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ON WEIGHTED (0,1,3)-INTERPOLATION

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. The weighted $(0,1,3)$-interpolation with additional Hermite-type conditions is studied in a unified way with respect to the existence, uniqueness and representation (explicit formulae). Sufficient conditions are given on the nodes and the weight function, for the problem to be regular. Examples are presented on the zeros of the classical orthogonal polynomials, including examples for regular weighted $(0,1,3)$-interpolation polynomials without any additional condition.

1. Introduction

The weighted $(0,1,3)$-interpolation is a lacunary interpolation problem, when beside the function and first derivative values weighted third derivative values are prescribed. It is an extension of the weighted $(0,2)$-interpolation, initiated by J. Balázs [1] and studied by several authors (see for references L. Szili [13] and M. Lénárd [6], [7]).

For given $n \in \mathbf{N}$, on the finite or infinite interval (a,b) let $\{x_{i,n}\}_{i=1}^{n}$ be a set of distinct points, called *nodes*. Let $w \in C^{(3)}(a,b)$ be a given function, called *weight function*. The problem is to find a polynomial R_n of degree $< 3n$ satisfying the $3n$ conditions

(1.1) $R_n(x_{i,n}) = y_{i,n}, \;\; R_n'(x_{i,n}) = y_{i,n}', \;\; (wR_N)'''(x_{i,n}) = y_{i,n}''' \;\; (i = 1,\dots,n),$

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where $y_{i,n}$, $y'_{i,n}$ and $y'''_{i,n}$ $(i = 1, \ldots, n)$ are arbitrary given real numbers.

The problem is called *regular* on the nodes $\{x_{i,n}\}_{i=1}^{n}$ with respect to the weight function w, if for any choice of the values $y_{i,n}$, $y'_{i,n}$, $y'''_{i,n}$ there exists a unique polynomial R_n of degree at most $3n-1$ satisfying the conditions in (1.1). The weighted $(0,1,3)$-interpolation is not regular in general (see e.g. A. Krebsz [5]). Therefore the question is how to choose the nodal points and the weight function for the problem to be regular. Furthermore, in order to prove convergence theorems a simple explicit form of the interpolation polynomial R_n is needed in each regular case.

Prescribing some additional conditions, and hence increasing the degree of the polynomial R_n, the problem can be made regular. Several authors studied the weighted $(0,1,3)$-interpolation with additional conditions. In the regular cases they gave the explicit formulae for the interpolation polynomial and in special cases they proved convergence theorems. The considered additional condition was either a Balázs-type condition or interpolatory condition(s) at additional node(s).

The Balázs-type condition is an extra condition for $R_n(0)$, if 0 is not a nodal point, namely

$$R_n(0) = \sum_{k=1}^{n} \left[\left(1 + 3x_{k,n}\ell'_{k,n}(x_{k,n})\right) y_{k,n} - x_{k,n}y'_{k,n} \right] \ell^3_{k,n}(0), \tag{1.2}$$

where $\ell_{k,n}$ $(k = 1, \ldots, n)$ are the fundamental polynomials of Lagrange interpolation on the nodes. The Balázs-type condition was considered on the zeros of the ultraspherical polynomials by P. Bajpai [2], on the zeros of the Hermite polynomials by K. K. Mathur and R. B. Saxena [8], and in general on the zeros of the classical orthogonal polynomials it was studied by A. Krebsz [5].

The weighted $(0,1,3)$-interpolation with additional interpolatory conditions were studied on the zeros of the Hermite polynomials by S. Datta and P. Mathur [4], and on the zeros of Chebyshev polynomials of second kind by P. Bajpai [3]. In [9] K. K. Mathur and A. K. Srivastava investigated the Pál-type weighted $(0;0,1,3)$-interpolation on the zeros of Hermite polynomials, which is a Lagrange interpolation problem on the zeros of the Hermite polynomial H_n and weighted $(0,1,3)$-interpolation on the zeros of H'_n with an additional Balázs-type condition. Recently, in [10] P. Mathur studied the problem on an arbitrary set of nodes with special additional interpolation type conditions, namely, at two nodes the weighted third derivatives are replaced by the second derivatives at those points.

In this paper we study the problem with additional Hermite-type conditions in a unified way with respect to the existence, uniqueness and representation (explicit formulae). In this general situation we construct polynomials in Theorem 2.3 which will serve as fundamental polynomials in special regular cases, when the free parameters are determined from the additional conditions. Hence, in the special cases one has to solve only 2×2 linear systems for the parameters in order to get the fundamental polynomials associated with the weighted interpolation conditions of the problem. The fundamental polynomials in the references are all special cases of this method. In Section 3 we give sufficient conditions on the nodes and the weight function for the problem to be regular with different additional interpolatory conditions. In Section 4 we present some special cases on the zeros of the classical orthogonal polynomials and also examples for regular weighted $(0,1,3)$-interpolation (without any additional condition).

