THE VOICE TRANSFORM ON THE BLASCHKE GROUP II.

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. In this paper we present results connected to the voice transform of the Blaschke group generated by a representation of the group.

Sections 1 and 2 contain the basic notations, definitions and results. In Section 3 the matrix elements of the representation are computed. It is showed that they can be given by the Zernike functions which play an important role in expressing the wavefront data in optical tests. An important consequence of this connection is the addition formulae for Zernike functions.

Theorem 2 shows that the voice transform can be expressed as a sum of infinite series. This permits the reconstruction of the function from its voice transform. A consequence of this result is the injectivity of the voice transform. If the parameter function is a trigonometric polynomial, then the voice transform can be represented as a differential operator and using this result admissibility conditions for the parameter can be given.

Sections 4 contains the proofs.

1. The voice transform

In signal processing and image reconstruction the wavelet-, Gábor-trans-

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Mathematics Subject Classification: 43A32, 42C40, 33C47, 43A65 forms play an important role. There exists a common generalization of these transforms, the so-called *voice-transform.* In this section we summarize the basic notations and notions used in the definition of voice-transform, we also present the definition and the most important properties of this transform.

In construction of voice-transform the starting point will be a locally compact topological group $(G,\cdot)$. It is known that every locally compact topological group has nontrivial left- and right-translation invariant Borel-measures, called left invariant and right invariant Haar measures. Let m be a left-invariant Haar measure of G. Let $f : G \to \mathbb{C}$ be a Borel-measurable function which is integrable with respect to the left invariant Haar measure m, the integral of f will be denoted by $\int\limits_G f\,dm = \int\limits_G f(x)\,dm(x)$. Because of left-translation invariance of the measure m it follows that

$$\int\limits_G f(x)\,dm(x) = \int\limits_G f(a^{-1}\cdot x)\,dm(x) \quad (a \in G).$$

There exist groups whose left invariant Haar measure is not right invariant. If the left invariant Haar measure of G is at the same time right invariant then we say that G is *unimodular* . Such measure will be called Haar measure of G. On a given group, Haar measure is unique only up to constant multiples. It is trivial that the commutative groups are unimodular. Furthermore it can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is also unimodular.

In the definition of voice-transform a *unitary representation of the group* $(G,\cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I, the identity operator on H and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1}, which is equal to the adjoint of U : $U^{-1} = U^*$. The homomorphism U of the group $(G,\cdot)$ on the group $(\mathcal{U},\circ)$ satisfying

$$i)\ \ U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G),$$

$$ii)\ \ G \ni x \to U_x f \in H \ \text{ is continuous for all } f \in H, \tag{1.1}$$

is called unitary representation of $(G,\cdot)$ on H.

The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x\rho\rangle \quad (x \in G, f, \rho \in H). \tag{1.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G.

The set of continuous bounded functions defined on the group G with the supremum norm form a Banach space and $V_\rho : H \to C(G)$ is a bounded linear operator. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$,

$$|(V_\rho f)(x)| = |\langle f, U_x \rho\rangle| \leq \|f\|\|U_x\rho\| = \|f\|\|\rho\|,$$

consequently $\|V_\rho\| \leq \|\rho\|$.

Taking as starting point (not necessarily commutative) locally compact groups we can construct in this way important transformations in signal processing and control theory. For example the affine wavelet transform and the Gábor-transform are all special voice transforms (see [6], [10]).

2. The voice transform of the Blaschke group

The affine wavelet transform is a voice transform of the affine group which is a subgroup of the Möbius group (i.e. the group of linear fractional transformations with the composition operation). In this section we will study the voice transform of another subgroup of the Möbius group, namely the voice transform of the Blaschke group.

2.1. The Blaschke group

Let us denote by

$$B_a(z) := \epsilon \frac{z - b}{1 - \bar{b}z} \quad (z \in \mathbb{C}, \quad a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \quad \bar{b}z \neq 1) \tag{2.1}$$

the so called *Blaschke functions*, where

$$\mathbb{D}_+ := \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}, \ \mathbb{D}_- := \{z \in \mathbb{C} : |z| > 1\}. \tag{2.2}$$

If $a \in \mathbb{B}$, then B_a is an 1-1 map on $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{D}_-$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group

$(\{B_a, a \in \mathbb{B}\}, \circ)$. If we use the notations $a_j := (b_j, \epsilon_j)$, $j \in \{1, 2\}$ and $a := := (b, \epsilon) =: a_1 \circ a_2$ then

$$\text{(2.3)} \quad b = \frac{b_1\bar{\epsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2,1)}(b_1\bar{\epsilon}_2), \quad \epsilon = \epsilon_1 \frac{\epsilon_2 + b_1\bar{b}_2}{1 + \epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\epsilon_1)}(\epsilon_2).$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \bar{\epsilon})$.

