Annales Univ. Sci. Budapest., Sect. Comp. 28 (2008) 303-325

IDENTITIES IN THE CONVOLUTION ARITHMETIC OF NUMBER THEORETICAL FUNCTIONS

K.-H. Indlekofer (Paderborn, Germany)

Dedicated to Professor Imre Kátai on the occasion of his 70th birthday

1. Introduction

Let $\mathcal{A} = \{f : \mathbb{N} \to \mathbb{C}\}$ be the set of arithmetic functions. There are many number theoretic investigations (for example, prime number theorem, mean behaviour of arithmetic functions) that are based on identities between various special functions. These identities result from manipulations of arithmetic functions like Möbius and von Mangoldt function, divisor function, etc. and they are evidence of a more formal structure surrounding the arithmetic functions. The setting for this structure is that of the ring $(\mathcal{A}, +, *)$ where the addition $+$ and the convolution $*$ are defined, for $f, g \in \mathcal{A}$, by

$$(f+g)(n) = f(n) + g(n) \qquad (n \in \mathbb{N})$$

and

$$(f * g)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) \qquad (n \in \mathbb{N}),$$

respectively. This point of view often provides simplicity and elegance to proofs.

In addition, for a given arithmetic function $f \in \mathcal{A}$ we can form the symbolism

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Such an object is called a *formal Dirichlet* series. In the case where the Dirichlet series converges absolutely for a given $s \in \mathbb{C}$, rearrangement of terms is permissible and multiplication yields

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{g(n)}{n^s} = \sum_{n,m=1}^{\infty}\frac{f(n)g(m)}{(nm)^s} = \sum_{l=1}^{\infty}\frac{1}{l^s}\sum_{nm=l} f(n)g(m) = \\ = \sum_{l=1}^{\infty}\frac{(f*g)(l)}{l^s}.$$

This then is taken as a definition for *multiplication* of formal Dirichlet series, namely,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{g(n)}{n^s} = \sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s}.$$

The set $\mathcal{D}$ of formal Dirichlet series forms a ring under addition and multiplication, and the map T of $\mathcal{A}$ into $\mathcal{D}$ defined by

$$f \mapsto T(f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

describes an isomorphism between the rings $\mathcal{A}$ and $\mathcal{D}$. For whatever s domain, the series

$$F(s) = T(f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

converges, the function F defined thereby is called *generating function of* f. (It should be noted, however, that not all $f \in \mathcal{A}$ have generating functions. For example, if $f(n) = 2^n$, the corresponding formal Dirichlet series converges nowhere.)

Thus the statement of identities and inequalities in convolution arithmetic may in many cases be expressed by means of generating functions. This motivation is the starting point of our investigations.

2. Results

The simple fact that multiplying by the log-function L_0 ($L_0(n) = \log n$ for $n \in \mathbb{N}$) acts as an derivation on $(\mathcal{A}, +, *)$ and the von Mangoldt function Λ defined by $L_0(n) = \sum_{d|n}\Lambda(d)$ plays an important role in our proofs.

Theorem 1. *Let* $f : \mathbb{N} \to \mathbb{C}$ *be a multiplicative function, and put*

$$M(x) = \sum_{n\le x} f(n) \quad \textit{for } x \ge 1.$$

Define a completely multiplicative function $\tilde{f}$ by $\tilde{f}(p) = f(p)$ for all primes p, and define g by $f = g * \tilde{f}$. Then, for all $x \geq 1$

$$M(x)\log^2 x = \sum_{n\leq x} M\left(\frac{x}{n}\right)\tilde{f}(n)\left\{\sum_{dd'=n}\Lambda(d)\Lambda(d') + \Lambda(n)\log n\right\} + \\ + \sum_{n\leq x}\left\{R_1\left(\frac{x}{n}\right) + R_2\left(\frac{x}{n}\right)\right\}\tilde{f}(n)\,\Lambda(n) + \\ + \{R_1(x) + R_2(x)\}\log x,$$

where

$$R_1(x) = \sum_{n\leq x} f(n)\log\frac{x}{n},$$

$$R_2(x) = \sum_{n\leq x}\left(\sum_{m\leq \frac{x}{n}}\tilde{f}(m)\right) g(n)\log n.$$