2. The problem

For given $n, m \in \mathbf{N}$, on the finite or infinite interval $[a,b]$ let $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ be disjoint sets of distinct points, the *nodal points*, and let $w \in C^3(a,b)$ be a given function, called *weight function.* (For the sake of simplicity we omit the use of double indices.) Find a minimal degree polynomial R_N satisfying the weighted $(0,1,3)$-interpolation conditions on $\{x_i\}_{i=1}^n$

$$(2.1)\qquad R_N(x_i) = y_i, \quad R_N'(x_i) = y_i', \quad (wR_N)'''(x_i) = y_i''' \qquad (i = 1, \dots, n)$$

with the additional Hermite-type interpolation conditions on $\{\bar{x}_i\}_{i=1}^m$

$$(2.2)\qquad R_N^{(j)}(\bar{x}_i) = \bar{y}_i^{(j)} \qquad (j = 0, \dots, j_i - 1;\ i = 1, \dots, m),$$

where y_i, y_i', y_i''' and $\bar{y}_i^{(j)}$ are arbitrary given real numbers. (For $m = 0$ the problem is the weighted $(0,1,3)$-interpolation.)

As the number of conditions is $N = 3n + M$, where $M = j_1 + \ldots + j_m$, the problem is called *regular*, if for any choice of the values y_i, y_i', y_i''' and $\bar{y}_i^{(j)}$ there exists a unique polynomial R_N of degree at most $N - 1$.

The problem is not regular in general, because such a minimal degree polynomial might not exist (Theorem 4.1), or if it exists, the uniqueness might fail. Therefore we study the problem with additional interpolatory conditions and we find sufficient conditions for the problem to be regular. In regular cases we give simple explicit forms for R_N. Finally we present examples on the zeros of the classical orthogonal polynomials.

In what follows, let p_n and q be polynomials of degree n and $M = j_1 + \dots + j_m$, respectively, associated with the interpolation conditions (2.1)-(2.2), that is

$$\begin{aligned} p_n(x_i) &= 0 \qquad (i = 1, \dots, n), \\ q^{(j)}(\bar{x}_i) &= 0 \qquad (j = 0, \dots, j_i - 1;\ i = 1, \dots, m). \end{aligned} \tag{2.3}$$

If only the weighted interpolation conditions are prescribed, that is $m = 0$, let $q(x) \equiv 1$. Furthermore, let ℓ_k denote the fundamental polynomials of Lagrange interpolation on the nodes $\{x_i\}_{i=1}^n$, that is

$$\ell_k(x) = \frac{p_n(x)}{p_n'(x_k)(x - x_k)} \qquad (k = 1, \dots, n), \tag{2.4}$$

hence $\ell_k(x_i) = \delta_{ki}$ $(i, k = 1, \dots, n)$.

The following statements can be verified by an easy calculation.

Lemma 2.1. *If $q = q_1 q_2$, $w = w_1 w_2$ and on the nodes $\{x_i\}_{i=1}^n$*

$$(q_1 w_1 p_n^2)'''(x_i) = 0 \quad \text{and} \quad (q_2 w_2)'(x_i) = 0 \quad (i = 1, \dots, n), \tag{2.5}$$

then for any $Q \in C^3(a, b)$

$$(q w p_n^2 Q)'''(x_i) = 6(q w p_n'^2 Q')(x_i) \qquad (i = 1, \dots, n). \tag{2.6}$$

Lemma 2.2. *If $q_1 w_1 \geq 0$ on $[a, b]$ and $w_1(x_i) \neq 0$ on the nodes $\{x_i\}_{i=1}^n$, then*

$$(q_1 w_1 p_n^2)'''(x_i) = 0 \qquad \text{iff} \qquad (\sqrt{q_1 w_1} p_n)''(x_i) = 0 \qquad (i = 1, \dots, n).$$

In the next theorem we give explicit forms of the polynomials which we will use in regular cases to construct the fundamental polynomials associated with the weighted $(0, 1, 3)$-interpolatory conditions.

Theorem 2.3. *If on the nodes $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ the weight function w satisfies (2.5) and $w(x_i) \neq 0$ $(i = 1, \dots, n)$, then the polynomials of degree at most $3n + M + 1$, which fulfill the conditions*

$$\begin{aligned} &C_k(x_i) = 0, \quad C_k'(x_i) = 0, \quad (w C_k)'''(x_i) = \delta_{ki}, && (i = 1, \dots, n), \\ &C_k^{(j)}(\bar{x}_i) = 0, && (j = 0, \dots, j_i - 1;\ i = 1, \dots, m), \end{aligned}$$

can be written in the form

$$C_k(x) = \frac{\left(qp_n^2\right)(x)}{6\left(qwp_n'^2\right)(x_k)} \left\{ \bar{\bar{c}}_k + \int\limits_{x_0}^{x} \left[\ell_k(t) + \bar{\bar{b}}_k p_n(t)\right] dt \right\} \tag{2.7}$$

