Annales Univ. Sci. Budapest., Sect. Comp. 29 (2008) 239-243

GROUPS ADMITTING ONLY QUASI–PERIODIC FACTORIZATIONS

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Dedicated to Imre Kátai on his 70th birthday

Abstract. We will show that if a finite abelian group is a direct product of two subgroups of relatively prime orders and one of the subgroups has prime order, then each normalized factorization of the group must be quasi-periodic.

1. Introduction

Let G be a finite abelian group and let A, B be a subsets of G. If each $g \in G$ can be written in the form

$$(1) \qquad g = ab, \quad a \in A, \quad b \in B,$$

then the product AB is equal to G. If each $g \in G$ can be written uniquely in the form (1), then we say that the product AB is direct and we refer to the equation $G = AB$ as a factorization of G.

If the identity element e belongs to the subset e A of G, then we say that A is a normalized subset of G. If both factors A and B of a factorization $G = AB$ are normalized subsets we call the factorization a normalized factorization.

A subset A of G is defined to be periodic if there is a $g \in G \subset \{e\}$ such that $Ag = A$. Note that if g and h are periods of A, then so is gh provided $gh \neq e$. It follows then that the periods of A together with the identity element

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Mathematics Subject Classification: 20K01 (05B45, 52C22, 68R05) form a subgroup H of G. We will call H the subgroup of periods of A. The reader can verify that there is a subset A_1 of G such that the product A_1H is direct and is equal to A. If A is normalized then $H \subset A$ and A_1 can be chosen to be a subset of A. If one of the factors in the factorization $G = AB$ is periodic, then we call the factorization periodic.

We say that a partition $B = B_1 \cup \cdots \cup B_n$ is a regular partition of the subset B of G if there is a periodic subset $C = \{c_1, \ldots, c_n\}$ of G such that $c_1 = e$, $|C| \geq 2$ and $AB_i = AB_1c_i$ for each i, $1 \leq i \leq n$. According to [2] we call a normalized factorization $G = AB$ quasi-periodic if one of the factors has a regular partition.

In [3] one can find another definition of the quasi-periodic factorization. Namely, a partition $B = B_1 \cup \cdots \cup B_n$ of the subset B of G is defined to be regular if there is a subgroup $H = \{h_1, \ldots, h_n\}$ of G such that $h_1 = e$, $|H| \geq 2$ and $AB_i = AB_1h_i$ for each i, $1 \leq i \leq n$. The normalized factorization $G = AB$ is called quasi-periodic if at least one of the factors A, B has a regular partition. The two definitions are equivalent. A proof can be found in [7].

In 1950 G. Hajós [2] asked if each factorization of each finite abelian group is quasi-periodic. In 1974 A.D. Sands [4] showed that an earlier factorization construction of N.G. de Bruijn [1] is a counter-example for the Hajós question. It was verified in [8] that another factorization construction of N.G. de Bruijn is not quasi-periodic either. In [7] a large family of nonquasi-periodic factorizations was exhibited.

One might wonder if each normalized factorization of a finite abelian group G is quasi-periodic, then does it hold for each subgroup H of G. A.D. Sands [6] showed that for this question the answer is "no". Using Sands' method in this paper we will show that if a finite abelian group G is a direct product of its subgroups H and K such that $|H|$, $|K|$ are relatively prime and $|K|$ is a prime, then G admits only quasi-periodic normalized factorizations.

2. The result

Let G be a finite abelian group and let H be a subgroup of G. If $f_1, \ldots, f_n$ is a complete set of representatives of cosets modulo H, then the cosets $Hf_1, \ldots, Hf_n$ form a partition of G. Therefore for a given subset A of G the sets $A_1 = A \cap Hf_1, \ldots, A_n = A \cap Hf_n$ form a partition of A. We will use this observation in the proof of the next theorem. The other observation we will use is that $G = AB$ is a factorization of G if and only if the sets Ab, $B \in B$ form a partition of G.

Theorem 1. *Let p be a prime. Let G be a finite abelian group that is the direct product of its subgroups H and K. If p does not divide $|H|$ and $|K| = p$, then each normalized factorization of G is quasi-periodic.*