For a given arithmetical function $w : \mathbb{N} \to \mathbb{C}$ with $w(1) \neq 0$ we define Λ_w by

$$L_0 w = w * \Lambda_w.$$

Then we prove

Theorem 2. *Let $f, \tilde{f}$ and g be defined as in Theorem 1, and let w be an arithmetical function with $w(1) \neq 0$. Put*

$$M(x) = \sum_{n\leq x}\left(f(n) - w(n)\right).$$

Then

$$M(x)\log^2 x = \sum_{n\leq x} M\left(\frac{x}{n}\right)\tilde{f}(n)\left\{\sum_{dd'=n}\Lambda(d)\Lambda(d') + \Lambda(n)\log n\right\} + \\ + \sum_{n\leq x}\left\{R_1\left(\frac{x}{n}\right) + R_2\left(\frac{x}{n}\right) + R_3\left(\frac{x}{n}\right)\right\}\tilde{f}(n)\,\Lambda(n) + \\ + \{R_1(x) + R_2(x) + R_3(x)\}\log x,$$

where

$$R_1(x) = \sum_{n\le x} (f(n) - w(n)) \log \frac{x}{n},$$

$$R_2(x) = \sum_{n\le x} \left(\sum_{m\le \frac{x}{n}} \tilde{f}(m) \right) g(n) \log n,$$

$$R_3(x) = -\sum_{n\le x} \left(\sum_{m\le \frac{x}{n}} w(m) \right) \left(\Lambda_w(n) - \Lambda(n)\tilde{f}(n) \right).$$

We shall apply these identities to multiplicative functions of modulus smaller or equal to one. For this we define, for a given $a \in \mathbb{R}$, the completely multiplicative function $\mathbf{1}_a$ by

$$\mathbf{1}_a(n) = \begin{cases} 1 & \text{if } n = 1, \\ n^{ia} & \text{if } n > 1. \end{cases}$$

Choosing $w = A\mathbf{1}_a$ with some constant $A \in \mathbb{C}$, then $\Lambda_w = \Lambda \mathbf{1}_a$ (if $A \ne 0$), and Theorems 1 and 2 lead to

Theorem 3. *Let f be multiplicative and $|f| \le 1$. Let $A \in \mathbb{C}$ and $a \in \mathbb{R}$. Then*

$$\frac{1}{x} \left| \sum_{n\le x} (f(n) - An^{ia}) \right| \le \frac{2}{\log x} \int_1^x \frac{\left| \sum_{n\le u} (f(n) - An^{ia}) \right|}{u^2} du + O\left(|A| \frac{1}{\log x} \sum_{p\le x} \frac{|f(p) - p^{ia}|}{p} \log p \right) + O\left(\frac{1}{\log x} \right)$$

as $x \to \infty$.

For an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ we define the generating function F of f by

$$F(s) := \sum_{n=1}^{\infty} f(n) n^{-s}, \tag{1}$$

where $s = \sigma + it$ and assume that $F(s)$ converges absolutely for $\sigma > 1$. Then, integration by parts shows for $\sigma > 1$

$$s^{-1}F(s) = \int_0^\infty e^{-\omega} \left(\sum_{n \le e^\omega} f(n) \right) e^{-\omega(\sigma-1)} e^{-i\omega t} d\omega$$

and, by Parseval's formula,

$$\int_{-\infty}^{\infty} \left| \frac{F(s)}{s} \right|^2 dt = 2\pi \int_0^\infty |e^{-\omega} \sum_{n \le e^\omega} f(n)|^2 e^{-2\omega(\sigma-1)} d\omega . \tag{2}$$