for $k = 1, \ldots, n$. *The polynomials of degree at most* $3n + M + 1$, *which fulfill the conditions*

$$\begin{aligned} &B_k(x_i) = 0, \quad B_k'(x_i) = \delta_{ki}, \quad (wB_k)'''(x_i) = 0, && (i = 1, \ldots, n), \\ &B_k^{(j)}(\bar{x}_i) = 0, && (j = 0, \ldots, j_i - 1; \ \ i = 1, \ldots, m), \end{aligned}$$

can be written in the form

$$B_k(x) = \frac{q(x)}{q(x_k)}(x - x_k)\ell_k^3(x) + \tag{2.8}$$

$$+ \frac{\left(qp_n^2\right)(x)}{\left(qp_n'^2\right)(x_k)} \left\{ \bar{c}_k + \int\limits_{x_0}^{x} \left[\frac{\ell_k'(x_k)\ell_k(t) - \ell_k'(t)}{t - x_k} + \bar{a}_k \ell_k(t) + \bar{b}_k p_n(t) \right] dt \right\},$$

where

$$\bar{a}_k = -\frac{(qw\ell_k^3)''(x_k)}{2(qw)(x_k)} + \ell_k''(x_k) - \ell_k'^2(x_k) \qquad (k = 1, \ldots, n). \tag{2.9}$$

Finally, the polynomials of degree at most $3n{+}M{+}1$, *which fulfill the conditions*

$$\begin{aligned} &A_k(x_i) = \delta_{ki}, \quad A_k'(x_i) = 0, \quad (wA_k)'''(x_i) = 0, && (i = 1, \ldots, n), \\ &A_k^{(j)}(\bar{x}_i) = 0, && (j = 0, \ldots, j_i - 1; \ \ i = 1, \ldots, m), \end{aligned}$$

can be written in the form

$$A_k(x) = \frac{q(x)}{q(x_k)}\,\ell_k^3(x) + \alpha_k B_k(x) + \frac{\left(qp_n^2\right)(x)}{\left(qp_n'^2\right)(x_k)} \times \tag{2.10}$$

$$\times \left\{ c_k + \int\limits_{x_0}^{x} \left[\frac{[\ell_k'(x_k) + \beta_k(t - x_k)]\ell_k(t) - \ell_k'(t)}{(t - x_k)^2} + a_k \ell_k(t) + b_k p_n(t) \right] dt \right\},$$

where for $k = 1, \dots, n$

$$
\begin{aligned}
&\alpha_k = -\frac{q'(x_k)}{q(x_k)} - 3\ell'_k(x_k), \\
(2.11) \qquad &\beta_k = \ell''_k(x_k) - \ell'^2_k(x_k), \\
&a_k = -\frac{(qw\ell_k^3)'''(x_k)}{6(qw)(x_k)} + \frac{1}{2}\left(\ell'''_k(x_k) + 2\ell'^3_k(x_k) - 3\ell'_k(x_k)\ell''_k(x_k)\right)
\end{aligned}
$$

and b_k, $\bar{b}_k$, $\bar{\bar{b}}_k$, c_k, $\bar{c}_k$, $\bar{\bar{c}}_k$ *are arbitrary real numbers.*

Proof. On using Lemma 2.1, the properties of the polynomials C_k $(1, \dots, n)$ in (2.7) can be verified by an easy calculation.

The polynomials B_k $(k = 1, \dots, n)$ are to be found in the form

$$B_k(x) = \frac{q(x)}{q(x_k)} \left\{(x - x_k)\ell_k^3(x) + p_n^2(x)\bar{Q}_k(x)\right\},$$

where $\bar{Q}_k$ are polynomials of degree at most $n+1$. It is clear, that $B_k(x_i) = 0$, $B'_k(x_i) = \delta_{ki}$ for $i = 1, \dots, n$ and B_k fulfill the additional Hermite-type interpolation conditions, as well. Applying Lemma 2.1 we get that $(wB_k)'''(x_i) = 0$ for $i \neq k$ if and only if

$$\bar{Q}'_k(x_i) = \frac{-1}{p'^2_n(x_k)} \frac{\ell'_k(x_i)}{x_i - x_k}$$

and hence the polynomials $\bar{Q}_k$ can be defined by

$$\bar{Q}'_k(x) = \frac{1}{p'^2_n(x_k)} \left\{\frac{\ell'_k(x_k)\ell_k(x) - \ell'_k(x)}{x - x_k} + \bar{a}_k\ell_k(x) + \bar{b}_k p_n(x)\right\},$$

where the parameters $\bar{a}_k$ in (2.9) are determined by the conditions

$$(wB_k)'''(x_k) = 0 \qquad\qquad (k = 1, \dots, n).$$

In a similar way, we are looking for the polynomials A_k $(k = 1, \dots, n)$ in the form

$$A_k(x) = \frac{q(x)}{q(x_k)} \left\{\ell_k^3(x) + p_n^2(x)Q_k(x)\right\} + \alpha_k B_k(x),$$

