Annales Univ. Sci. Budapest., Sect. Comp. 29 (2008) 83-91

A NOTE ON THE POLLACZEK–KHINCHIN FORMULA

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. By using the theory of regenerative processes one derives the generating function of number of present customers in the M/G/1 system with single and bulk arrivals.

0. Notations

λ - arrival rate;

$B(x)$ - the distribution function of service time for one customer; $b(s) = \int_0^\infty e^{-sx} dB(x)$ its Laplace-Stieltjes transform; $\tau = \int_0^\infty x dB(x)$ - its mean value;

$\rho = \lambda\tau$;

$\zeta = \dfrac{\tau}{1-\rho}$ - the mean value of length of a busy period;

a_i - the probability of appearance of i ones for the service of a customer in the simple M/G/1 system, $A(z) = \sum_{i=0}^{\infty} a_i z^i = b(\lambda(1-z))$;

The research was supported by the Hungarian-Romanian Intergovernmental Cooperation for Science and Technology under grant RO-40/2005 and by the Hungarian National Foundation for Scientific Research under grant OTKA K60698/2005.

g_i - the probability of appearance of i customers in a group in case of bulk arrivals, $G(z) = \sum_{i=1}^{\infty} g_i z^i$ the corresponding generating function, $\alpha = \sum_{i=1}^{\infty} i g_i$;

c_i - the probability of appearance of i ones for the service of a customer in the M/G/1 system with bulk arrivals, $C(z) = \sum_{i=0}^{\infty} c_i z^i = b(\lambda(1 - G(z)))$;

ξ_i - the mean value of time spent in the i-th state for a busy period in the M/G/1 system;

$\zeta' = \dfrac{\alpha\tau}{1 - \alpha\rho}$ - the mean value of length of a busy period in case of bulk arrivals;

ξ_{1i} - the mean value of time spent in the i-th state for a busy period in case of bulk arrivals on condition the busy period begins with the presence of one customer;

ξ_i' - the mean value of time spent in the i-th state for a busy period in case of bulk arrivals;

$P(z) = \sum_{i=0}^{\infty} p_i z^i$ and $P^*(z) = \sum_{i=0}^{\infty} p_i^* z^i$ - the generating functions of ergodic distributions of number of customers for the simple M/G/1 system and for the system with bulk arrivals.

1. Introduction

For the equilibrium description of the M/G/1 system one usually uses the embedded Markov chain technique leading to the Pollaczek-Khinchin equation, the generating function of ergodic distribution. From mathematical viewpoint it gives the solution of problem, but to find the values of probabilities from it requires additional work. The most natural way would be its differentiation, unfortunately it gives very complicated expressions. Some authors considered this problem from another point of view, the inversion of generating function was realized by comparing the coefficients at the corresponding powers of z, a recursive algorithm was obtained for different concrete service time distributions. [1] presents a unified approach for the numerical solution of the M/G/1 queue. The authors give a possible method in case the service time distribution has a rational Laplace-Stieltjes transform, under such assumptions explicit closed-form expressions are derived in terms of roots of the associated characteristic equation (it corresponds to the denominator of generating function). Sometimes the FFT method can be useful.

In [9] Tijms prefers to use the regenerative approach, it directly leads to a numerically stable recursion scheme for the state probabilities and allows in a natural way for generalizations to more complex queueing models. In this recursion scheme he uses the coefficents

$$A_{kj} = \int_0^\infty [1 - B(t)] \frac{\lambda t^{j-k}}{(j-k)!} e^{-\lambda t} dt,$$

the inconvenience is that they have no direct probabilistic meaning. The method is based on the Theorems 1.3.2 and 1.3.3 of [9].

In [6-7] we followed a different way. The functioning of M/G/1 system may be described by a regenerative process, according to [3, 9] the stationary distribution for such a process is determined by the relation of mean value of time spent in different states to the mean value of length of the regenerative cycle. The arrival rate, the mean value of service time and the probabilities of appearance of a given number of customers for the service time are the primary information about the functioning of the system, the desired probabilities were obtained directly from them avoiding the generating functions, even we had not to know the concrete service time distribution.

For the sake of completeness we mention that the expression Pollaczek-Khinchin formula is used in different meaning in different places. [2] calls so the generating function of ergodic distribution, in [3, 4] the generating function is mentioned as Pollaczek-Khinchin transform equation, the mean value of queue length derived from it as Pollaczek-Khinchin formula. In this paper we will use it for the generating function of number of customers in the system.

