NOTE ON $t$–QUASIAFFINE FUNCTIONS

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Dedicated to the 70th birthday of Professor Imre Kátai

Abstract. Given a convex subset $D$ of a vector space and a constant $0 < t < 1$, a function $f : D \rightarrow \mathbb{R}$ is called $t$-quasiaffine if, for all $x, y \in D$,

$$\min\{f(x), f(y)\} \leq f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$  

If, furthermore, both of these inequalities are strict for $f(x) \neq f(y)$, $f$ is called strictly $t$-quasiaffine. The main results of the paper show that $t$-quasiaffinity implies $Q$-quasiaffinity (i.e. $t$-quasiaffinity for every rational number $t$ in $[0, 1]$). An analogous result is established for strict $t$-quasiaffinity.

1. Introduction

Let $\mathbb{F}$ be fixed subfield of the set of real numbers $\mathbb{R}$ and let $X$ be a vector space over $\mathbb{F}$ throughout this paper. The two most important particular settings of our investigations are when either $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$.

In what follows, we briefly recall the terminology related to $t$-convexity of sets and to $t$-quasiconvexity and $t$-quasiaffinity of functions.

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Given a constant \( t \in [0, 1] \), a subset \( D \subseteq X \) is called \( t \)-convex if \( tx + (1 - t)y \in D \) holds whenever \( x, y \in D \). For a given collection \( T \subseteq [0, 1] \) of numbers, the set \( D \) is called \( T \)-convex if it is \( t \)-convex for all \( t \in T \) (cf. [2], [3], [8]). Observe, that \( T \)-convexity is simply equivalent to convexity in the standard sense if \( T = [0, 1] \). We say that \( D \) is \( \mathbb{F} \)-convex if it is \( \mathbb{F} \cap [0, 1] \)-convex.

The set of all numbers \( t \in [0, 1] \) such that \( D \) is \( t \)-convex will be denoted by \( T(D) \) in the sequel. Obviously, \( 0, 1 \in T(D) \) for any set \( D \subseteq X \).

Given a \( t \)-convex set \( D \subseteq X \), a real-valued function \( f : D \to \mathbb{R} \) is called \( t \)-quasiconvex (cf. [2], [3], [8]) if

\[
(1) \quad f(tx + (1 - t)y) \leq \max \{ f(x), f(y) \}, \quad x, y \in D.
\]

If \( -f \) is \( t \)-quasiconvex then \( f \) is said to be \( t \)-quasiconcave. When \( f \) is both \( t \)-quasiconvex and \( t \)-quasiconcave, i.e. when

\[
(2) \quad \min \{ f(x), f(y) \} \leq f(tx + (1 - t)y) \leq \max \{ f(x), f(y) \}, \quad x, y \in D,
\]

holds, then \( f \) is termed a \( t \)-quasiaffine function.

We say that \( f \) is strictly \( t \)-quasiconvex if it satisfies (1), furthermore,

\[
(3) \quad f(tx + (1 - t)y) < \max \{ f(x), f(y) \} \quad \text{if} \quad f(x) \neq f(y).
\]

If \( f \) and \( -f \) are strictly \( t \)-quasiconvex, i.e. if (2) and

\[
(4) \quad \min \{ f(x), f(y) \} < f(tx + (1 - t)y) < \max \{ f(x), f(y) \} \quad \text{if} \quad f(x) \neq f(y)
\]

are satisfied then \( f \) is called strictly \( t \)-quasiaffine.

If \( D \) is \( T \)-convex and (1), (2), (3) and (4) hold for all \( t \in T \), then \( f \) is said to be \( T \)-quasiconvex, \( T \)-quasiaffine, strictly \( T \)-quasiconvex and strictly \( T \)-quasiaffine, respectively.

The collection of all numbers \( t \in [0, 1] \) such that \( f \) is \( t \)-quasiconvex is denoted by \( T(f) \).