where Q_k are polynomials of degree at most $n+1$ and B_k are defined by (2.8). It is clear, that A_k fulfill the additional Hermite-type interpolation conditions, as well as $A_k(x_i) = \delta_{ki}$ for $i = 1, \dots, n$ and $A'_k(x_i) = 0$ for $i \neq k$. From the conditions $A_k'(x_k) = 0$ we obtain α_k in (2.11). Applying Lemma 2.1 it follows that $(wA_k)'''(x_i) = 0$ for $i \neq k$ if and only if

$$Q_k'(x_i) = \frac{-1}{p_n'^2(x_k)} \frac{\ell_k'(x_i)}{(x_i - x_k)^2}$$

and hence the polynomials Q_k can be defined by

$$Q_k'(x) = \frac{1}{p_n'^2(x_k)} \left\{ \frac{[\ell_k'(x_k) + \beta_k(x - x_k)]\ell_k(x) - \ell_k'(x)}{(x - x_k)^2} + a_k\ell_k(x) + b_k p_n(x) \right\},$$

where the parameters β_k assure that Q_k are polynomials and a_k are determined from the conditions $(wA_k)'''(x_k) = 0 \quad (k = 1, \dots, n)$. Their values are given in (2.11).

2.1. Pál-type weighted $(0, 1, 3\,; 0, 1, \dots, r - 1)$-interpolation

For $r \in \mathbf{N}$, let us consider weighted $(0, 1, 3)$-interpolation on the zeros of

$$p_n(x) = c(x - x_1) \dots (x - x_n)$$

with Hermite-type interpolation on the zeros of p_n', where the derivatives up to the $(r - 1)$-st order are prescribed. If $r = 0$, the problem is the weighted (0,1,3)-interpolation problem (1.1). For $r \geq 1$ in (2.2) we have $m = n - 1$ and $\bar{x}_1, \dots, \bar{x}_{n-1}$ are the zeros of p_n'.

Theorem 2.4. *If on the zeros of* p_n

$$(q_1 w_1 p_n^2)'''(x_i) = 0 \quad \text{and} \quad w_1(x_i) \neq 0 \qquad (i = 1, \dots, n), \tag{2.12}$$

then the fundamental polynomials associated with the weighted $(0, 1, 3)$*-interpolation conditions of the Pál-type weighted* $(0, 1, 3\,; 0, 1, \dots, r - 1)$*-interpolation w.r.t. the weight function* $w = w_1(q_1 w_1)^r$ *can be constructed by Theorem 2.3, with* $q = q_1(p_n')^r$*.*

Proof. Let $q_2 = (p_n')^r$ and $w_2 = (q_1 w_1)^r$, hence on using $p_n(x_i) = 0$ we get

$$(q_1 w_1 p_n^2)'''(x_i) = 6p_n'(x_i)(p_n' q_1 w_1)'(x_i) \qquad (i = 1, \dots, n)$$

and by $p_n'(x_i) \neq 0$ it follows

$$(q_2 w_2)'(x_i) = \left((p_n' q_1 w_1)^r\right)'(x_i) = r(p_n' q_1 w_1)^{r-1}(x_i)(p_n' q_1 w_1)'(x_i) = 0$$

for $i = 1, \dots, n$, which is the hypothesis of Theorem 2.3.

2.2. Pál-type weighted $(0, 1, \dots, r-1; 0, 1, 3)$-interpolation

Let us consider Hermite-type interpolation problem on the zeros of

$$\bar{p}(x) = c(x - \bar{x}_1) \dots (x - \bar{x}_n),$$

where the derivatives up to the $(r-1)$-st order are prescribed and weighted $(0, 1, 3)$-interpolation on the zeros of $\bar{p}\,' = p_{n-1}$. Now in (2.2) $m = n$ and $n := n - 1$.

Theorem 2.5. *If on the zeros of* $p_{n-1} = \bar{p}\,'$ *the weight function* w *satisfies*

$$(q_1 w \bar{p}\,'^2)'''(x_i) = 0, \qquad w(x_i) \neq 0, \qquad (i = 1, \dots, n-1), \tag{2.13}$$

then the fundamental polynomials of the Pál-type weighted $(0, 1, \dots, r-1; 0, 1, 3)$*-interpolation associated with the weighted* $(0, 1, 3)$*-interpolatory conditions w.r.t. the weight function* w *can be constructed by Theorem 2.3, with* $q = q_1 \bar{p}^{\,r}$.

Proof. With $w_2 = 1$ and $q_2 = \bar{p}^{\,r}$

$$(w_2 q_2)'(x_i) = \left(\bar{p}^{\,r}\right)'(x_i) = r\left(\bar{p}^{\,r-1}\right)(x_i) p_{n-1}(x_i) = 0 \qquad (i = 1, \dots, n-1),$$

and we can apply Theorem 2.3.

