } 0 < \gamma < \frac{1}{2} \end{cases}$$

uniformly in $x \in [0,1]$ and $n \in \mathbf{N}$.

If $\alpha \geq 0$ then

$$B_n^{(1)}(x) \leq C\left\{\frac{j^{2\gamma+1}\log(j+1)}{n^{2\gamma+2}} + \frac{j^{2(\gamma-\alpha)-1}}{n^{2(\gamma-\alpha)}}\right\} \leq$$

$$\leq C \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha)}}, & \text{if } 0 \leq \alpha < \gamma < \alpha + \frac{1}{2}. \end{cases}$$

Summarizing the above three formulas we obtain that

$$B_n^{(1)}(x) \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \end{cases} \tag{3.16}$$

$$\big(x \in [0,1],\ n \in \mathbf{N},\ \alpha, \beta > -1,\ \gamma, \delta > 0\big),$$

where the constant $C > 0$ depends only on parameters α, β, γ and δ.

Now let $[0,1] \ni x \approx y_{j,n}$ and $y_{k,n} \in (-1,0)$ moreover $K := n+1-k$. Consider the sum

$$B_n^{(2)}(x) = (1-x)^\gamma (1+x)^\delta \sum_{y_{k,n}<0} |x - y_{k,n}| \ell_{k,n}^2(x).$$

For $x \in [0,1]$ and $n \in \mathbf{N}$ then we have

$$B_n^{(2)}(x) \leq C \left(\frac{j}{n}\right)^{2\gamma} \left\{ \sum_{y_{k,n}<0} n^{2(\alpha-\beta)} \frac{K^{2\beta+3}}{j^{2\alpha+3}} \frac{|x-y_{j,n}|^2}{|x-y_{k,n}|} \right\} \leq$$

$$\leq C \left(\frac{j}{n}\right)^{2\gamma} \frac{n^{2(\alpha-\beta-2)}}{j^{2\alpha+1}} \sum_{y_{k,n}<0} \frac{K^{2\beta+3}}{|x-y_{k,n}|}.$$

Now let $\frac{1}{2} \leq x \leq 1$, then $|x - y_{k,n}| \sim 1$, thus

$$B_n^{(2)}(x) \leq C \left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1} \frac{1}{n^{2\beta+5}} \sum_{K=1}^{[c_2 n]} K^{2\beta+3} \leq C \frac{1}{n} \left(\frac{j}{n}\right)^{2(\gamma-\alpha)-1};$$

if $0 \leq x \leq \frac{1}{2}$, then

$$j \sim n, \quad k \sim n, \quad j \neq k \quad \text{and} \quad \frac{|j^2 - k^2|}{n^2} \sim \frac{|j-k|}{n},$$

so we get

$$B_n^{(2)}(x) \leq \frac{C}{n^{2\beta+5}} \sum_{y_{k,n}<0} \frac{K^{2\beta+3}}{|y_{j,n} - y_{k,n}|} \leq$$

$$\leq \frac{C}{n^{2\beta+5}} \left\{ \sum_{y_{k,n} \leq -\frac{1}{2}} K^{2\beta+3} + \sum_{-\frac{1}{2} \leq y_{k,n} < 0} \frac{K^{2\beta+3}}{(|j^2 - k^2|/n^2)} \right\} \leq$$

$$\leq \frac{C}{n^{2\beta+5}} \left\{ \sum_{K=1}^{[c_2 n]} K^{2\beta+3} + n^{2\beta+4} \sum_{-\frac{1}{2} \leq y_{k,n} < 0} \frac{1}{|j-k|} \right\} \leq C \left\{ \frac{1}{n} + \frac{1}{n} \sum_{l=1}^{n} \frac{1}{l} \right\} \leq$$

$$\leq C \frac{\log n}{n}.$$

Therefore by (3.11) and (3.12) we get

$$B_n^{(2)}(x) \leq C \left\{ \frac{1}{n} \left(\frac{j}{n} \right)^{2(\gamma-\alpha)-1} + \frac{\log n}{n} \right\} \leq \tag{3.17}$$

$$\leq C \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \end{cases}$$

$$\big(x \in [0,1], \quad \alpha, \beta > -1, \quad \gamma, \delta > 0, \quad n \in \mathbf{N} \big),$$

where C is independent of x and n.