Because of $B_a : \mathbb{T} \to \mathbb{T}$ is bijection it follows the existence of a function $\beta_a : \mathbb{R} \to \mathbb{R}$ such that $B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R})$, where β_a can be expressed in an explicit form. Namely, let us introduce the function

$$\text{(2.4)} \qquad \gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2r\cos s + r^2}\, ds \quad (t \in \mathbb{R}, 0 \le r \le 1).$$

Then
(2.5)

$$\beta_a(t) := \theta + \varphi + \gamma_r(t - \varphi), \quad (a = (re^{i\varphi}, e^{i\theta}) \in \mathbb{B},\ t \in \mathbb{R},\ \theta, \varphi \in \mathbb{I} := [-\pi, \pi)).$$

The integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the left invariant Haar-measure m of the group $(\mathbb{B}, \circ)$, is given by

$$\text{(2.6)} \qquad \int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt,$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $a \to a_0 \circ a$ and under the inverse transformation $a \to a^{-1}$, so this group is unimodular.

We will study the voice transform of the Blaschke group. In the construction it will be used a class of unitary representations of the Blaschke group on the Hilbert space $H = L^2(\mathbb{T})$.

2.2. The voice transform on $L^2(\mathbb{T})$

In this section we summarize the results obtained in [9]. In this paper the voice transform on the Hilbert space $H = L^2(\mathbb{T})$ was constructed, where the inner product is given by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\, dt \quad (f, g \in H).$$

The trigonometric system $\epsilon_n(t) = e^{int}(t \in \mathbb{I}, n \in \mathbb{Z})$ is orthonormal and complete with respect to this scalar product.

It was proved that

$$(U_a f)(e^{it}) := f(e^{i\beta_{a^{-1}}(t)})(\beta'_{a^{-1}}(t))^{1/2} e^{i(\beta_{a^{-1}}(t)-t)/2} \quad (a \in \mathbb{B}) \tag{2.7}$$

is a unitary representation of the Blaschke group on $L^2(\mathbb{T})$.

Denote by $H^2(\mathbb{T})$ the closure in $L^2(\mathbb{T})$-norm of the set

$$span\{\epsilon_n, n \in \mathbb{N}\}.$$

The functions which belong to $H^2(\mathbb{T})$ can be obtained as boundary limits of the functions from Hardy space $H^2(\mathbb{D})$ (see [15]).

The restriction of this representation to $H^2(\mathbb{T})$ can be expressed in the following form:

(2.8)

$$(U_{a^{-1}} f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\bar{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\bar{b}z}\right) \quad \big(z = e^{it} \in \mathbb{T},\ a = (b, e^{i\theta}) \in \mathbb{B}\big).$$

The voice transform generated by U_a $(a \in \mathbb{B})$ is given by

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \quad (f, \rho \in H^2(\mathbb{T})). \tag{2.9}$$

Let consider the shift operator

$$(S\varphi)(z) = z\varphi(z), \quad \varphi \in H^2(\mathbb{T}),$$

and let be $\varphi = 1 \in H^2(\mathbb{T})$. Then the *discrete Laguerre* functions can be generated by the shift operator and by the representation operator in the following way:

$$(U_{a^{-1}} S^m \varphi)(z) = \frac{\sqrt{\epsilon(1-|b|^2)}}{(1-\bar{b}z)} \left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^m \quad (z \in \mathbb{T}). \tag{2.10}$$

Thus the discrete Laguerre functions can be considered as a wavelet generated by the mother wavelet $\varphi = 1$. It is known that $((U_{a^{-1}} S^m \varphi)(z),\ m \in \mathbb{N})$ forms an orthogonal basis in $H^2(\mathbb{T})$, for all $a \in \mathbb{B}$. Extending the functions in (2.10) to $\mathbb{D}$ let define $\varphi_{a,m}(z)$ by (2.10) for $z \in \mathbb{D}$.

Let $V_{\epsilon_m} f(a^{-1}) = \langle f, U_{a^{-1}}\epsilon_m\rangle$ and let define the following projection operator

$$Pf(a,z) := \sum_{m=0}^{\infty} (V_{\epsilon_m} f)(a^{-1})\varphi_{a,m}(z) \quad (a \in \mathbb{B},\ z \in \mathbb{D}), \tag{2.11}$$

where the infinite series is absolute convergent for $z \in \mathbb{D}$.