For $r = 1$ we get the Pál-type weighted $(0; 0, 1, 3)$-interpolation problem, where Lagrange interpolation is prescribed at the zeros of $\bar{p}$ and weighted $(0, 1, 3)$-interpolation at the zeros of $\bar{p}'$. This problem was studied in [9] by K. K. Mathur and A. K. Srivastava on the zeros of the Hermite polynomials ($\bar{p} = H_n$) with a Balázs-type condition for the second derivative at $x_0 = 0$.

3. Additional interpolatory conditions and regularity

The interpolation problem (2.1)-(2.2) is not regular in general (see Lemma 4.1), therefore we study the problem with different additional interpolatory conditions prescribed at the points x_0 and $x_{n+1} \in [a, b]$. The conditions on the nodes and the weight function w for the problem to be regular are summarized in Table 1.

Theorem 3.1. *If on the nodes* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^m$ *the weight function* w *satisfies (2.5),* $w(x_i) \neq 0$ $(i = 1, \dots, n)$*, then the interpolation problem*

(2.1)-(2.2) is regular under the additional condition(s) (i)-(x) if and only if the condition(s) in the third column of Table 1 is(are) fulfilled.

Proof. For the sake of simplicity we will prove only the case (iv), the other cases are similar. We study the homogeneous problem: Find a polynomial of degree at most $N = 3n + M$ for which

$$\begin{aligned} &\bar{R}_N(x_i) = 0, \;\; \bar{R}_N'(x_i) = 0, \;\; (w\bar{R}_N)'''(x_i) = 0 \qquad (i = 1, \ldots, n), \\ &\bar{R}_N^{(j)}(\bar{x}_i) = 0 \qquad\qquad\qquad\qquad (j = 0, \ldots, j_i - 1; \;\; i = 1, \ldots, m). \end{aligned}$$

It is obvious that the polynomial $\bar{R}_N$ can be written in the form

$$\bar{R}_N(x) = \left(qp_n^2 Q\right)(x),$$

where p_n and q are defined in (2.3), and Q is a polynomial of degree at most n. On using Lemma 2.1 the weighted (0,1,3)-interpolatory conditions are

$$\left(w\bar{R}_N\right)'''(x_i) = 6\left(qwp_n'^2 Q'\right)(x_i) = 0 \qquad (i = 1, \ldots, n),$$

where $q(x_i) \neq 0$, $w(x_i) \neq 0$, $p_n'(x_i) \neq 0$. Hence $Q'(x_i) = 0$ for $i = 1, \ldots, n$, that is $Q'(x) \equiv 0$ and $Q(x) \equiv c$. From $\left(w\bar{R}_N\right)'''(x_0) = c(wqp_n^2)'''(x_0) = 0$ it follows that $\bar{R}_N(x) \equiv 0$ if and only if $(wqp_n^2)'''(x_0) \neq 0$, which completes the proof.

4. Special cases on the zeros of the classical orthogonal polynomials

The classical orthogonal polynomials (Hermite-, Laguerre-, Jacobi-polynomials) can be characterized as unique solutions of degree n of the second-order linear homogeneous differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0, \tag{4.1}$$

where σ and τ are polynomials of degree at most 2 and 1, respectively. The polynomials σ and τ are independent of n, while λ_n is independent of x. It is known that in this case