Proof. Let $G = AB$ be a normalized factorization of G. Since the order of G is equal to the product of the orders of A and B we may assume that $p \not| |A|$ and $p||B|$. Let $K = \langle f \rangle$. The factor A can be written in the form

$$A = A_0 \cup A_1 \cup \cdots \cup A_{p-1},$$

where $A_i = A \cap Hf^i$. Further there are subsets $D_0, \ldots, D_{p-1} \subset H$ such that $A_i = D_i f^i$. Thus

$$A = D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1}.$$

Similarly B can be written in the form

$$B = B_0 \cup B_1 \cup \cdots \cup B_{p-1},$$

where $B_i = B \cap Hf^i$. There are subsets $C_0, \ldots, C_{p-1} \subset H$ such that $B_i = C_i f^i$. Therefore

$$B = C_0 \cup C_1 f^1 \cup \cdots \cup C_{p-1} f^{p-1}.$$

We claim that $AB_v = AB_0 f^v$ holds for each $v,\ 1 \leq v \leq p-1$. Using the above notations the claim is equivalent to

$$(D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1})(C_v f^v) = (D_0 \cup D_1 f^1 \cup \cdots \cup D_{p-1} f^{p-1})(C_0) f^v,$$

that is, equivalent to

$$D_0 C_v \cup D_1 C_v f^1 \cup \cdots \cup D_{p-1} C_v f^{p-1} = D_0 C_0 \cup D_1 C_0 f^1 \cup \cdots \cup D_{p-1} C_0 f^{p-1}.$$

In fact we will verify that $D_u C_v = D_u C_0$ holds for each $u,\ v,\ 0 \leq u \leq p-1,\ 1 \leq$ $\leq v \leq p-1$.

Choose an integer $t(i)$ such that

$$t(i) \equiv 1 \pmod{|H|},$$

$$t(i) \equiv i \pmod{|K|}.$$

Such a $t(i)$ exists for each $i,\ 0 \leq i \leq p-1$. By Proposition 3 of [5], in the factorization $G = AB$ the factor A can be replaced by $A^{t(i)}$ to get the normalized factorization $G = A^{t(i)} B$. Note that

$$\begin{aligned} A^{t(i)} &= D_0^{t(i)} \cup D_1^{t(i)} f^{t(i)} \cup \cdots \cup D_{p-1}^{t(i)} f^{(p-1)t(i)} = \\ &= D_0 \cup D_1 f^i \cup \cdots \cup D_{p-1} f^{(p-1)i}. \end{aligned}$$

The factorization

$$G = (D_0 \cup D_1 f^i \cup \cdots \cup D_{p-1} f^{(p-1)i})B$$

provides that the sets

$$(2) \qquad D_0B,\ D_1Bf^i, \ldots, D_{p-1}Bf^{(p-1)i}$$

form a partition of G. The $i=0$ and $i=1$ special cases of (2) give that

$$(3) \qquad D_1B \cup D_2B \cup \cdots \cup D_{p-1}B = D_1Bf \cup D_2Bf^2 \cup \cdots \cup D_{p-1}f^{p-1}B.$$

Choose an integer $u,\ 1 \leq u \leq p-1$. Note that

$$D_uB \cap D_wBf^w = \emptyset$$

for each $w,\ 1 \leq w \leq p-1,\ w \neq u$. In order to verify the claim assume on the contrary that

$$D_uB \cap D_wBf^w = \emptyset$$

for some u and w. Multiplying by f^{ui} we get

$$(4) \qquad D_uBf^{ui} \cap D_wBf^{w+iu} = \emptyset.$$

The congruence

$$wi \equiv w + iu \pmod{p}$$

is solvable for i and so (4) contradicts to (2). This contradiction proves our claim.

From (3) it follows that $D_uB = D_uBf^u$. Using this equation repeatedly we get that

$$D_uB = D_uBf = D_uBf^2 = \cdots = D_uBf^{p-1}.$$

Choose an integer $v,\ 1 \leq v \leq p-1$. The equation $D_uB = D_uBf^{p-v}$ can be written in the form

$$D_u(C_0 \cup C_1f^1 \cup \cdots \cup C_{p-1}f^{p-1}) = D_u(C_0f^{p-v} \cup C_1f^{p-v+1} \cup \cdots \cup C_{p-1}f^{p-v+p-1}),$$

that is, in the form

$$D_uC_0 \cup D_uC_1f^1 \cup \cdots \cup D_uC_{p-1}f^{p-1} =$$

$$D_uC_0f^{p-v} \cup D_uC_1f^{p-v+1} \cup \cdots \cup D_uC_{p-1}f^{p-v+p-1}.$$

Intersecting by H we get $D_uC_0 = D_uC_v$. This holds for each $u, v,\ 1 \leq u \leq$ $\leq p-1,\ 1 \leq v \leq p-1$. (The $u = 0$ case is not covered.)

It remains to verify that $D_0C_v = D_0C_0$ holds for each $v,\ 1 \leq v \leq p-1$. In fact we will show that $D_uC_0D_uC_v$ holds for each $u, v,\ 0 \leq u \leq p-2,\ 1 \leq$ $\leq v \leq p-1$. Multiplying the factorization $G = AB$ by f we get the normalized factorization $G = (Af)B$. Here

$$Af = D_0f \cup D_1f^2 \cup \cdots \cup D_{p-2}f^{p-1} \cup D_{p-1}.$$

(Now D_{p-1} will play the earlier role of D_0.) Repeating the whole argument in this setting we get the required result.

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