The \( \frac{1}{2} \)-quasiaffine functions defined on an interval \( I \subset \mathbb{R} \), that is functions satisfying

\[
(5) \quad \min \{ f(x), f(y) \} \leq f \left( \frac{x + y}{2} \right) \leq \max \{ f(x), f(y) \}, \quad x, y \in I,
\]

were introduced in 1949 by A. Császár [4], [5]. (They were also called midpoint-quasiaffine or internal.) Observe that monotone functions \( f : I \to \mathbb{R} \) as well as Jensen functions, i.e. solutions of the Jensen functional equation
are always midpoint-quasiaffine. It is known that Jensen functions may be very irregular (see [1], [10]). However, measurable Jensen functions are of the form $f(x) = ax + b$, $x \in I$, hence they are also monotone. Midpoint-quasiaffine functions enjoy a similar property. Namely, Császár [5] proved that if a function $f : I \rightarrow \mathbb{R}$ satisfies (5) then it is either monotone or nonmeasurable. Other results of this type were also obtained by Déak [7] and Marcus [13].

Midpoint-quasiaffine functions in a more general setting were investigated by the authors in [14]. It was proved, among others, that under some regularity assumptions every strictly midpoint-quasiaffine function $f : X \rightarrow \mathbb{R}$ is of the form $f = g \circ \alpha$ where $\alpha : X \rightarrow \mathbb{R}$ is an additive function and $g : \mathbb{R} \rightarrow \mathbb{R}$ is monotone. This characterization gives some insight into the result of Császár.

Recently Lewicki [12] extended this characterization to strictly $t$-quasiaffine functions. The basic role in his paper is played by a theorem stating that if $f : X \rightarrow \mathbb{R}$ is strictly $t$-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous, i.e.

$$\lim_{r \in \mathbb{Q}, \ r \to 0^+} \sup f(rx + (1 - r)y) \leq f(y), \quad x, y \in X,$$

then it is strictly midpoint-quasiaffine. This result is analogous to the result of Kuhn [11] which states that every $t$-convex function is midpoint-convex. In the proof of Lewicki's theorem the assumptions that the domain of $f$ is the whole space $X$ and $f$ is $\mathbb{Q}$-radially upper semicontinuous are essential. However, using a more sophisticated method we can prove much stronger version of this statement. Namely, we show that every $t$-quasiaffine function $f : D \rightarrow \mathbb{R}$, where $D$ is an $F$-convex set and $t \in F \cap ]0, 1[$, is $\mathbb{Q}$-quasiaffine. An analogous result is also established for strictly $t$-quasiaffine functions.

2. $t$-convexity and complementarity of sets

**Lemma 1.** Let $D \subseteq X$ be a nonempty set. Then, for the set $T(D)$, we have the following properties:

(i) if $t \in T(D)$, then $1 - t \in T(D)$;

(ii) if $r, s, t \in T(D)$, then $tr + (1 - t)s \in T(D)$. 


Proof. Implication (i) is obvious. To prove (ii), assume that \( r, s, t \in T(D) \) and fix \( x, y \in D \). Then \( rx + (1-r)y, sx + (1-s)y \in D \) and, consequently,

\[
(tr + (1-t)s)x + (1-tr - (1-t)s)y = t(rx + (1-r)y) + (1-t)(sx + (1-s)y) \in D,
\]

which shows that \( tr + (1-t)s \in T(D) \).

As a consequence of the next lemma, we obtain that \( T(D) \) is also dense in \([0, 1]\) provided that \( T(D) \cap [0, 1] \neq \emptyset \).

Lemma 2. Let \( S \subseteq [0, 1] \) be a nonempty set with the following properties:

(i) \( 0, 1 \in S \) and \( S \cap [0, 1] \neq \emptyset \);

(ii) if \( r, s, t \in S \), then \( tr + (1-t)s \in S \).

Then \( S \) is dense in \([0, 1]\).

Proof. Let \( t \in S \cap [0, 1] \) be fixed and suppose, contrary to our claim, that \( S \) is not dense in \([0, 1]\). Then there exists an open interval \([a, b] \subseteq [0, 1] \setminus S \). Let

\[
r = \sup (S \cap [0, a]) \quad \text{and} \quad s = \inf (S \cap [b, 1]).
\]

One can see that \( S \cap [r, s] = \emptyset \). Take sequences \((r_n), (s_n)\) such that \( r_n, s_n \in S \) and \( r_n \nearrow r, s_n \searrow s \). Since

\[
tr_n + (1-t)s_n \longrightarrow tr + (1-t)s \in [r, s],
\]

for a sufficiently large \( n_0 \), we have \( tr_{n_0} + (1-t)s_{n_0} \in [r, s] \). On the other hand, by property (ii), \( tr_{n_0} + (1-t)s_{n_0} \in S \), which contradicts the fact that \( S \cap [r, s] = \emptyset \).