$$\lambda_n = -n\left(\frac{\tau' + (n-1)\sigma''}{2}\right), \tag{4.2}$$

	Additional Interpolatory Condition(s)	Condition(s) for Regularity
(i)	$R_N(x_0) = y_0$	$(qp_n)(x_0) \neq 0$
(ii)	$R'_N(x_0) = y'_0$	$\left(qp_n^2\right)'(x_0) \neq 0$
(iii)	$R''_N(x_0) = y''_0$	$\left(qp_n^2\right)''(x_0) \neq 0$
(iv)	$(wR_N)'''(x_0) = y'''_0$	$p_n(x_0) \neq 0,\ (wqp_n^2)'''(x_0) \neq 0$
(v)	$R_N(x_0) = y_0,$ $R'_N(x_0) = y'_0$	$(qp_n)(x_0) \neq 0$
(vi)	$R'_N(x_0) = y'_0,$ $R''_N(x_0) = y''_0$	$\left(qp_n^2\right)'(x_0)\left(qp_n^2\int_{x_0}^x p_n(t)dt\right)''(x_0)$ $-(qp_n^2)''(x_0)(qp_n^3)(x_0) \neq 0$
(vii)	$R'_N(x_0) = y'_0,$ $(wR_N)'''(x_0) = y'''_0$	$\left(qp_n^2\right)'(x_0)\left(wqp_n^2\int_{x_0}^x p_n(t)dt\right)'''(x_0)$ $-(wqp_n^2)'''(x_0)(qp_n^3)(x_0) \neq 0$
(viii)	$R_N(x_0) = y_0,$ $R_N(x_{n+1}) = y_{n+1}$	$(qp_n)(x_0) \neq 0, \quad (qp_n)(x_{n+1}) \neq 0,$ $\int_{x_0}^{x_{n+1}} p_n(t)dt \neq 0$
(ix)	$R'_N(x_0) = y'_0,$ $R'_N(x_{n+1}) = y'_{n+1}$	$(qp_n^2)'(x_0)(qp_n^2)'(x_{n+1})\int_{x_0}^{x_{n+1}} p_n(t)dt$ $+(qp_n^2)'(x_0)(qp_n^3)(x_{n+1})$ $-(qp_n^2)'(x_{n+1})(qp_n^3)(x_0) \neq 0$
(x)	$(R_N)''(x_0) = y''_0,$ $(R_N)''(x_{n+1}) = y''_{n+1}$	$(qp_n^2)''(x_{n+1})\left(qp_n^2\int_{x_0}^x p_n(t)dt\right)''(x_0)$ $-(qp_n^2)''(x_0)\left(qp_n^2\int_{x_0}^x p_n(t)dt\right)''(x_{n+1}) \neq 0$
(xi)	$(wR_N)'''(x_0) = y'''_0,$ $(wR_N)'''(x_{n+1}) = y'''_{n+1}$	$p_n(x_0) \neq 0, \quad p_n(x_{n+1}) \neq 0,$ $(wqp_n^2)'''(x_{n+1})\left(wqp_n^2\int_{x_0}^x p_n(t)dt\right)'''(x_0)$ $-(wqp_n^2)'''(x_0)\left(wqp_n^2\int_{x_0}^x p_n(t)dt\right)'''(x_{n+1})$ $\neq 0$

Table 1.

furthermore, with an appropriate weight function ω and function f the differential equation (4.1) can be transformed into

$$(\omega y)'' + f \cdot (\omega y) = 0. \tag{4.3}$$

Hence, based on Lemma 2.2 the zeros of the classical orthogonal polynomials fulfill the conditions of Lemma 2.1 with appropriate weight functions. The special cases of the functions σ, τ and ω are summarized in the following table ([12]).

y	$\sigma(x)$	$\tau(x)$	$\omega(x)$
Hermite $H_n(x)$	1	$-2x$	$e^{-\frac{x^2}{2}}$
Laguerre $L_n^{(\alpha)}(x)$ $(\alpha > -1)$	x	$\alpha + 1 - x$	$e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}}$
Jacobi $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$	$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$	$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}$

Table 2.

Theorem 4.1. *If the nodes $\{x_i\}_{i=1}^n$ are the zeros of the classical orthogonal polynomial $p_n(x)$ and $q'(x) = p_n(x)$, then the problem (2.1)-(2.2) is not regular with respect to the weight function $w(x) = \omega^2(x)$ in (4.3).*

Proof. Let $\{x_i\}_{i=1}^n$ be the zeros of the classical orthogonal polynomial p_n, $q'(x) = p_n(x)$ and let $w = \omega^2$ given in (4.3). In this case $w(x_i) \neq 0$, $q(x_i) \neq 0$, $p_n'(x_i) \neq 0$ for $i = 1, \dots, n$ and $m = n+1$. We show that there is no polynomial of degree at most $3n + M - 1$ which satisfies the conditions

$$
\begin{aligned}
&R_N(x_i) = 0, \quad R_N'(x_i) = 0, \quad (wR_N)'''(x_i) = \delta_{ik} \qquad (i = 1, \dots, n), \\
&R_N^{(j)}(\bar{x}_i) = 0 \qquad\qquad\qquad\qquad\qquad (j = 0, \dots, j_i - 1;\ i = 1, \dots, m),
\end{aligned}
\tag{4.4}
$$

where k is a fixed number $(1 \le k \le n)$.

If $\bar{R}_N$ is a polynomial of degree at most $3n + M - 1$ which satisfies the conditions (4.4), then it can be written in the form

$$\bar{R}_N(x) = (qp_n^2\, Q)(x),$$

where Q is a polynomial of degree at most $n - 1$. Based on Lemma 2.2 we can apply Lemma 2.1 with $q_1 = w_2 = 1$, and we have

$$(wR_N)'''(x_i) = (qwp_n^2 Q)'''(x_i) = 6q(x_i)w(x_i)p_n'(x_i)Q'(x_i) = 0$$

for $i \neq k$ ($i \in \{1,\dots,n\}$). Thus $Q'(x_i) = 0$ for $i \neq k$ and hence $Q'(x) \equiv 0$ which contradicts the condition $(wR_N)'''(x_k) = 1$.