This leads to

Theorem 4. *Let $A \in \mathbb{C}, a \in \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{C}$. Assume that the generating function F of f converges absolutely for $\sigma > 1$. Then*

$$\frac{1}{\log x} \int_1^x \frac{\left| \sum_{n \le u} (f(n) - An^{ia}) \right|}{u^2} du \ll \left(\frac{1}{\log x} \int_{-\infty}^{\infty} \left| \frac{F(s) - A\zeta(s - ia)}{s} \right|^2 dt \right)^{\frac{1}{2}}$$

and

$$\frac{1}{\log x} \int_1^x \frac{\left| \sum_{n \le u} (f(n) - An^{ia}) \right|}{u^2} du \ll \frac{1}{\log x} \left(\int_{-\infty}^{\infty} \left| \frac{F'(s) - A\zeta'(s - ia)}{s} \right|^2 dt \right)^{\frac{1}{2}},$$

where $s = 1 + \frac{1}{\log x} + it$.

For $\sigma > 1$ and $a \in \mathbb{R}$ we put

$$F_a(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^{ia}} n^{-s}$$

and obtain

$$\frac{F(s - ia)}{\zeta(s)} = \frac{F_a(s)}{\zeta(s)} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{k(s+ia)}} \right) \left(1 - \frac{1}{p^s} \right).$$

From this we deduce the representation

$$\frac{F_a(s)}{\zeta(s)} = \exp \left(\sum_p \frac{f(p)p^{-ia} - 1}{p^s} + h(s) \right),$$

where

$$h(s)=\sum_p\left\{\log\left(1+\sum_{k=1}^{\infty}\frac{f(p^k)}{p^{iak}}\cdot p^{-ks}\right)-\frac{f(p)}{p^{ia}}p^{-s}+\log\left(1-p^{-s}\right)+p^{-s}\right\}.$$

In view of $|f(p^k)p^{-iak}|\le 1$ the function h is uniformly continuous and bounded for $\sigma\ge 1$. Especially we have

$$\begin{aligned}|h(s)-h(1)|\ &\le\sum_p\sum_{k=2}^{\infty}|p^{-ks}-p^{-k}|\le\\ &\le\sum_p\sum_{k=2}^{\infty}p^{-k}\,|1-\exp\left(k(1-s)\log p\right)|\le\\ &\le\sum_p\sum_{k=2}^{\infty}\frac{k\log p}{p^k}\,|s-1\ll\\ &\ll|s-1|\,.\end{aligned}$$

Then, putting

$$A=\exp\left(\sum_{p\le x}\frac{f(p)p^{-ia}-1}{p}+h(1)\right)$$

we shall prove

Theorem 5. *Let f be multiplicative and $|f|\le 1$. Assume that*

$$\sum_p\frac{|f(p)-p^{ia}|^2}{p}\le c<\infty. \tag{3}$$

for some $a\in\mathbb{R}$.

Let $(\log x)^{-1}\log\log x\le\delta_0(x)$ and $\delta_0(x)\to 0$ as $x\to\infty$ such that, if we put $y=y(x)=x^{\delta_0(x)}$

$$\sum_{y(x)<p\le x}\frac{|f(p)-p^{ia}|^2}{p}=:\delta_1(x)\le\delta_0(x)$$

for $x \geq x_0$. *Then*

$$\frac{1}{x}\sum_{n\leq x} f(n) - A\frac{x^{ia}}{1+ia} =$$

$$= O\left(\exp\left(\sum_{p\leq x}\frac{\mathrm{Re}f(p)p^{-ia}-1}{p}\right)\cdot\frac{1}{\log x}\sum_{p\leq x}\frac{\log p}{p}\left|f(p)-p^{ia}\right|\right)+$$

$$+O(\delta_0(x))^{1/10} =$$

$$= O((\delta_0(x))^{1/10}).$$

As a corollary we obtain

Corollary 1. *Assume that (3) holds. Then, with the notations of Theorem 5*

$$\frac{1}{x}\sum_{n\leq x} f(n) = \frac{x^{ia}}{1+ia}\prod_{p\leq x}\left(1-\frac{1}{p}\right)\left(1+\sum_{k=1}^{\infty} f(p^k)p^{-k(1+ia)}\right)+$$

$$+O\left((\delta_0(x))^{\frac{1}{10}}\right)$$

as $x \longrightarrow \infty$.