In general, given a rational number \( r \in [0, 1] \), the \( r \)-convexity of a set \( D \) does not imply its midpoint-convexity, and conversely, the midpoint-convexity of a set does not imply its \( r \)-convexity for an arbitrary rational number \( r \in [0, 1] \). For instance, the set of diadic rational numbers \( \{ \frac{k}{2^n} : k \in \mathbb{Z}, n \in \mathbb{N} \} \) is midpoint-convex but not \( \frac{1}{3} \)-convex. Similarly, the set of triadic rational numbers \( \{ \frac{k}{3^n} : k \in \mathbb{Z}, n \in \mathbb{N} \} \) is \( \frac{1}{3} \)-convex but not midpoint-convex. It is therefore surprising that the midpoint-convexity of a set and its complementary (with respect to a \( Q \)-convex set) is equivalent to \( Q \)-convexity.

The key for such kind of implications is contained in our next result.

Theorem 3. Let \( D \) be an \( F \)-convex set, let \( s, t \in F \cap [0, 1] \) with \( t(s+1) \leq 1 \) and assume that \( A \) and \( B \) are disjoint \( \{s, t\} \)-convex sets such that \( D = A \cup B \). Then \( A \) and \( B \) are also \( \frac{1-t}{1-ts} \)-convex.
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**Proof.** To verify the \(\frac{1-t}{1-ts}\)-convexity of \(A\), let \(x, y \in A\) be arbitrary points and define

\[
u := \frac{1-t}{1-ts} x + \frac{t-ts}{1-ts} y, \quad v := \frac{1-t-ts}{1-ts} x + \frac{t}{1-ts} y.
\]

The point \(u\) is trivially a convex combination of \(x\) and \(y\) with coefficients belonging to \(F\). By the assumption \(t(s+1) \leq 1\), it follows that \(v\) is also a convex combination of \(x\) and \(y\) with coefficients in \(F\). Therefore \(u, v \in D = A \cup B\).

To complete the proof, we have to show that \(u \in A\). We distinguish two cases.

In the case \(v \in A\), the easy-to-check identity \(u = sx + (1-s)v\) and the \(s\)-convexity of \(A\) yields that \(u \in A\).

In the case \(v \in B\), we use the identity

\[(6) \quad tu + (1-t)v = (1-t)x + ty\]

which is also easy to see. By the \(t\)-convexity of \(A\), the right hand side of (6) is an element of \(A\). Hence \(tu + (1-t)v \in A\). If the point \(u\) were in \(B\), then, by the \(t\)-convexity of \(B\), \(tu + (1-t)v\) would be an element of \(B\) contradicting the disjointness of \(A\) and \(B\). Thus, \(u\) cannot be in \(B\), i.e. \(u\) must belong to \(A\) in this case, too.

The proof of the \(\frac{1-t}{1-ts}\)-convexity of \(B\) is analogous.

**Remark.** If \(D\) is an \(F\)-affine set (i.e. \(tx + (1-t)y \in D\) for all \(x, y \in D\) and \(t \in F\)), then the condition \(t(s+1) \leq 1\) of Theorem 3 can be removed. Indeed, due to the \(F\)-affinity, the point \(v\) constructed in the proof is contained in \(D\) (even if it is not in the convex hull of \(x\) and \(y\)). It seems to be an important question if this remains valid also in the case when \(D\) is only an \(F\)-convex set.

**Theorem 4.** Let \(D\) be an \(F\)-convex set, \(t \in \mathbb{F} \cap [0,1]\) and assume that \(A\) and \(B\) are disjoint \(t\)-convex sets such that \(D = A \cup B\). Then \(A\) and \(B\) are also \(Q\)-convex.