Theorem 1. *For every $f \in H^2(\mathbb{T})$, for every $z = r_1 e^{it} \in \mathbb{D}$ and for every $a \in \mathbb{B}$*

$$\lim_{r_1 \to 1} Pf(a, z) = f(e^{it}), \tag{2.12}$$

a.e. $t \in \mathbb{I}$ and in H^2 norm. If $f \in C(\mathbb{T})$, then the convergence is uniform.

If $a = e = (0, 1)$ then $V_{\epsilon_m} f(e) = \langle f, \epsilon_m \rangle$ and $U_{e^{-1}} \epsilon_m(z) = z^m$, consequently in this special case (2.12) is the Abel summation for the trigonometric series.

3. New results

3.1. The matrix of the representation U

In this section we compute the matrix elements of the representation U given by formulae (2.8). It is showed that they can be expressed by the Zernike functions which play an important role in expressing the wavefront data in optical tests. An important consequence of this connection is the addition formulae for Zernike functions.

The matrix elements $v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}} \epsilon_m \rangle$ of representation U with respect to the basis $\{\epsilon_n : n \in \mathbb{N}\}$ can be expressed using the trigonometric system $\epsilon_n(\varphi) := e^{in\varphi}$ $(n \in \mathbb{Z}, \varphi \in \mathbb{I})$ and using the associated Legendre polynomials:

(3.1)

$$P_n^{\ell}(x) := \frac{x^{-\ell}}{n!}[(1-x)^n x^{n+\ell}]^{(n)}, \quad P_n^{-\ell}(x) := (-1)^{\ell} P_n^{\ell}(x) \quad (x \in [0,1], \ n, \ell \in \mathbb{N}),$$

which are orthogonal on $[0, 1]$ with respect to the weight function x^{ℓ} for a fix ℓ:

$$\int_0^1 P_m^{\ell}(x) P_n^{\ell}(x) x^{\ell} \, dx = \delta_{mn} \frac{1}{2n + |\ell| + 1} \qquad (n, m \in \mathbb{N}, \ell \in \mathbb{Z}). \tag{3.2}$$

From (2.8) it follows that for $a = (re^{i\varphi}, e^{i\psi})$

$$v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}} \epsilon_m \rangle =$$

(3.3) $$= \frac{e^{-i(m+1/2)\psi}\sqrt{1-r^2}}{2\pi}\int_{-\pi}^{\pi}\frac{(e^{-it}-re^{-i\varphi})^m}{(1-re^{i(-t+\varphi)})^{m+1}}e^{int}\,dt.$$

Making the change of variables $t = s + \varphi$, we obtain that

$$v_{mn}(a^{-1}) = \frac{e^{-i(m+1/2)\psi}e^{i(n-m)\varphi}\sqrt{1-r^2}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{ins}(e^{-is}-r)^m}{(1-re^{-is})^{m+1}}\,dt =$$

(3.4) $$= \sqrt{1-r^2}e^{-i(m+1/2)\psi}e^{i(n-m)\varphi}\alpha_{mn}(r),$$

where
(3.5)

$$\alpha_{mn}(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(e^{-is}-r)^m}{(1-re^{-is})^{m+1}}e^{ins}\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-re^{is})^m}{(e^{is}-r)^{m+1}}e^{i(n+1)s}\,ds.$$

In this last integral making the change of variables $\zeta = e^{is}$ and applying the Cauchy integral formula we get that

$$\alpha_{mn}(r) := \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(1-r\zeta)^m}{(\zeta-r)^{m+1}}\zeta^n\,d\zeta =$$

(3.6) $$= \frac{r^{-n}}{m!}\frac{d^m}{dz^m}[(1-rz)^m(rz)^n]_{z=r} = \frac{r^{-n+m}}{m!}\frac{d^m}{dx^m}[(1-x)^m x^n]_{x=r^2}.$$

If $n \geq m$ let us denote $n = m + \ell$, then $\alpha_{mn}(r)$ can be expressed by the associated Legendre polynomials, namely

(3.7) $$\alpha_{mn}(r) = P_m^\ell(r^2) = (-1)^m r^\ell P_m^{(0,\ell)}(2r^2-1).$$

Consequently

$$v_{mn}(a^{-1}) = \sqrt{1-r^2}e^{-i(m+1/2)\psi}e^{i(n-m)\varphi}(-1)^m r^\ell P_m^{(0,\ell)}(2r^2-1) =$$

$$= \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}}e^{-i(m+1/2)\psi}(-1)^m Y_m^{n-m}(r,\varphi),$$

where $Y_m^{n-m}(r,\varphi)$ are the complex Zernike polynomials (see [8]). If $n<m$, then

$$v_{mn}(a^{-1}) := \langle \epsilon_n, U_{a^{-1}}\epsilon_m\rangle = \langle U_a\epsilon_n, \epsilon_m\rangle = \overline{\langle \epsilon_m, U_a\epsilon_n\rangle} = \overline{v_{nm}(a)} =$$

$$= \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}} e^{-i(m+1/2)\psi}(-1)^m Y_n^{m-n}(r,\varphi).$$