If the series (3) diverges for all $a \in \mathbb{R}$ we choose $A = 0$, and as in [6] we obtain

Corollary 2. *Assume that (3) diverges for all* $a \in \mathbb{R}$. *Then, if* f *is multiplicative and* $|f| \leq 1$,

$$\frac{1}{x}\sum_{n\leq x} f(n) = o(1)$$

as $x \longrightarrow \infty$.

As an immediate consequence of the above results we have

Corollary 3. *Let* f *be multiplicative and* $|f| \leq 1$. *Put*

$$M(x) = \sum_{n\leq x} f(n).$$

Then the following assertions are equivalent.

(i) $\lim_{x\to\infty} x^{-1}\, M(x) = 0$,

(ii) $\lim_{x\to\infty}\frac{1}{\log x}\int_1^x \frac{|M(u)|}{u^2}du = 0$,

(iii) $\lim_{x\to\infty} \frac{1}{\log x} \int_1^x \frac{|M(u)|^2}{u^3} du = 0,$

(iv) $\lim_{\sigma\to 1^+} (\sigma - 1) \int_{-\infty}^{\infty} \left| \frac{F(\sigma+it)}{\sigma+it} \right|^2 dt = 0.$

3. Simple properties of convolution

Our treatment of this topic follows that of Shapiro's book [10].

The classes of functions that are distinguished are denoted by $\mathcal{S}$ and $\mathcal{A}$, and are defined as follows

$$\mathcal{S} := \{f : \mathbb{R} \to \mathbb{C}, f(x) = 0 \; for \; x < 1\},$$

$$\mathcal{A} := \{f \in \mathcal{S} : \; f(x) = 0 \; for \; x \notin \mathbb{N}\}.$$

Then, for $f, g \in \mathcal{S}$, the *convolution* $f * g$ in $\mathcal{S}$ is defined by

$$(f * g)(x) = \sum_{1\le n\le x} f\left(\frac{x}{n}\right) g(n) \; . \tag{4}$$

The "action" of this definition on functions of $\mathcal{A}$ is given by the following: If $f \in \mathcal{A}$, $g \in \mathcal{S}$ then $f * g \in \mathcal{A}$ and for $n \in \mathbb{N}$,

$$(f * g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) \; . \tag{5}$$

In general the binary operation $*$ is not commutative in $\mathcal{S}$, but if $f, g \in \mathcal{A}$ then $f * g = g * f$.

Consider the function ε defined by

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\varepsilon \in \mathcal{A}$, and

$$f * \varepsilon = f \qquad \text{for } f \in \mathcal{S}$$

and

$$(\varepsilon * f)(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } f \in \mathcal{S}. \tag{6}$$

Thus ε serves as a *right identity under convolution* for all of $\mathcal{S}$, but is a *left idendity* only in $\mathcal{A}$.

The relation (6) suggests that for each $f \in \mathcal{S}$ we define an image $f_0 \in \mathcal{A}$ by

$$f_0 = \varepsilon * f \qquad \text{for } f \in \mathcal{S}.$$

This definition leads to

$$(f * g) * h = f * (g_0 * h) \quad \text{for } f, g, h \in \mathcal{S}$$

which implies

$$(f * g) * h = f * (g * h) \quad \text{for } f, g, h \in \mathcal{A}.$$

An element $f \in \mathcal{S}$ is called a *left unit* in $\mathcal{S}$ if there exists a $g \in \mathcal{S}$ such that

$$f * g = \varepsilon \,. \tag{7}$$

It is called a *right unit* if there exists a $g \in \mathcal{S}$ such that

$$g * f = \varepsilon.$$

As a comparison terminology, if (7) holds g is called a *right inverse* for f, and f a *left inverse* for g.