4.1. Pál-type weighted $(0,1,3;0,1,\dots,r-1)$-interpolation on the zeros of Hermite-polynomials

On $(-\infty,\infty)$ let the nodes $\{x_i\}_{i=1}^n$ be the zeros of the Hermite-polynomial of degree n, that is $p_n(x) = H_n(x)$. Let us consider weighted $(0,1,3)$-interpolation with respect to the weight function $w(x) = e^{-(r+1)x^2}$ on the zeros of the Hermite polynomial H_n and Hermite interpolation on the zeros of H_n'. Let the additional node be $x_0 = 0$. Based on (4.3), the conditions of Theorem 2.4 are fulfilled with

$$w_1(x) = e^{-x^2}, \quad w_2(x) = e^{-rx^2}, \quad q_1(x) = 1, \quad q_2(x) = {H_n'}^r(x),$$

hence the fundamental polynomials of this problem associated with the weighted (0,1,3)-interpolation can be written in the form

$$C_k(x) = \frac{{H_n'}^r(x) H_n^2(x) e^{(r+1)x_k^2}}{6{H_n'}^{r+2}(x_k)} \left\{ \int_0^x \ell_k(t)dt + \bar{\bar{c}}_k \right\}, \tag{4.4}$$

$$B_k(x) = \frac{{H_n'}^r(x) H_n(x) \ell_k^2(x)}{{H_n'}^{r+1}(x_k)} + \frac{{H_n'}^r(x) H_n^2(x)}{{H_n'}^{r+2}(x_k)} \times \tag{4.5}$$

$$\times \left\{ \int_0^x \frac{x_k \ell_k(t) - \ell_k'(t)}{t - x_k}\, dt + \frac{(3r+1)n + 2 - 2x_k^2}{3} \int_0^x \ell_k(t)dt + \bar{c}_k \right\},$$

$$A_k(x) = \frac{{H_n'}^r(x)}{{H_n'}^r(x_k)} \ell_k^3(x) - (2r+3)x_k B_k(x) + \tag{4.6}$$

$$+ \frac{{H_n'}^r(x) H_n^2(x)}{{H_n'}^{r+2}(x_k)} \times \left\{ c_k + \int_0^x \frac{[x_k + \beta_k(t - x_k)]\ell_k(t) - \ell_k'(t)}{(t - x_k)^2}\, dt + \right.$$

$$\left. + \frac{(3+r)nx_k - 2x_k^3}{3} \int_0^x \ell_k(t)dt \right\},$$

where $\beta_k = \frac{x_k^2+2-2n}{3}$ for $k = 1, \dots, n$. In regular cases the parameters c_k, $\bar{c}_k$ and $\bar{\bar{c}}_k$ are to be determined from the additional condition at $x_0 = 0$.

4.2. Weighted $(0, 1, 3)$-interpolation on the zeros of Hermite-polynomials

Theorem 4.2. *On the zeros of the Hermite-polynomial H_n the weighted $(0, 1, 3)$-interpolation is regular with respect to the weight function $w = e^{-x^2}$ with the additional condition (i) in Table 1 at $x_0 = 0$ only for even n, and it is regular for all n with the condition (iii), but it is not regular for any n if the condition is (ii) or (iv).*

Proof. For the sake of simplicity we discuss only the case (i), when the additional condition is

$$R_N(0) = y_0, \qquad y_0 \in R. \tag{4.7}$$

Under this additional condition the regularity is assured by $H_n(0) \neq 0$, therefore the problem is regular for even n. The parameters c_k, $\bar{c}_k$ and $\bar{\bar{c}}_k$ are determined from the conditions $C_k(0) = 0$, $B_k(0) = 0$ and $A_k(0) = 0$, respectively, and

$$\bar{\bar{c}}_k = 0, \quad \bar{c}_k = \frac{\ell_k(0)}{x_k}, \quad c_k = -\frac{\ell_k(0)}{x_k^2} \qquad (k = 1, \dots, n). \tag{4.8}$$

Hence the polynomial

$$R_N(x) = \sum_{k=1}^{n} \Big[A_k(x) y_k + B_k(x) y_k' + C_k(x) y_k''' \Big] + \frac{H_n^2(x)}{H_n^2(0)} y_0$$

is of degree at most $3n$ and it satisfies (1.1) with the additional condition (4.7) on the zeros of H_n for even n. The fundamental polynomials A_k, B_k and C_k are given in (4.4)–(4.6) with $r = 0$ and with the parameters in (4.8).

4.3. Weighted $(0, 1, 3)$-interpolation on the zeros of Laguerre-polynomials

On $[0, \infty)$ let the nodes $\{x_i\}_{i=1}^n$ be the zeros of Laguerre-polynomial of degree n, namely for $\alpha > -1$

$$p_n(x) = L_n^{(\alpha)}(x) \qquad \text{with} \qquad L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}.$$

Theorem 4.3. *On the zeros of the Laguerre-polynomial $L_n^{(\alpha)}$ the weighted $(0,1,3)$-interpolation is regular for all $\alpha > -1$ with respect to the weight function $w = e^{-x}x^{\alpha+1}$ under the additional condition (i), (ii), (iii) or (v) in Table 1 at $x_0 = 0$.*

Proof. Based on (4.3) the conditions of Lemma 2.1 are fulfilled with

$$w_1(x) = e^{-x}x^{\alpha+1}, \qquad w_2(x) = q_1(x) = q_2(x) = 1.$$

It can be verified easily that the conditions for regularity are fulfilled for all α in these cases and the fundamental polynomials can be constructed by Theorem 2.3.