Consequently

$$v_{mn}(a^{-1}) = \frac{\sqrt{1-r^2}}{\sqrt{m+n+1}} e^{-i(m+1/2)\psi}(-1)^m Y_{\min\{n,m\}}^{|m-n|}(r,\varphi). \tag{3.8}$$

It is known that the matrix elements of the representations satisfy the following so called addition formula

$$v_{mn}(a_1\circ a_2) = \sum_k v_{mk}(a_1)v_{kn}(a_2) \quad (a_1,a_2\in\mathbb{B}).$$

From this relation we obtain the following addition formulae for Zernike functions:

$$\frac{\sqrt{1-r^2}}{\sqrt{(n+m+1)(1-r_1^2)(1-r_2^2)}} e^{-i(m+1/2)\psi} Y_{\min\{m,n\}}^{|n-m|}(r,\varphi) = \tag{3.9}$$

$$= \sum_k \frac{(-1)^k e^{-i(m+1/2)\psi_1} e^{-i(k+1/2)\psi_2}}{\sqrt{(m+k+1)(n+k+1)}} Y_{\min\{m,k\}}^{|k-m|}(r_1,\varphi_1) Y_{\min\{k,n\}}^{|n-k|}(r_2,\varphi_2),$$

where $a_j := (r_je^{i\varphi_j}, e^{i\psi_j})$, $j\in\{1,2\}$ and $a := (re^{i\varphi}, e^{i\psi}) = a_1\circ a_2$.

Another way to determine $v_{mn}(a)$ occurs from the definition. We observe that they are equal by the m-th Fourier coefficients of the function $\varphi_{a^{-1},n}(z)$ in (2.10), namely

$$v_{mn}(a) = \frac{1}{m!}\frac{d^m}{d^m z}(U_a(z))_{z=0} =$$

$$= \frac{1}{m!} e^{-i\psi(1/2+n)} \sqrt{(1-|b|^2)} \frac{d^m}{d^m z}\left[(z+be^{i\psi})^n(1+\overline{b}e^{-i\psi}z)^{-(n+1)}\right]_{z=0}.$$

Using the Leibniz formulae we obtain that

$$v_{mn}(a) = \frac{1}{m!} e^{-i\psi(1/2+n)} \sqrt{(1-|b|^2)} \times$$

$$\times \sum_{k=0}^{\min\{m,n\}} C_m^k (-1)^{m-k} (n+1)(n+2)...(n+m-k)(\bar{b}e^{-i\psi})^{m-k} \cdot$$

$$\cdot n(n-1)...(n-k+1)(be^{i\psi})^{n-k} =$$

$$= e^{-i\psi(1/2+m)} \sqrt{(1-|b|^2)} \sum_{k=0}^{\min\{m,n\}} C_m^k (-1)^{m-k} C_{n+m-k}^{n-k} (\bar{b})^{m-k} b^{n-k} =$$

(3.10)

$$= e^{-i\psi(1/2+m)} e^{i\varphi(n-m)} \sqrt{(1-r^2)} \sum_{k=0}^{\min\{m,n\}} (-1)^{m-k} C_m^k C_{n+m-k}^{n-k} r^{n+m-2k}.$$

Comparing this with (3.4) we deduce that the radial part of complex Zernike functions (see [1]) can be expressed in the following form

$$\alpha_{mn}(r) = \sum_{k=0}^{\min\{m,n\}} (-1)^{m-k} C_m^k C_{n+m-k}^{n-k} r^{n+m-2k}. \tag{3.11}$$

For the special case when $b \in (-1, 1)$ the expression of $v_{mn}(a)$ was obtained in [4].

