

GROUPS ADMITTING
ONLY QUASI–PERIODIC FACTORIZATIONS

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Dedicated to Imre Kátai on his 70th birthday

Abstract. We will show that if a finite abelian group is a direct product of two subgroups of relatively prime orders and one of the subgroups has prime order, then each normalized factorization of the group must be quasi-periodic.

1. Introduction

Let $G$ be a finite abelian group and let $A, B$ be subsets of $G$. If each $g \in G$ can be written in the form

$$g = ab, \quad a \in A, \quad b \in B;$$

then the product $AB$ is equal to $G$. If each $g \in G$ can be written uniquely in the form (1), then we say that the product $AB$ is direct and we refer to the equation $G = AB$ as a factorization of $G$.

If the identity element $e$ belongs to the subset $e A$ of $G$, then we say that $A$ is a normalized subset of $G$. If both factors $A$ and $B$ of a factorization $G = AB$ are normalized subsets we call the factorization a normalized factorization.

A subset $A$ of $G$ is defined to be periodic if there is a $g \in G \setminus \{e\}$ such that $Ag = A$. Note that if $g$ and $h$ are periods of $A$, then so is $gh$ provided $gh \neq e$. It follows then that the periods of $A$ together with the identity element

Mathematics Subject Classification: 20K01 (05B45, 52C22, 68R05)
form a subgroup $H$ of $G$. We will call $H$ the subgroup of periods of $A$. The reader can verify that there is a subset $A_1$ of $G$ such that the product $A_1H$ is direct and is equal to $A$. If $A$ is normalized then $H \subset A$ and $A_1$ can be chosen to be a subset of $A$. If one of the factors in the factorization $G = AB$ is periodic, then we call the factorization periodic.

We say that a partition $B = B_1 \cup \cdots \cup B_n$ is a regular partition of the subset $B$ of $G$ if there is a periodic subset $C = \{c_1, \ldots, c_n\}$ of $G$ such that $c_1 = e$, $|C| \geq 2$ and $AB_i = AB_1c_i$ for each $i$, $1 \leq i \leq n$. According to [2] we call a normalized factorization $G = AB$ quasi-periodic if one of the factors has a regular partition.

In [3] one can find another definition of the quasi-periodic factorization. Namely, a partition $B = B_1 \cup \cdots \cup B_n$ of the subset $B$ of $G$ is defined to be regular if there is a subgroup $H = \{h_1, \ldots, h_n\}$ of $G$ such that $h_1 = e$, $|H| \geq 2$ and $AB_i = AB_1h_i$ for each $i$, $1 \leq i \leq n$. The normalized factorization $G = AB$ is called quasi-periodic if at least one of the factors $A$, $B$ has a regular partition. The two definitions are equivalent. A proof can be found in [7].


One might wonder if each normalized factorization of a finite abelian group $G$ is quasi-periodic, then does it hold for each subgroup $H$ of $G$. A.D. Sands [6] showed that for this question the answer is "no". Using Sands’ method in this paper we will show that if a finite abelian group $G$ is a direct product of its subgroups $H$ and $K$ such that $|H|$, $|K|$ are relatively prime and $|K|$ is a prime, then $G$ admits only quasi-periodic normalized factorizations.

2. The result

Let $G$ be a finite abelian group and let $H$ be a subgroup of $G$. If $f_1, \ldots, f_n$ is a complete set of representatives of cosets modulo $H$, then the cosets $Hf_1, \ldots, Hf_n$ form a partition of $G$. Therefore for a given subset $A$ of $G$ the sets $A_1 = A \cap Hf_1, \ldots, A_n = A \cap Hf_n$ form a partition of $A$. We will use this observation in the proof of the next theorem. The other observation we will use is that $G = AB$ is a factorization of $G$ if and only if the sets $Ab$, $B \in B$ form a partition of $G$. 
Theorem 1. Let \( p \) be a prime. Let \( G \) be a finite abelian group that is the direct product of its subgroups \( H \) and \( K \). If \( p \) does not divide \(|H|\) and \(|K| = p\), then each normalized factorization of \( G \) is quasi-periodic.

Proof. Let \( G = AB \) be a normalized factorization of \( G \). Since the order of \( G \) is equal to the product of the orders of \( A \) and \( B \) we may assume that \( p \nmid |A| \) and \( p|B| \). Let \( K = \langle f \rangle \). The factor \( A \) can be written in the form

\[
A = A_0 \cup A_1 \cup \cdots \cup A_{p-1},
\]

where \( A_i = A \cap Hf^i \). Further there are subsets \( D_0, \ldots, D_{p-1} \subset H \), thus

\[
A = D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1}.
\]

Similarly \( B \) can be written in the form

\[
B = B_0 \cup B_1 \cup \cdots \cup B_{p-1},
\]

where \( B_i = B \cap Hf^i \). There are subsets \( C_0, \ldots, C_{p-1} \subset H \) such that \( B_i = C_i f^i \). Therefore

\[
B = C_0 \cup C_1 f^1 \cup \cdots \cup C_{p-1} f^{p-1}.
\]