2. Previous results

In [6, 7] we proved that in the M/G/1 system the mean values of times ξ_i spent in the different states for a busy period satisfy the relations

$$\xi_0 = \tau, \qquad \xi_1 = \frac{1-a_0}{a_0}, \qquad \xi_2 = \frac{1-a_0-a_1}{a_0}(\xi_0+\xi_1), \tag{1}$$

$$\xi_k = \sum_{i=1}^{k-2} \frac{1-a_0-\ldots-a_i}{a_0}\xi_{k-i} + \frac{1-a_0-\ldots-a_{k-1}}{a_0}(\xi_0+\xi_1), \qquad k \geq 3.$$

The ergodic probabilities are obtained from these expressions dividing them by the mean value of busy period, i.e. $p_i = \xi_i/\zeta$ $(i = 0, 1, 2, \ldots)$. This approach, similarly to that of Tijms, is based again on Theorems 1.3.2 and 1.3.3 of [9] or Corollary 3.6 [3, p.49].

In the M/G/1 system with bulk arrivals (see [8])

$$\xi_0' = \tau, \quad \xi_1' = \frac{\tau}{c_0} - g_1\tau, \quad \xi_2' = (1-g_1)(\xi_{10}+\xi_{11}) + \frac{1-c_0-c_1}{c_0}(\xi_{10}+\xi_{11}) - g_2\tau,$$

$$\xi_k' = \sum_{i=1}^{k-2}(1-g_1-\ldots-g_i)\xi_{1,k-i} + \sum_{i=1}^{k-2}\frac{1-c_0-\ldots-c_i}{c_0}\xi_{1,k-i} + \tag{2}$$

$$+(1-g_1-\ldots-g_{k-1})(\xi_{10}+\xi_{11}) + \frac{1-c_0-\ldots-c_{k-1}}{c_0}(\xi_0+\xi_1) - g_k\tau,$$

where

$$\xi_{10} = \tau, \qquad \xi_{11} = \frac{\tau}{c_0} - \tau, \qquad \xi_{12} = \frac{1-c_0-c_1}{c_0}(\xi_{10}+\xi_{11}), \tag{3}$$

$$\xi_{1k} = \sum_{i=1}^{k-2}\frac{1-c_0-\ldots-c_i}{c_0}\xi_{1,k-i} + \frac{1-c_0-\ldots-c_{k-1}}{c_0}(\xi_{10}+\xi_{11}), \qquad k \geq 3.$$

The ergodic probabilities are obtained from these expressions dividing them by the mean value of busy period, i.e. $p_i = \xi_i'/\zeta_i'$ $(i = 0, 1, 2\ldots)$.

3. Results

Theorem. *The generating function of present customers in the M/G/1 system*

$$P(z) = \frac{(1-\rho)(1-z)b(\lambda(1-z))}{b(\lambda(1-z))-z}, \tag{4}$$

and in the M/G/1 system in case of bulk arrivals

$$P^*(z) = \frac{(1-\alpha\rho)[1-G(z)]b(\lambda(1-G(z)))}{\alpha[b(\lambda(1-G(z)))-z]} \tag{5}$$

may be obtained from (1) and (2) by using the theory of regenerative processes.

Remark 1. (4) is the classical formula for the generating function of present customers in the M/G/1 system. We underline that in the M/G/1 system under the regenerative cycle one has to understand the busy period, according to the definition of embedded chain the service time of last customer corresponds to the free state.

Remark 2. In (5) appears the expression $C(z) = b(\lambda(1 - G(z)))$ for the generating function of number of ones occuring for the service of a customer, it is derived in [8] (see also Exercise 5.11 in [4, 5]).

Remark 3. We mention that the formula for the generating function of present customers in case of bulk arrivals given in [4] is not correct, it was improved in the Russian translation [5]. As example (see Example 5.12 in [4, 5]) it was proposed to obtain it by using the corresponding Markov chain. In order to have a complete description of the question, we give the system of equations determining it, in our notations it has the form

$$\begin{aligned} p_0 =& p_0 g_1 c_0 + p_1 c_0, \\ p_j =& p_0 \sum_{i=1}^{j+1} g_i c_{j-i+1} + \sum_{i=1}^{j+1} p_i c_{j-i+1}, \quad j = 1, 2, \ldots \end{aligned}$$

from which one can obtain it by using standard tools.