3.2. The representation of voice transform as a sum of infinite series and differential operator

In what follows we show that the voice transform can be expressed as a sum of infinite series which is absolutely convergent and is convergent in norm, induced by the following inner product

$$\langle\langle F, G \rangle\rangle := \frac{1}{2\pi} \int_0^1 \int_{\mathbb{I}} \frac{rF(re^{i\varphi})\overline{G(re^{i\varphi})}}{1-r^2} d\varphi dr. \tag{3.12}$$

We also give a representation of the voice transform as differential operator, which is easy to handel. This representation can be used in the construction of the so called *admissible functions.*

Theorem 2. *Let us consider $\rho \in H^2(\mathbb{T})$, let us denote $b_n := \langle \rho, \epsilon_n \rangle$ and suppose that $\sum_{n=0}^{\infty} |b_n| < \infty$, then for all $f \in H^2(\mathbb{T})$ and $a = (re^{i\varphi}, 1) \in \mathbb{B}$*

$$V_\rho f(a^{-1}) = (V_\rho f)(re^{i\varphi}) = \sqrt{1-r^2} \sum_{\ell=-\infty}^{\infty} r^{|\ell|} e^{-i\ell\varphi} \sum_{n=0}^{\infty} c_n^\ell P_n^\ell(r^2), \tag{3.13}$$

where the infinite series is absolute convergent and convergent in norm induced by (3.12) and the coefficients c_n^ℓ can be expressed using the trigonometric Fourier coefficients of f and ρ :

$$c_n^\ell := \langle f, \epsilon_n \rangle \overline{\langle \rho, \epsilon_{n+\ell} \rangle} \quad (\ell \geq 0), \quad c_n^\ell := \langle f, \epsilon_{n-\ell} \rangle \overline{\langle \rho, \epsilon_n \rangle} \quad (\ell < 0), \tag{3.14}$$

furthermore

$$\langle \langle V_\rho f, V_\rho f \rangle \rangle = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|\langle f, \epsilon_n \rangle|^2 |\langle \rho, \epsilon_m \rangle|^2}{n+m+1}. \tag{3.15}$$

From the formulae (3.15) it follows that the voice transform is injective for every $\rho \neq 0$, $\rho \in H^2(\mathbb{T})$ satisfying $\sum_{n=0}^{\infty} |b_n| < \infty$. Indeed if $\rho \neq 0$, $\rho \in H^2(\mathbb{T})$, then there exists $m \in \mathbb{N}$ such that $\langle \rho, \epsilon_m \rangle \neq 0$. From (3.15) it follows that if $V_\rho f = 0 \quad (f \in H^2(\mathbb{T}))$, then $\langle f, \epsilon_n \rangle = 0 \quad (n \in \mathbb{N})$, which implies that $f = 0$. This implies the injectivity of $V_\rho f$.

Taking into account the orthogonality of trigonometric system and the orthogonality of polynomials P_n^ℓ, these coefficients can be expressed by voice transform in the following way:

(3.16)

$$c_n^\ell = \frac{2n+\ell+1}{2\pi} \int_0^1 \int_{\mathbb{I}} (V_\rho f)(re^{i\varphi}) r^{|\ell|} e^{i\ell\varphi} P_n^\ell(r^2) \frac{r}{\sqrt{1-r^2}} d\varphi dr \quad (\ell \in \mathbb{Z}, n \in \mathbb{N}).$$

On the base of (3.16) if we know the voice transform $V_\rho f$ the coefficients c_n^ℓ can be computed and from (3.14) the Fourier coefficients of f can be obtained. Consequently beside (2.12) this is a way for the reconstruction of f .

Let fix a polynomial

$$\kappa(z) := c_0 + c_1 z + \cdots + c_N z^N \quad (z \in \mathbb{C})$$

and a complex number $b \in \mathbb{C}$ and let denote by A the set of analytic functions on $\mathbb{D}$. Denote by $\alpha_b(z) := 1 - \overline{b} z \quad (z \in \mathbb{C})$. For every $f \in \mathcal{A}$ let be

$$L_\kappa^b f := \sum_{n=0}^{N} \frac{\overline{c_n}}{n!} (\alpha_b^n f)^{(n)}. \tag{3.17}$$

It is trivial that for $b = 0$ this is a differential operator with constant coefficients, whose characteristic polynomial is $\overline{\kappa(\overline{z})}$ $(z \in \mathbb{C})$.

To represent the voice transform

$$(3.18)\quad (V_\rho f)(a^{-1}) := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{it})\frac{\sqrt{1-|b|^2}}{1-be^{-it}}\overline{\rho(B_a(e^{it}))}\,dt \quad (a=(b,1)\in\mathbb{B})$$

as a differential operator in connection of $f\in L^2(\mathbb{T})$ we introduce two functions belonging to $H^2(\mathbb{D})$. For an arbitrary function

$$f(e^{it}) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad (t\in\mathbb{I})$$

let denote by

$$(3.19)\qquad f^*(z) := \sum_{n=0}^{\infty} a_n z^n, \quad f_*(z) = \sum_{n=0}^{\infty} a_{-n-1} z^n \quad (z\in\mathbb{D}).$$