**Proof.** Let \(\tau \in T(A) \cap T(B)\) be fixed. With the notation \(t := s := 1 - \tau\), we can see that the inequality \(t(s+1) \leq 1\) holds if \(\tau \geq \frac{2}{5}\). Therefore, applying Theorem 3, it follows that

\[
\frac{1}{2-\tau} = \frac{1}{1+t} = \frac{1-t}{1-ts} \in T(A) \cap T(B), \quad \tau \in T(A) \cap T(B) \cap \left[\frac{2}{5}, 1\right].
\]

By Lemma 1, the set \(S := T(A) \cap T(B)\) satisfies property (ii) of Lemma 2. Hence \(S\) is dense in \([0,1]\). Thus, we can choose the element \(\tau \in T(A) \cap T(B)\) such that \(\frac{2}{5} \leq \tau \leq \frac{1}{2}\) also holds.
Now define $t := \frac{1}{2} - \tau$ and $s := \tau$. The inequality $t(s + 1) \leq 1$ is equivalent to $\tau \leq \frac{1}{2}$ which is valid by the choice of $\tau$. Thus, using Theorem 3 again, we get that

$$\frac{1}{2} = \frac{(2 - \tau) - 1}{(2 - \tau) - \tau} = 1 - \frac{1}{1 - 2 \tau} = \frac{1 - t}{1 - ts} \in T(A) \cap T(B),$$

i.e. $A$ and $B$ are $\frac{1}{2}$-convex sets.

Finally, we prove the $Q$-convexity of $A$ and $B$. For a fixed number $n \geq 1$, denote by $(S_n)$ the following statement: For all $k \in \{1, \ldots, n\}$, the inclusion $\frac{k}{n+1} \in T(A) \cap T(B)$ holds. Clearly, $(S_1)$ is equivalent to $\frac{1}{2} \in T(A) \cap T(B)$ (what we have already proved).

Assume that $(S_n)$ has been verified. Then we have $\frac{n}{n+1} \in T(A) \cap T(B) \cap \left[\frac{2}{n}, 1\right]$. Therefore, by the first assertion of this proof,

$$\frac{n+1}{n+2} = \frac{1}{2} - \frac{n}{n+1} \in T(A) \cap T(B).$$

Using the second statement of Lemma 1, it follows that

$$\frac{k}{n+2} = \left(1 - \frac{k}{n+1}\right) \cdot 0 + \frac{k}{n+1} \cdot \frac{n+1}{n+2} \in T(A) \cap T(B)$$

for all $k \in \{1, \ldots, n+1\}$, which proves the validity of $(S_{n+1})$ and completes the proof.

As an obvious consequence of this theorem, we get the following result which was established in the particular case $D = X$ in [15].

**Corollary 5.** Let $D$ be a $Q$-convex set and assume that $A$ and $B$ are disjoint sets such that $D = A \cup B$. Then $A$ and $B$ are midpoint convex if and only if they are also $Q$-convex.

3. $t$-quasiaffine functions

For a given function $f : D \subset X \to \mathbb{R}$, we define the upper and lower level sets of $f$ by

$$A(f, c) = \{x \in D \mid f(x) < c\}, \quad \overline{A}(f, c) = \{x \in D \mid f(x) \leq c\}$$
and

\[ B(f, c) = \{ x \in D \mid f(x) > c \}, \quad \overline{B}(f, c) = \{ x \in D \mid f(x) \geq c \}. \]

The \( t \)-convexity property of these sets is related to \( t \)-quasiconvexity and \( t \)-quasiconcavity of the function \( f \) by the following lemma.

**Lemma 6.** Let \( t \in [0, 1] \), \( D \) be a \( t \)-convex set and \( f : D \to \mathbb{R} \). Then the following three properties are equivalent:

(i) \( f \) is a \( t \)-quasiconvex function;

(ii) for all \( c \in \mathbb{R} \), the level set \( A(f, c) \) is \( t \)-convex;

(iii) for all \( c \in \mathbb{R} \), the level set \( \overline{A}(f, c) \) is \( t \)-convex.

The proof of this lemma is elementary, therefore, it is omitted. The following lemma describes the basic properties of the set \( T(f) \) of a given function \( f \). Its statement is analogous to Lemma 1.

**Lemma 7.** Let \( D \subseteq X \) be a nonempty \( F \)-convex set and \( f : D \to \mathbb{R} \). Then, for the set \( T(f) \), we have the following properties:

(i) if \( t \in T(f) \), then \( 1 - t \in T(f) \);

(ii) if \( r, s, t \in T(f) \), then \( tr + (1 - t)s \in T(f) \).