The investigation of these concepts may be initiated with the following

Further properties:

(i) A necessary and sufficient condition for $f \in \mathcal{S}$ to have a left inverse is that $f(1) \neq 0$.

(ii) If $f(1) \neq 0$, the left inverse of f is in $\mathcal{A}$.

(iii) If $f(1) \neq 0$, and $f \in \mathcal{A}$, then f has a unique two-sided inverse in $\mathcal{A}$.

(iv) Let $h \in \mathcal{A}$ be completely multiplicative (i.e. $h(nm) = h(n)h(m)$ for all $n, m \in \mathbb{N}$) then

$$h(f * g) = (hf) * (hg) \quad \text{for all } f, g \in \mathcal{A},$$

especially

$$h * h\mu = \varepsilon.$$

Here the Möbius function μ is defined by

$$\mathbf{1}_0 * \mu = \varepsilon,$$

where $\mathbf{1}_0 = \varepsilon * \mathbf{1}$ and $\mathbf{1} \in \mathcal{S}$ with

$$\mathbf{1}(x) = \begin{cases} 1 & x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The wellknown *Möbius inversion formula* says that if $f, g \in \mathcal{S}$ then $f = g * \mathbf{1}_0$ if and only if $g = f * \mu$.

Examples:

(i) Let $g = \mathbf{1}$. Then $f(x) = [x]$ and $\sum_{n \leq x} \left[\frac{x}{n}\right] \mu(n) = 1$ which implies $x \sum_{n \leq x} \frac{\mu(n)}{n} = O(x)$, i.e.

$$\sum_{n \leq x} \frac{\mu(n)}{n} = O(1). \tag{8}$$

(ii) Let $g(x) = x$ for $x \geq 1$. Then

$$f(x) = \sum_{n \leq x} \frac{x}{n} = x \log x + c_1 x + O(1)$$

and

$$\begin{aligned} x &= g(x) = \sum_{n \leq x} \mu(n) \left\{\frac{x}{n} \log \frac{x}{n} + c_1 \frac{x}{n}\right\} + O(x) = \\ &= x \sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} + c_1 x \sum_{n \leq x} \frac{\mu(n)}{n} + O(x) \end{aligned}$$

which implies

$$\sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} = O(1). \tag{9}$$

The constant c_1 equals Euler's constant γ.

(iii) Let $g(x) = x \log x$. By a straightforward calculation (partial summation) we deduce

$$\begin{aligned} f(x) &= \sum_{n \leq x} \frac{x}{n} \log \frac{x}{n} = \\ &= x \log x \sum_{n \leq x} \frac{1}{n} - x \sum_{n \leq x} \frac{\log n}{n} = \\ &= \frac{1}{2} x \log^2 x + c_1 x \log x - c_2 x + O(\log x) \end{aligned}$$

since with some constant c_2

$$\sum_{n\le x}\frac{\log n}{n}=\frac{1}{2}\log^2 x+c_2+O\left(\frac{\log x}{x}\right).$$

This implies, by (8) and (9)

$$x\log x=g(x)=\frac{1}{2}x\sum_{n\le x}\frac{\mu(n)}{n}\log^2\frac{x}{n}+O(x)$$

and

$$\sum_{n\le x}\frac{\mu(n)}{n}\log^2\frac{x}{n}=2\log x+O(1). \tag{10}$$

Let $L\in\mathcal{S}$ denote the logarithm function. Then obviously L acts as a derivation on $\mathcal{S}$, that is

$$L\cdot(f*g)=(L\cdot f)*g+f*L\cdot g\quad\text{for all } f,g\in\mathcal{S}. \tag{11}$$

Further, we introduce the von Mangoldt function $\Lambda\in\mathcal{A}$ by

$$\varepsilon*L=L_0=\Lambda*\mathbf{1}_0, \tag{12}$$

i.e.