Then $f^*, f_* \in H^2(\mathbb{D})$ and

$$f(e^{it}) = f^*(e^{it}) + e^{-it} f_*(e^{-it}) \quad \text{(for almost every } t\in\mathbb{I}).$$

Theorem 3. *For every function $f\in L^2(\mathbb{T})$ and for every trigonometric polynomial $\rho\in L^2(\mathbb{T})$ the voice transform $V_\rho f$ of f can be represented as*

$$(3.20)\qquad V_\rho f(a^{-1}) = \sqrt{1-|b|^2}[(L^b_{\rho^*} f^*)(b) + (L^{\overline{b}}_{\rho_*} f_*)(\overline{b})] \quad (a=(b,1)\in\mathbb{B}).$$

From Theorem 3 it follows that V_ρ represents the space $L^2(\mathbb{T})$ in the space $\mathcal{A}$ of analytic functions. If $f\in\mathcal{A}$, then $f_*=0$, consequently in this case

$$(3.21)\qquad V_\rho f(a^{-1}) = \sqrt{1-|b|^2}(L^b_{\rho^*} f)(b)) \quad (a=(b,1)\in\mathbb{B}).$$

In the special case when $\rho = 1$,

$$(3.22)\qquad V_1 f(a^{-1}) = \sqrt{1-|b|^2} f(b) \quad (a=(b,1)\in\mathbb{B}).$$

Let

$$\rho(e^{it}) = \frac{e^{it}+e^{-it}}{2} = \cos t \quad (t\in\mathbb{I}).$$

Then

$$\rho^*(z) = z,\ \rho_*(z) = 1 \quad (z\in\mathbb{C}),$$

and the voice transform is equal by the following differential operator for every $(a = (b, 1) \in \mathbb{B})$:

$$(3.23)\quad (V_\rho f)(a^{-1}) = \frac{\sqrt{1-|b|^2}}{2}(\alpha_b f)'(b) = \frac{\sqrt{1-|b|^2}}{2}[(1-|b|^2)f'(b) - \bar{b}f(b)].$$

On the base of (3.20) we can give a sufficient condition for the admissibility of $\rho \in L^2(\mathbb{T})$, namely when ρ satisfies the condition $V_\rho\rho \in L^2_m(\mathbb{B})$. If $a = (b, 1) \in$ $\in \mathbb{B}$ does not depend on θ, then the function $F : \mathbb{B} \to \mathbb{C}$ belongs to $L^2_m(\mathbb{B})$ if

$$\int_{\mathbb{D}} |F(b)|^2 \frac{db_1 db_2}{(1-|b|^2)^2} < \infty.$$

From (3.22) it follows easily that if $\rho = 1$ then $V_\rho\rho \notin L^2_m(\mathbb{B})$, consequently this function is not admissible.

If $\rho(e^{it}) = \cos t$, then on the base of (3.20)

$$(V_\rho\rho)(a^{-1}) = (1-|b|^2)^{3/2},$$

and

$$\int_{\mathbb{B}} |V_\rho\rho|^2\, dm = \frac{1}{2\pi}\int_{\mathbb{D}} (1-|b|^2)\, db_1 db_2 = \int_0^1 (1-r^2)r\, dr = \frac{1}{4},$$

which means that $\rho(e^{it}) = \cos t$ is admissible.The following theorem refers on the square integrability of $V_\rho\rho$.

Theorem 4. *Let suppose that ρ is a real trigonometric polynomial and ρ^* is an odd or even algebraic polynomial which vanishes in* 0, *namely*

$$b_0 = 0, \quad b_k = \overline{b_{-k}} \quad (N \in \mathbb{N}^*, k \le N), \qquad \rho^*(-z) = \pm\rho^*(z) \quad (z \in \mathbb{D}).$$

Then ρ is an admissible function for the representation U_a which means that $V_\rho\rho \in L^2_m(\mathbb{B})$.

4. Proofs

In this section the proofs of previous results will be presented.

Proof of Theorem 2. Let denote by $a_n := \langle f, \epsilon_n \rangle$ and $b_n := \langle \rho, \epsilon_n \rangle$. Then using (2.9) and (3.4), for $a = (re^{i\varphi}, 1)$, the voice transform can be expressed with the following series:

$$\begin{aligned}(V_\rho f)(a^{-1}) &= \left\langle \sum_{n=0}^{\infty} a_n \epsilon_n, \sum_{m=0}^{\infty} b_m U_{a^{-1}} \epsilon_m \right\rangle = \\ &= \sum_{m=0}^{\infty} \overline{b_m} \sum_{n=0}^{\infty} a_n \langle \epsilon_n, U_{a^{-1}} \epsilon_m \rangle = \\ &= \sum_{m=0}^{\infty} \overline{b_m} \sum_{n=0}^{\infty} a_n v_{mn}(a).\end{aligned}$$