Proof. I. Let us consider the case of single arrivals and write the formulae of ξ_i for the first values. We have

$$\begin{aligned} \xi_0 =& \xi_0, \\ \xi_1 =& \frac{1 - a_0}{a_0} \xi_0, \\ \xi_2 =& \frac{1 - a_0 - a_1}{a_0} \xi_1 + \frac{1 - a_0 - a_1}{a_0} \xi_0, \\ \xi_3 =& \frac{1 - a_0 - a_1}{a_0} \xi_2 + \\ & + \frac{1 - a_0 - a_1 - a_2}{a_0} \xi_1 + \frac{1 - a_0 - a_1 - a_2}{a_0} \xi_0, \\ \xi_4 =& \frac{1 - a_0 - a_1}{a_0} \xi_3 + \frac{1 - a_0 - a_1 - a_2}{a_0} \xi_2 + \\ & + \frac{1 - a_0 - a_1 - a_2 - a_3}{a_0} \xi_1 + \frac{1 - a_0 - a_1 - a_2 - a_3}{a_0} \xi_0, \end{aligned}$$

$$\begin{aligned}\xi_5 =&\frac{1-a_0-a_1}{a_0}\xi_4+\frac{1-a_0-a_1}{a_0}\xi_4+\\&+\frac{1-a_0-a_1-a_2}{a_0}\xi_3+\frac{1-a_0-a_1-a_2-a_3}{a_0}\xi_2+\\&+\frac{1-a_0-a_1-a_2-a_3-a_4}{a_0}\xi_1+\frac{1-a_0-a_1-a_2-a_3-a_4}{a_0}\xi_0,\end{aligned}$$

....

Let us multiply the expression for ξ_i by z^i and sum up them from the third row excluding the last term (containing ξ_0). We have

$$\begin{aligned}&\frac{1-a_0-a_1}{a_0}z(\xi_1 z+\xi_2 z^2+\dots)+\frac{1-a_0-a_1-a_2}{a_0}z^2(\xi_1 z+\xi_2 z^2+\dots)+\\&\qquad+\frac{1-a_0-a_1-a_2-a_3}{a_0}z^3(\xi_1 z+\xi_2 z^2+\dots)+\dots=\\&=\left(\sum_{i=1}^{\infty}\xi_i z^i\right)\left\{\frac{1-a_0-a_1}{a_0}z+\frac{1-a_0-a_1-a_2}{a_0}z^2+\right.\\&\qquad\left.+\frac{1-a_0-a_1-a_2-a_3}{a_0}z^3+\dots\right\}=\\&=\left(\sum_{i=1}^{\infty}\xi_i z^i\right)\frac{1}{a_0}\left\{\frac{z}{1-z}-\frac{a_0 z}{1-z}-\frac{a_1 z}{1-z}-\frac{a_2 z^2}{1-z}-\frac{a_3 z^3}{1-z}-\dots\right\}=\\&=\left(\sum_{i=1}^{\infty}\xi_i z^i\right)\frac{1}{a_0(1-z)}\left\{z(1-a_0)-[A(z)-a_0]\right\}=\\&=\left[\overline{P}(z)-\xi_0\right]\frac{1}{a_0(1-z)}\left\{z(1-a_0)-[A(z)-a_0]\right\},\end{aligned}\tag{6}$$

where $\overline{P}(z)=\sum\limits_{i=0}^{\infty}\xi_i z^i$. For the term containing ξ_0, we have

$$\xi_0 z\sum_{i=1}^{\infty}\frac{1-a_0-\ldots-a_i}{a_0}z^i=\xi_0 z\frac{1}{a_0(1-z)}\{z(1-a_0)-[A(z)-a_0]\}.\tag{7}$$

Adding (6), (7), the first row and the second one multiplied by z, we obtain

$$\begin{aligned}\overline{P}(z)=[\overline{P}(z)-\xi_0]&\frac{1}{a_0(1-z)}\{z(1-a_0)-[A(z)-a_0]\}+\\+\xi_0 z&\frac{1}{a_0(1-z)}\{z(1-a_0)-[A(z)-a_0]\}+\xi_0+\frac{1-a_0}{a_0}\xi_0 z.\end{aligned}$$

From it

$$\overline{P}(z) = \frac{(1-z)A(z)}{A(z)-z}\xi_0.$$

Dividing it by the mean value of busy period $\dfrac{\tau}{1-\rho}$ and taking into account that $\xi_0 = \tau$, finally

$$P(z) = \frac{(1-\rho)(1-z)A(z)}{A(z)-z}.$$

II. Now we are going to consider the case of bulk arrivals. (2) can be rewritten in the form

$$\xi_k' = \xi_{1k} + \sum_{i=1}^{k-2}(1-g_1-\ldots-g_i)\xi_{1,k-i} + (1-g_1-\ldots-g_{k-1})(\xi_{10}+\xi_{11}) - g_k\tau.$$