Using (3.16) and (3.17) we obtain that

$$H_n(x) = (1-x)^\gamma (1+x)^\delta \sum_{k=1}^{n} |x - y_{k,n}| \ell_{k,n}^2(x) \leq \tag{3.18}$$

$$\leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \geq \alpha^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \end{cases}$$

$$\big(x \in [0,1], \ n \in \mathbf{N}, \alpha, \beta > -1, \ \gamma, \delta > 0 \big),$$

where the constant $C > 0$ depends only on α, β, γ and δ.

Case 2. Let us consider (3.15) for $x \approx y_{j,n} \in [-1, 0]$. Then by symmetry of Jacobi polynomials and (3.18) we have

$$H_n(x) \leq C \cdot \begin{cases} \dfrac{\log n}{n}, & \text{if } \delta \geq \beta^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\delta-\beta^+)}}, & \text{if } \beta^+ < \delta < \beta^+ + \frac{1}{2} \end{cases} \tag{3.19}$$

$$\big(x \in [-1,0], \ n \in \mathbf{N}, \ \alpha, \beta > -1, \ \gamma, \delta > 0 \big),$$

where the constant $C > 0$ depends only on α, β, γ and δ.

Finally collecting Cases 1 and 2 (see (3.18) and (3.19)) we obtain that

$$H_n(x) \le C \begin{cases} \dfrac{\log n}{n}, & \text{if } \gamma \ge \alpha^+ + \frac{1}{2} \text{ and } \delta \ge \beta^+ + \frac{1}{2}, \\ \dfrac{1}{n^{2(\gamma-\alpha^+)}}, & \text{if } \alpha^+ < \gamma < \alpha^+ + \frac{1}{2} \text{ and } \delta - \beta^+ \ge \gamma - \alpha^+, \\ \dfrac{1}{n^{2(\gamma-\beta^+)}}, & \text{if } \beta^+ < \delta < \beta^+ + \frac{1}{2} \text{ and } \gamma - \alpha^+ \ge \delta - \beta^+ \end{cases}$$

$$\big(x \in [-1,1],\ n \in \mathbf{N},\ \alpha, \beta > -1,\ \gamma, \delta > 0\big),$$

where the constant $C > 0$ depends only on α, β, γ and δ, which proves Lemma 2.

3.5. Finally using (3.6), Lemmas 1 and 2 we obtain (2.4), as it was stated.

References

[1] **Balázs J.,** Megjegyzések a stabil interpolációról, *Matematikai Lapok*, **11** (1960), 280-293.

[2] **Grünwald G.,** On the theory of interpolation, *Acta Math.*, **75** (1943), 219-245.

[3] **Háy B. and Vértesi P.,** Interpolation in spaces of weighted maximum norm, *Studia Sci. Math. Hungar.*, **13** (1978), 1-9.

[4] **Joó I.,** On interpolation on the roots of Jacobi polynomials, *Annales Univ. Sci. Budapest. Sect. Math.*, **17** (1974) 119-124.

[5] **Joó I.,** Stabil interpolációról, *MTA III. Osztály Közleményei*, **23** (1974), 329–363. (*Joó I.,* On stable interpolation (in Hungarian))

[6] **Mastroianni G. and Vértesi P.,** Some applications of generalized Jacobi weights, *Acta Math. Hungar.*, **74** (1997), 323-357.

[7] **Szabados J.,** Weighted Lagrange and Hermite-Fejér interpolation on the real line, *J. of Inequal. and Appl.*, **1** (1997), 99-123.

[8] **Szabados J. and Vértesi P.,** *Interpolation of functions*, World Sci. Publ., 1990.

[9] **Szabó V.E.S.,** Weighted interpolation: the L_∞ theory I., *Acta Math. Hungar.*, **83** (1999), 131-159.

[10] **Szegő G.,** *Orthogonal polynomials*, AMS Coll. Publ. **23**, Providence, 1978.

[11] **Vértesi P.,** On Lagrange interpolation, *Periodica Math. Hungar.*, **12** (1981), 103-112.

[12] **Vértesi P.,** Hermite and Hermite-Fejér interpolations of higher order II. (Mean convergence), *Acta Math. Hungar.*, **56** (1990), 369-380.

[13] **Vértesi P.,** Classical (unweighted) and weighted interpolation, *A Panorama of Hungarian Mathematics in the Twentieth Century I.*, Bolyai Society Mathematical Studies **14**, 2005, 71-117.

L. Szili
Department of Numerical Analysis
Eötvös Loránd University
Pázmány Péter S. 1/C
H-1117 Budapest, Hungary