This double infinite series is absolutely and uniformly convergent if $a = (re^{i\varphi}, 1) \in \mathbb{B}$, $f \in H^2(\mathbb{T})$ and if ρ satisfies the assumptions of the Theorem 3. Indeed, $f \in H^2(\mathbb{T})$ implies that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, the unitarity of the representation U_a implies that $\sum_{n=0}^{\infty} |v_{mn}(a)|^2 = 1$ $(m \in \mathbb{N})$, then using the Cauchy- Schwartz inequality we obtain that

$$\sum_{m=0}^{\infty} |\overline{b_m}| \sum_{n=0}^{\infty} |a_n||v_{mn}(a)| \leq \sum_{m=0}^{\infty} |\overline{b_m}| \sqrt{\sum_{n=0}^{\infty} |a_n|^2} \sqrt{\sum_{n} |v_{mn}(a)|^2} < \infty.$$

Rearranging the series and using (3.4) we obtain that

$$\begin{aligned}(V_\rho f)(a^{-1}) =& \sqrt{1-r^2} \sum_{m,n=0}^{\infty} a_n \overline{b_m} e^{i(n-m)\varphi} \alpha_{mn}(r) = \\ =& \sqrt{1-r^2} \sum_{\ell=0}^{\infty} e^{-i\ell\varphi} \sum_{n=0}^{\infty} a_n \overline{b_{n+\ell}} \alpha_{n+\ell,n}(r) + \\ &+ \sqrt{1-r^2} \sum_{\ell=1}^{\infty} e^{i\ell\varphi} \sum_{n=0}^{\infty} a_{m+\ell} \overline{b_m} \alpha_{m,m+\ell}(r).\end{aligned}$$

For an arbitrary index $n \in \mathbb{N}$ let introduce the notations

$$c_n^\ell := a_n \overline{b_{n+\ell}} \ \ (\ell \geq 0), \quad c_n^\ell := a_{n-\ell} \overline{b_n}, \quad P_n^\ell := (-1)^\ell P_n^{-\ell} \ (\ell < 0),$$

then using (3.7) we obtain that

$$(V_\rho f)(a^{-1}) = \sqrt{1-r^2} \sum_{\ell=-\infty}^{\infty} r^{|\ell|} e^{-i\ell\varphi} \sum_{n=0}^{\infty} c_n^\ell P_n^\ell(r^2).$$

Using (3.2) the norm of $V_\rho f$ induced by (3.12) is equal to

$$\begin{aligned}\langle\langle V_\rho f, V_\rho f\rangle\rangle &= \int_0^1 r \sum_{\ell=-\infty}^{\infty} r^{2|\ell|} |\sum_{n=0}^{\infty} c_n^\ell P_n^\ell(r^2)|^2 \, dr = \\ &= \frac{1}{2} \int_0^1 \sum_{\ell=-\infty}^{\infty} x^{|\ell|} \Big| \sum_{n=0}^{\infty} c_n^\ell P_n^\ell(x) \Big|^2 \, dx = \\ &= \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{|c_n^\ell|^2}{2n+\ell+1} = \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|a_n b_m|^2}{n+m+1}.\end{aligned}$$

Proof of Theorem 3. The voice transform $V_\rho f$ generated by an arbitrary trigonometric polynomial of the form

$$\rho(e^{it}) := \sum_{n=-N}^{N} b_n e^{int}$$

can be expressed as

$$\frac{(V_\rho f)(a^{-1})}{\sqrt{1-|b|^2}} = \sum_{n=-N}^{N} \frac{\overline{b_n}}{2\pi} \int_{\mathbb{I}} f(e^{it}) \frac{1}{1-be^{-it}} \left(\frac{e^{-it} - \overline{b}}{1-be^{-it}} \right)^n dt = \sum_{n=-N}^{N} \overline{b_n} J_n,$$

where for J_n – using the decomposition (3.19) and making the change of variables $t = -s$ in the second therm – we obtain that:

$$J_n = \frac{1}{2\pi} \int_{\mathbb{I}} f(e^{it}) \frac{1}{1-be^{-it}} \left(\frac{e^{-it} - \overline{b}}{1-be^{-it}} \right)^n dt =$$

$$= \frac{1}{2\pi} \int_{\mathbb{I}} f^*(e^{it}) \frac{(1-\overline{b}e^{it})^n}{(e^{it}-b)^{n+1}} e^{it} \, dt + \frac{1}{2\pi} \int_{\mathbb{I}} f_*(e^{is}) e^{is} \frac{(e^{is}-\overline{b})^n}{(1-be^{is})^{n+1}} \, ds = J_n^1 + J_n^2.$$