**Proof.** By Lemma 6, we have that

\[ T(f) = \bigcap_{c \in \mathbb{R}} T(A(f, c)). \]

Thus, the statement directly follows from Lemma 1.

As a consequence of Lemma 2 and Lemma 7, we can see that \( T(f) \) is dense in \([0, 1]\) provided that \( T(f) \cap [0, 1] \neq \emptyset \). However, the density of \( T(f) \) in \([0, 1]\) does not imply that \( \frac{1}{2} \in T(f) \). In other words, \( t \)-quasiconvex functions, in general, need not be midpoint-\( t \)-quasiconvex (in contrast to the theorem of Kuhn [11] stating that \( t \)-convex functions are always midpoint-convex). For instance, consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[ f(x) = \begin{cases} 0 & \text{if } x = \frac{k}{3^n}, \ k \in \mathbb{Z}, n \in \mathbb{N}, \\ 1 & \text{otherwise}. \end{cases} \]

Then \( \frac{1}{3} \in T(f) \), but \( \frac{1}{2} \notin T(f) \). Note also that for some \( t \in [0, 1] \) the \( t \)-quasiconvexity implies midpoint-\( t \)-quasiconvexity. For instance, if \( f \) is \( (2^{-\frac{t}{2}}) \)-quasiconvex for some \( n \in \mathbb{N} \), then it is midpoint-\( t \)-quasiconvex. Indeed, if \( t \in \)
∈ T(f), then, by Lemma 7, $t^2 ∈ T(f)$, and, by induction, $t^n ∈ T(f)$ for every $n ∈ \mathbb{N}$. Therefore, if $2^{-\frac{1}{2}} ∈ T(f)$, then $\frac{1}{2} = (2^{-\frac{1}{2}})^n ∈ T(f)$.

The following result describes the algebraic structure of the set $T(f) ∩ T(−f)$. It is an easy consequence of Theorem 3 and Lemma 6.

**Theorem 8.** Let $D ⊆ X$ be a nonempty $F$-convex set and $f : D → \mathbb{R}$. Let $s, t ∈ F ∩ [0, 1]$ with $t(s + 1) ≤ 1$ such that $f$ is $(s, t)$-quasiaffine. Then $f$ is also $1 − t$-1−ts-quasiaffine.

**Proof.** By Lemma 6 and the $t$-quasiaffinity of $f$, we have, for all $c ∈ \mathbb{R}$ that $A(f, c)$ and $B(f, c) = \overline{A}(−f, −c)$ are $(s, t)$-convex subsets of $D$. Clearly, $A(f, c)$ and $B(f, c)$ are disjoint and $D = A(f, c) ∪ B(f, c)$. Thus, applying Theorem 3, it follows that the sets $A(f, c)$ and $\overline{B}(f, c)$ are $\frac{1−t}{1−ts}$-convex for all $c ∈ \mathbb{R}$. Now, by Lemma 6 again, we obtain that $f$ and $−f$ are $\frac{1−t}{1−ts}$-quasiconvex, i.e. $f$ is $\frac{1−t}{1−ts}$-quasiaffine.

Applying Lemma 6 and Theorem 4, we can state and prove the main result of this paper, which generalizes Theorem 1 in [14].

**Theorem 9.** Let $D$ be an $F$-convex set and $t ∈ F ∩ [0, 1]$. Assume that $f : D → \mathbb{R}$ is a $t$-quasiaffine function. Then $f$ is also $Q$-quasiaffine.

**Proof.** Arguing in the same way as in the proof of Theorem 8, the $t$-quasiaffinity of $f$ and Theorem 4 yield that the sets $A(f, c)$ and $\overline{B}(f, c)$ are $Q$-convex for all $c ∈ \mathbb{R}$. Now, by Lemma 6 again, it follows that $f$ is $Q$-quasiaffine.

In view of Theorem 9, we immediately obtain the following

**Corollary 10.** Let $D$ be a $Q$-convex set and $f : D → \mathbb{R}$. Then $f : D → \mathbb{R}$ is midpoint-quasiaffine if and only if it is $Q$-quasiaffine.