$$\Lambda=L_0*\mu. \tag{13}$$

The relation (12) and (13) immediately show

$$\begin{aligned} L_0^2 &= L_0\cdot(\mathbf{1}_0*\Lambda)= \\ &= L_0*\Lambda+\mathbf{1}_0*L_0\Lambda= \\ &= \mathbf{1}_0*(\Lambda*\Lambda+L_0\Lambda) \end{aligned}$$

and

$$\mu*L_0^2=\Lambda*\Lambda+L_0\Lambda. \tag{14}$$

On the other hand, by (8) and (9)

$$\begin{aligned} \mathbf{1}*(\mu*L_0^2)(x) &= \sum_{n\le x}\sum_{d|n}\mu(d)\log^2\frac{n}{d}=\sum_{dd'\le x}\mu(d)\log^2 d'= \\ &=\sum_{d\le x}\mu(d)\sum_{d'\le\frac{x}{d}}\log^2 d'= \\ &=x\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d}-2x\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}+2x\sum_{d\le x}\frac{\mu(d)}{d}+O(x)= \\ &=x\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d}+O(x) \end{aligned}$$

since

$$\sum_{n\le y}\log^2 n = \int_1^y \log^2 t\,dt + O(\log^2 y) = $$
$$= y\log^2 y - 2y\log y + 2y + O(\log^2 y).$$

Considering (14) and (10) produces

$$\mathbf{1}*(L_0\Lambda + \Lambda*\Lambda)(x) = \sum_{n\le x}\Lambda(n)\log n + \sum_{dd'\le x}\Lambda(d)\Lambda(d') = \tag{15}$$
$$= 2x\log x + O(x).$$

which is known as *Selberg's Symmetry Formula.*

4. Proof of Theorem 1 and Theorem 2

Let

$$M(x) = \sum_{n\le x} f(n),$$

i.e.

$$M = \mathbf{1}*f = \mathbf{1}*(g*\tilde{f})$$

with the notations of Theorem 1.
Then

$$LM = L*f + \mathbf{1}*L_0 f. \tag{16}$$

Putting $R_1 = L*f$ leads to

$$LM = \mathbf{1}*L_0 f + R_1.$$

Observing

$$L_0 f = L_0 g*\tilde{f} + g*\left(\Lambda\tilde{f}*\tilde{f}\right) =$$
$$= f*\Lambda\tilde{f} + L_0 g*\tilde{f}$$

gives

$$\mathbf{1}*L_0 = M*\Lambda\tilde{f} + R_2, \tag{17}$$

where $R_2 = \mathbf{1}*\left(L_0\,g*\tilde{f}\right)$.

Collecting (16) and (17) shows

$$LM = M * \Lambda\tilde{f} + R_1 + R_2 \tag{18}$$

with

$$R_1 = L * f, \; R_2 = \mathbf{1} * \left(L_0\, g * \tilde{f}\right). \tag{19}$$

We multiply (18) with L and obtain

$$L^2M = \left(LM * \Lambda\tilde{f}\right) + M * L_0\Lambda\tilde{f} + L(R_1 + R_2). \tag{20}$$

Then, by substituting (18) in (20) we arrive at

$$\begin{aligned} L^2M &= \left(M * \Lambda\tilde{f} + R_1 + R_2\right) * \Lambda\tilde{f} + \\ &\quad + M * L_0\Lambda\tilde{f} + L(R_1 + R_2) = \\ &= M * \left(\Lambda\tilde{f} * \Lambda\tilde{f} + L_0\, \Lambda\tilde{f}\right) + \\ &\quad + (R_1 + R_2) * \Lambda\tilde{f} + L(R_1 + R_2) \end{aligned} \tag{21}$$

which leads immediately to Theorem 1.