Applying the Cauchy-formula for every index $n \in \mathbb{N}$ we obtain that

$$J_n^1 := \frac{1}{2\pi}\int_{\mathbb{I}} f^*(e^{it})\frac{(1-\overline{b}e^{it})^n}{(e^{it}-b)^{n+1}}e^{it}\,dt =$$

$$= \frac{1}{2\pi i}\int_{\mathbb{T}} f^*(\zeta)\alpha_b^n(\zeta)\frac{d\zeta}{(\zeta-b)^{n+1}} = \frac{1}{n!}(f^*\alpha_b^n)^{(n)}(b),$$

and for $n<0, n\in\mathbb{Z}$ is trivial that $J_n^1 = 0$. Similarly for $n \in \mathbb{N}$

$$J_{-n-1}^2 := \frac{1}{2\pi}\int_{\mathbb{I}} f_*(e^{is})\frac{(1-be^{is})^n}{(e^{is}-\overline{b})^{n+1}}e^{is}\,ds =$$

$$= \frac{1}{2\pi i}\int_{\mathbb{T}} f_*(\zeta)\alpha_{\overline{b}}^n(\zeta)\frac{d\zeta}{(\zeta-\overline{b})^{n+1}} = \frac{1}{n!}\left(f_*\alpha_{\overline{b}}^n\right)^{(n)}(\overline{b})$$

and is trivial that $J_{-n-1} = 0$, if $n<0$. Using this we obtain

$$\frac{(V_\rho f)(a^{-1})}{\sqrt{1-|b|^2}} = \sum_{n=-N}^{N}\overline{b_n}(J_n^1+J_n^2) = \sum_{n=0}^{N}\overline{b_n}J_n^1 + \sum_{k=-N}^{-1}\overline{b_k}J_k^2 =$$

$$= \sum_{n=0}^{N}\frac{\overline{b_n}}{n!}(f^*\alpha_b^n)^{(n)}(b) + \sum_{n=0}^{N-1}\frac{\overline{b_{-n-1}}}{n!}(f_*\alpha_{\overline{b}}^n)^{(n)}(\overline{b}) =$$

$$= (L_{\rho^*}^b f^*)(b) + (L_{\rho_*}^{\overline{b}} f_*)(\overline{b}).$$

Proof of Theorem 4. Applying the Leibniz formulae the differential operator L_κ^b can be written in the following form:

$$(L_\kappa^b f)(b) := \sum_{n=0}^{N}\frac{\overline{c_n}}{n!}(\alpha_b^n f)^{(n)}(b) =$$

$$= \sum_{n=0}^{N}\frac{\overline{c_n}}{n!}n!(-\overline{b})^n f(b) + \sum_{n=1}^{N}\sum_{k=1}^{n}((1-\overline{b}z)^n)_{z=b}^{(n-k)} f^{(k)}(b).$$

Let us denote the last sum by $(1-|b|^2)Q(b)$. If f is a polynomial, then Q is a continuous function of variable b and we obtain that

$$(L_\kappa^b f)(b) = \sum_{n=0}^{N}\overline{c_n}(-\overline{b})^n f(b) + (1-|b|^2)Q(b) =$$

$$= \overline{\kappa(-b)} f(b) + (1 - |b|^2) Q(b).$$

Using this we obtain that

$$\frac{V_\rho \rho(a^{-1})}{\sqrt{1-|b|^2}} = \overline{\rho^*(-b)}\rho^*(b) + \overline{\rho_*(-\bar{b})}\rho_*(\bar{b}) + (1-|b|^2)Q_1(b),$$

where Q_1 is a continuous function of variable b. If ρ satisfies the conditions of Theorem 4, then

$$\rho^*(z) = \sum_{n=1}^{N} b_n z^n = \sum_{n=0}^{N-1} \overline{b_{-n-1}} z^{n+1} = z\overline{\rho_*(\overline{z})}.$$

From this relation it follows that

$$\overline{\rho^*(-b)}\rho^*(b) + \overline{\rho_*(-\bar{b})}\rho_*(\bar{b}) = (|b|^2 - 1)|\rho_*(\bar{b})|^2,$$

$$\frac{V_\rho \rho(a^{-1})}{\sqrt{1-|b|^2}} = (1-|b|^2)(Q_1(b) - |\rho_*(\bar{b})|^2),$$

which implies that $V_\rho \rho \in L^2_m(\mathbb{B})$.

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