### 4. Strictly $t$-quasiaffine functions

The result of this section shows that strict $t$-quasiaffinity implies strict $Q$-quasiaffinity.

**Theorem 11.** Let $D$ be an $F$-convex set and $t ∈ F ∩ [0, 1]$. Assume that $f : D → \mathbb{R}$ is a strictly $t$-quasiaffine function. Then $f$ is also strictly $Q$-quasiaffine.

**Proof.** In view of Theorem 9, the function $f$ is $Q$-quasiaffine.
We show first that $f$ is strictly midpoint-quasiaffine. Let $x, y \in D$ such that $f(x) \neq f(y)$. Without loss of generality, we may assume that $f(x) < f(y)$. By the $\mathbb{Q}$-quasiaffinity, we have that (5) holds. We have to prove that both inequalities in (5) are strict. We will only show that both inequalities in (5) are strict. We will only show that $f\left(\frac{x + y}{2}\right) < f(y)$ (the proof of $f(x) < f\left(\frac{x + y}{2}\right)$ is analogous). Suppose, on the contrary, that $f\left(\frac{x + y}{2}\right) = f(y)$.

Define

$$u := tx + (1 - t)\frac{x + y}{2} \quad \text{and} \quad v := t\frac{x + y}{2} + (1 - t)y.$$  

Then, we have the following identity (cf. [6]):

$$\frac{x + y}{2} = (1 - t)u + tv.$$  

By (7) and the $t$-quasiaffinity of $f$,

$$f(v) = f\left(t\frac{x + y}{2} + (1 - t)y\right) = \max\left\{f\left(\frac{x + y}{2}\right), f(y)\right\} = f\left(\frac{x + y}{2}\right).$$  

Since $f(x) < f(y) = f\left(\frac{x + y}{2}\right)$, by (7) and the strict $t$-quasiaffinity of $f$, we get

$$f(u) = f\left(tx + (1 - t)\frac{x + y}{2}\right) < \max\left\{f(x), f\left(\frac{x + y}{2}\right)\right\} = f\left(\frac{x + y}{2}\right).$$  

Consequently, using the strict quasiaffinity once more, we obtain

$$f\left(\frac{x + y}{2}\right) = f\left(tv + (1 - t)u\right) < \max\{f(u), f(v)\} = f(v) = f\left(\frac{x + y}{2}\right),$$  

which is an obvious contradiction showing that $f$ is strictly midpoint-quasiaffine.

Now we will prove (similarly as in the case of Theorem 1 in [14]) that $f$ is strictly $\mathbb{Q}$-quasiaffine. By induction, we can get that

$$\min\{f(x), f(y)\} < f(dx + (1 - d)y) < \max\{f(x), f(y)\}$$  

if $f(x) \neq f(y)$ and $d \in [0, 1]$ is a dyadic rational number, that is $d = k/2^n$, where $k, n \in \mathbb{N}$, $0 < k < 2^n$. Let $r \in [0, 1] \cap \mathbb{Q}$ be arbitrary and $f(x) \neq f(y)$. There exist dyadic rational numbers $d', d''$ such that $0 < d' < r < d'' < 1$. Then $rx + (1 - r)y$ is a $\mathbb{Q}$-convex combination of $d'x + (1 - d')y$ and $d''x + (1 - d'')y$. Since, by Theorem 9, $f$ is $\mathbb{Q}$-quasiaffine, we have

$$\min\{f(d'x + (1 - d')y), f(d''x + (1 - d'')y)\} \leq f(rx + (1 - r)y) \leq$$
\[
\leq \max \{ f(d'x + (1 - d')y), f(d''x + (1 - d'')y) \}.
\]

On the other hand, we have (9) with \( d = d' \) and \( d = d'' \). These inequalities together with the previous one yield
\[
\min \{ f(x), f(y) \} < f(rx + (1 - r)y) < \max \{ f(x), f(y) \}.
\]

Hence \( f \) is strictly \( \mathbb{Q} \)-quasiaffine, which completes the proof.

In view of Theorem 11, we immediately obtain the following

**Corollary 12.** Let \( D \) be a \( \mathbb{Q} \)-convex set and \( f : D \to \mathbb{R} \). Then \( f : D \to \mathbb{R} \) is strictly midpoint-quasiaffine if and only if it is strictly \( \mathbb{Q} \)-quasiaffine.