Theorem 4.4. *On the zeros of the Laguerre-polynomial $L_n^{(\alpha)}$ the weighted $(0,1,3)$-interpolation is regular for all $\alpha > -1$ with respect to the weight function $w = e^{-x}x^{\alpha}$ under the additional condition(s) (ii) or (vi) in Table 1 at $x_0 = 0$, but it is regular only for $\alpha = 0$ and $\alpha = 1$ under the conditions (vii).*

Proof. Based on (4.3) the conditions of Lemma 2.1 are fulfilled with

$$w_1(x) = e^{-x}x^{\alpha}, \qquad q_1(x) = x, \qquad w_2(x) = q_2(x) = 1.$$

As $L_n^{(\alpha)}(0) \neq 0$, in the case (ii) the condition for regularity in Table 1 is

$$(qp_n^2)'(0) = \left([L_n^{(\alpha)}]^2\right)'(0) = 2L_n^{(\alpha)}(0)L_n^{(\alpha)\prime}(0) \neq 0.$$

In the case (vi) we have

$$(xp_n^2)'(0)\left(xp_n^2\int\limits_0^x p_n(t)dt\right)''(0) = 2p_n^5(0) = 2[L_n^{(\alpha)}]^5(0) \neq 0.$$

Finally, in the case of the additional conditions (vii) the condition for regularity is equivalent to

$$\left(wp_n^2\int\limits_0^x p_n(t)dt\right)''(0) \neq 0,$$

that is, it is equivalent to

$$2w'(0)p_n(0) + 5w(0)p_n'(0) \neq 0,$$

which is fulfilled only for $\alpha = 0$ and 1. The fundamental polynomials in the regular cases can be constructed by Theorem 2.3.

Theorem 4.5. *On the zeros of the Laguerre-polynomial $L_n^{(\alpha)}$ the weighted $(0,1,3)$-interpolation is regular only for $\alpha = 0, 1$ and 2 with respect to the weight function $w = e^{-x}x^{\alpha-1}$ under the condition (iv) in Table 1.*

Proof. Based on (4.3) the conditions of Lemma 2.1 are fulfilled with

$$w_1(x) = e^{-x}x^{\alpha-1}, \qquad q_1(x) = x^2, \qquad w_2(x) = q_2(x) = 1.$$

For the condition of regularity by Theorem 3.1 we have

$$(qwp_n^2)'''(0) = \left(x^{\alpha+1}e^{-x}L_n^{(\alpha)^2}\right)'''(0) \neq 0$$

if and only if $\alpha = 0, 1, 2$. The fundamental polynomials in the regular cases can be constructed by Theorem 2.3.

Corollary. *The weighted $(0,1,3)$-interpolation is regular on the zeros of $xL_n^{(\alpha)}$ with respect to the weight function*

$$w(x) = e^{-x}x^{\alpha} \qquad \text{for } \alpha = 0, 1;$$

and

$$w(x) = e^{-x}x^{\alpha-1} \qquad \text{for } \alpha = 0, 1, 2.$$

4.4. Weighted $(0,1,3)$-interpolation on the zeros of Jacobi-polynomials

On $[-1,1]$ let $P_n^{(\alpha,\beta)}$ denote the Jacobi-polynomial of degree n $(\alpha, \beta > -1)$ with

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Following the steps of the previous sections we can derive similar statements if the nodes are the zeros of Jacobi polynomials and the additional nodes are $x_0 = -1$ and $x_{n+1} = 1$. We obtain that the weighted $(0,1,3)$-interpolation is regular in the cases described in the Table 3.

Remark. The regular case, when the nodes are the zeros of the integrated Legendre polynomial $(1-x^2)P_n^{(1)}(x)$, is the generalization of P. Turán's $(0,2)$-interpolation problem studied by R. B. Saxena and A. Sharma [11].

$p_n(x)$	$w(x)$	α	β	n
$(1+x)P_n^{(\alpha,\beta)}$	$(1-x)^{\alpha+1}(1+x)^{\beta-1}$	>-1	$0,1,2$	
$(1+x)P_n^{(\alpha,\beta)}$	$(1-x)^{\alpha+1}(1+x)^{\beta}$	>-1	$0,1$	
$(1-x)P_n^{(\alpha,\beta)}$	$(1-x)^{\alpha-1}(1+x)^{\beta+1}$	$0,1,2$	>-1	
$(1-x)P_n^{(\alpha,\beta)}$	$(1-x)^{\alpha}(1+x)^{\beta+1}$	$0,1$	>-1	
$(1-x^2)P_n^{(\alpha)}$	$(1-x^2)^{\alpha-1}$	$0,1,2$		even

Table 3.

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