We claim that \( AB_v = AB_0 f^v \) holds for each \( v, 1 \leq v \leq p - 1 \). Using the above notations the claim is equivalent to

\[
(D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1})(C_v f^v) = (D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1})(C_0 f^v),
\]

that is, equivalent to

\[
D_0 C_v \cup D_1 C_v f^1 \cup \cdots \cup D_{p-1} C_v f^{p-1} = D_0 C_0 \cup D_1 C_0 f^1 \cup \cdots \cup D_{p-1} C_0 f^{p-1}.
\]

In fact we will verify that \( D_u C_v = D_u C_0 \) holds for each \( u, v, 0 \leq u \leq p - 1, 1 \leq v \leq p - 1 \).

Choose an integer \( t(i) \) such that

\[
t(i) \equiv 1 \pmod{|H|},
\]

\[
t(i) \equiv i \pmod{|K|}.
\]

Such a \( t(i) \) exists for each \( i, 0 \leq i \leq p - 1 \). By Proposition 3 of [5], in the factorization \( G = AB \) the factor \( A \) can be replaced by \( A^{t(i)} \) to get the normalized factorization \( G = A^{t(i)} B \). Note that

\[
A^{t(i)} = D_0^{t(i)} \cup D_1^{t(i)} f^{t(i)} \cup \cdots \cup D_{p-1}^{t(i)} f^{(p-1)t(i)} = D_0 \cup D_1 f^i \cup \cdots \cup D_{p-1} f^{(p-1)i}.
\]
The factorization
\[ G = (D_0 \cup D_1 f^i \cup \cdots \cup D_{p-1} f^{(p-1)i})B \]
provides that the sets
\[ (2) \quad D_0 B, D_1 B f^i, \ldots, D_{p-1} B f^{(p-1)i} \]
form a partition of \( G \). The \( i = 0 \) and \( i = 1 \) special cases of (2) give that
\[ (3) \quad D_1 B \cup D_2 B \cup \cdots \cup D_{p-1} B = D_1 B f \cup D_2 B f^2 \cup \cdots \cup D_{p-1} f^{p-1} B. \]

Choose an integer \( u \), \( 1 \leq u \leq p - 1 \). Note that
\[ D_u B \cap D_w B f^w = \emptyset \]
for each \( w \), \( 1 \leq w \leq p - 1 \), \( w \neq u \). In order to verify the claim assume on the contrary that
\[ D_u B \cap D_w B f^w = \emptyset \]
for some \( u \) and \( w \). Multiplying by \( f^{w+iu} \) we get
\[ (4) \quad D_u B f^{w+iu} \cap D_u B f^w = \emptyset. \]
The congruence
\[ w i \equiv w + i u \pmod{p} \]
is solvable for \( i \) and so (4) contradicts to (2). This contradiction proves our claim.

From (3) it follows that \( D_u B = D_u B f^u \). Using this equation repeatedly we get that
\[ D_u B = D_u B f = D_u B f^2 = \cdots = D_u B f^{p-1}. \]
Choose an integer \( v \), \( 1 \leq v \leq p - 1 \). The equation \( D_u B = D_u B f^{p-v} \) can be written in the form
\[ D_u (C_0 \cup C_1 f^{1} \cup \cdots \cup C_{p-1} f^{p-1}) = D_u (C_0 f^{p-v} \cup C_1 f^{p-v+1} \cup \cdots \cup C_{p-1} f^{p-v+p-1}), \]
that is, in the form
\[ D_u C_0 \cup D_u C_1 f^{1} \cup \cdots \cup D_u C_{p-1} f^{p-1} =\]
\[ D_u C_0 f^{p-v} \cup D_u C_1 f^{p-v+1} \cup \cdots \cup D_u C_{p-1} f^{p-v+p-1}. \]
Groups admitting only quasi-periodic factorizations

Intersecting by $H$ we get $D_uC_0 = D_uC_v$. This holds for each $u, v$, $1 \leq u \leq p - 1$, $1 \leq v \leq p - 1$. (The $u = 0$ case is not covered.)

It remains to verify that $D_0C_v = D_0C_0$ holds for each $v$, $1 \leq v \leq p - 1$. In fact we will show that $D_uC_0D_uC_v$ holds for each $u, v$, $0 \leq u \leq p - 2$, $1 \leq v \leq p - 1$. Multiplying the factorization $G = AB$ by $f$ we get the normalized factorization $G = (Af)B$. Here

$$Af = D_0f \cup D_1f^2 \cup \cdots \cup D_{p-2}f^{p-1} \cup D_{p-1}.$$

(Now $D_{p-1}$ will play the earlier role of $D_0$.) Repeating the whole argument in this setting we get the required result.

References


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