5. M-quasiaffinity

We can generalize the notion of \( t \)-quasiaffine functions by replacing the weighted arithmetic mean used in its definition by a more general mean. Given two points \( x, y \in X \), we define
\[
[x, y] := \{ tx + (1 - t)y : t \in [0, 1]\}, \quad [x, y] := \{ tx + (1 - t)y : t \in [0, 1]\}.
\]

Given a convex set \( D \subseteq X \), a function \( M : D \times D \to D \) is called a *strict mean* on \( D \) if
\[
M(x, y) \in [x, y], \quad x, y \in D, \ x \neq y
\]
and
\[
M(x, x) = x, \quad x \in D.
\]

Let \( D \) be a convex subset of \( X \). A function \( f : D \to \mathbb{R} \) is said to be \( M \)-quasiaffine if
\[
\min \{ f(x), f(y) \} \leq f(M(x, y)) \leq \max \{ f(x), f(y) \}, \quad x, y \in D.
\]

Of course if \( t \in [0, 1] \), then \( M(x, y) = tx + (1 - t)y \) is a strict mean and \( M \)-quasiaffinity of \( f \) coincides with its \( t \)-quasiaffinity. In this section, we consider the question if \( M \)-quasiaffinity implies midpoint-quasiaffinity. The following example shows that without any additional assumptions such implication does not hold.

**Example 2.** Let \( M : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a strict mean defined by
Note on \( t \)-quasiaffine functions

\[ M(x, y) = \begin{cases} \frac{3}{4}x + \frac{1}{4}y, & \text{if } x, y \notin \mathbb{Q} \text{ and } \frac{x+y}{2} \in \mathbb{Q}, \\ \frac{1}{2}x + \frac{1}{2}y, & \text{otherwise.} \end{cases} \]

The Dirichlet function

\[ D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \]

is \( M \)-quasiaffine (note that \( M(x, y) \in \mathbb{Q} \) if \( x, y \in \mathbb{Q} \), and \( M(x, y) \notin \mathbb{Q} \) if \( x, y \notin \mathbb{Q} \)). However, by the theorem of Császár, it is not midpoint-quasiaffine (because it is measurable and non-monotone).

This example shows that \( M \)-quasiaffine functions need not be midpoint-quasiaffine even if they are measurable. However, if an \( M \)-quasiaffine function is continuous, then it is quasiaffine, that is it satisfies (2) for every \( t \in [0,1] \).

**Theorem 13.** Let \( X \) be a Hausdorff topological-vector space, \( D \) be a convex subset of \( X \) and \( M \) be a strict mean on \( X \). If a function \( f : D \to \mathbb{R} \) is \( M \)-quasiaffine and continuous, then it is quasiaffine.

**Proof.** Let \( x, y \in D \), \( x \neq y \). Define

\[ C = \{ z \in D : \min\{f(x), f(y)\} \leq f(z) \leq \max\{f(x), f(y)\} \}. \]

Clearly, \( x, y \in C \) and \( C \) is closed (because \( f \) is continuous). To prove that \( f \) is quasiaffine, it is enough to show that \([x, y] \subset C\). Suppose, contrary to this claim, that there exists \( z_0 \in [x, y] \setminus C \). Then there exist points \( x', y' \in [x, y] \cap \cap C \) such that \([x', y'] \cap C = \emptyset \). Since \( x', y' \in C \), we have \( f(x') \leq \max\{f(x), f(y)\} \) and \( f(y') \leq \max\{f(x), f(y)\} \). Hence, by \( M \)-quasiaffinity,

\[ f(M(x', y')) \leq \max\{f(x'), f(y')\} \leq \max\{f(x), f(y)\}. \]

Analogously

\[ f(M(x', y')) \geq \min\{f(x), f(y)\}. \]

Consequently \( M(x', y') \in C \). On the other hand we have \( M(x', y') \in [x', y'] \), because \( M \) is a strict mean. This contradicts the fact that \([x', y'] \cap C = \emptyset \) and completes the proof.

In the case when \( X = \mathbb{R} \) and \( M \) is a strict mean which is continuous in both variables, the above result is a consequence of the characterization of quasiconvexity presented in [9].


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