For $k = 0, 1, 2, \ldots$ we have

$$\begin{aligned}
\xi_0' =& \xi_{10} = \tau,\\
\xi_1' =& (\xi_{10}+\xi_{11}) - g_1\tau,\\
\xi_2' =& \xi_{12} + (1-g_1)(\xi_{10}+\xi_{11}) - g_2\tau,\\
\xi_3' =& \xi_{13} + (1-g_1)\xi_{12} + (1-g_1-g_2)(\xi_{10}+\xi_{11}) - g_3\tau,\\
\xi_4' =& \xi_{14} + (1-g_1)\xi_{13} + (1-g_1-g_2)\xi_{12} + (1-g_1-g_2-g_3)(\xi_{10}+\xi_{11}) - g_4\tau,\\
\xi_5' =& \xi_{15} + (1-g_1)\xi_{14} + (1-g_1-g_2)\xi_{13} + (1-g_1-g_2-g_3)\xi_{12}+\\
& + (1-g_1-g_2-g_3-g_4)(\xi_{10}+\xi_{11}) - g_5\tau,
\end{aligned}$$

....

Let us multiply ξ_i' by z^i and sum up these values. The terms can be grouped on the following way:

(I) the first terms from each row (ξ_{1i}) excluding ξ_1', from which we take ξ_{11}, together give

$$\sum_{i=0}^{\infty}\xi_{1i}z^i = Q_1(z);$$

(II) the sum of terms starting from the expression for ξ_3', excluding the members with $\xi_{10}+\xi_{11}$ and $g_i\tau$, gives

$$(1-g_1)z[\xi_{12}z^2+\xi_{13}z^3+\ldots] + (1-g_1-g_2)z^2[\xi_{12}z^2+\xi_{13}z^3+\ldots]+$$

$$+(1-g_1-g_2-g_3)z^3[\xi_{12}z^2+\xi_{13}z^3+\ldots] =$$

$$= \left(\sum_{i=2}^{\infty} \xi_{1i} z^i\right) [(z+z^2+z^3+\ldots) - g_1(z+z^2+z^3+\ldots) - g_2(z^2+z^3+\ldots) - \ldots] =$$

$$= \left(\sum_{i=2}^{\infty} \xi_{1i} z^i\right) \left[\frac{z}{1-z} - \frac{g_1 z}{1-z} - \frac{g_2 z^2}{1-z} - \ldots\right] = \left(\sum_{i=2}^{\infty} \xi_{1i} z^i\right) \frac{z-G(z)}{1-z};$$

(III) the terms containing $\xi_{10}+\xi_{11}$ (starting from the expression for ξ_2') give

$$(1-g_1)z(\xi_{10}+\xi_{11})z + (1-g_1-g_2)z^2(\xi_{10}+\xi_{11})z +$$

$$+(1-g_1-g_2-g_3)z^3(\xi_{10}+\xi_{11})z + \ldots =$$

$$= (\xi_{10}+\xi_{11})z[(1-g_1)z + (1-g_1-g_2)z^2 + (1-g_1-g_2-g_3)z^3 + \ldots] =$$

$$= (\xi_{10}+\xi_{11})z \left[\frac{z}{1-z} - \frac{g_1 z}{1-z} - \frac{g_2 z^2}{1-z} - \ldots\right] = (\xi_{10}+\xi_{11})z\frac{z-G(z)}{1-z};$$

(IV) the terms $g_i\tau$ give

$$-g_1 z\tau - g_2 z^2\tau - g_3 z^3\tau - \ldots = -G(z)\tau = -G(z)\xi_{10};$$

(V) finally, the first term of expression for ξ_1' gives

$$\xi_{10} z.$$

Adding (I)-(V), we obtain

$$Q_1(z) + [Q_1(z) - \xi_{10} - \xi_{11} z]\frac{z-G(z)}{1-z} + (\xi_{10}+\xi_{11})z\frac{z-G(z)}{1-z} - G(z)\xi_{10} + \xi_{10} z =$$

$$= Q_1(z)\frac{1-G(z)}{1-z} = \overline{P}^*(z). \tag{8}$$

Since

$$Q_1(z) = \frac{C(z)(1-z)}{C(z)-z}\xi_{10}$$

(it is the generating function of service times for the M/G/1 system on condition the customers may enter by groups and the busy period begins with the entry of one customer, the transition probabilities are c_j), from (8)

$$\overline{P}^*(z) = \frac{C(z)(1-z)}{C(z)-z}\xi_{10}\frac{1-G(z)}{1-z} = \frac{[1-G(z)]C(z)}{C(z)-z}\xi_{10},$$

where $\overline{P}^*(z)$ is the generating function of time spent in different states for a busy period. Dividing it by the mean value of length of the busy period we come to the final expression

$$P^*(z) = \frac{1-\alpha\rho}{\alpha\tau}\frac{[1-G(z)]C(z)}{C(z)-z}\tau = \frac{(1-\alpha\rho)[1-G(z)]b(\lambda(1-G(z)))}{\alpha[b(\lambda(1-G(z)))-z]},$$

it coincides with (5).

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