For the proof of Theorem 2 we put $M = \mathbf{1} * (f - w)$. This leads to

$$LM = \mathbf{1} * L_0(f - w) + R_1, \tag{16'}$$

where $R_1 = L * (f - w)$. Since $w(1) \neq 0$ there exists Λ_w such that

$$L_0 w = w * \Lambda_w$$

holds, and, as above

$$\begin{aligned} \mathbf{1} * L_0(f - w) &= \mathbf{1} * f * \Lambda\tilde{f} - \mathbf{1} * w * \Lambda_w + R_2 = \\ &= M * \Lambda\tilde{f} - \mathbf{1} * w * (\Lambda_w - \Lambda\tilde{f}) + R_2, \end{aligned} \tag{17'}$$

where $R_2 = \mathbf{1} * (L_0\, g * \tilde{f})$. Collecting (16') and (17') shows

$$LM = M * \Lambda\tilde{f} + R_1 + R_2 + R_3 \tag{18'}$$

with

$$R_1 = L * (f - w), \quad R_2 = \mathbf{1} * (L_0\, g * \tilde{f}) \tag{19'}$$

and

$$R_3 = -\mathbf{1} * w * (\Lambda_w - \Lambda\tilde{f}). \tag{22}$$

We multiply (18') by L and obtain

$$L^2M = (LM * \Lambda\tilde{f}) + M * L_0\Lambda\tilde{f} + L(R_1 + R_2 + R_3). \tag{20'}$$

Then, by (18') and (20') we arrive at

$$\begin{aligned} L^2M &= M * (\Lambda\tilde{f} * \Lambda\tilde{f} + L_0\Lambda\tilde{f}) + \\ &\quad + (R_1 + R_2 + R_3) * \Lambda\tilde{f} + L(R_1 + R_2 + R_3) \end{aligned} \tag{21'}$$

which proves Theorem 2.

5. Proof of Theorem 3

Let f be multiplicative and $|f| \le 1$. We apply either Theorem 1 or Theorem 2 with the choice $w = A\mathbf{1}_a$ with some $A \in \mathbb{C}$ and $a \in \mathbb{R}$. In the second case $\Lambda_w = \Lambda\mathbf{1}_a$. In both cases we have

$$|R_1(x)| \ll \sum_{n\le x} \log\frac{x}{n} = O(x)$$

and

$$|R_2(x)| = O\left(x\sum_{n\le x}\frac{|g(n)|}{n}\log n\right) = O(x)$$

which implies

$$\left|(R_1 + R_2) * \Lambda\tilde{f}\right|(x) = O\left(x\sum_{n\le x}\frac{\Lambda(n)}{n}\right) = O(x\log x).$$

For the estimate of R_3 we observe $\Lambda_w = \Lambda\mathbf{1}_a$ and obtain

$$|R_3(x)| \ll |A|x\sum_{n\le x}\frac{\Lambda(n)}{n}|\tilde{f}(n) - n^{ia}|.$$

Since

$$\begin{aligned} |(R_3 * \Lambda\tilde{f})(x)| &\ll |A|\sum_{n\le x}\left(\sum_{m\le \frac{x}{n}}\log m\right)\Lambda(n)|\tilde{f}(n) - n^{ia}| \ll \\ &\ll |A|x(\log x)\sum_{n\le x}\frac{\Lambda(n)}{n}|\tilde{f}(n) - n^{ia}| \end{aligned}$$

we obtain the remainder terms in Theorem 2. The rest of the proof is based on a summation formula the proof of which can be found in [6].

Lemma 1. *Let $R \in S, R(x) \geq 0$ and $v \in \mathcal{A}$ such that*

$$\sum_{n \leq x} v(n) = cx(\log x)^m + O(x(\log x)^{m-1})$$

for some $m \geq 0$. Assume that there is a steadily increasing function $H \in \in S, H(x) = O(x)$ such that for $1 \leq t' \leq t$

$$|R(t) - R(t')| \leq H(t) - H(t').$$

Then

$$(R * v)(x) = \sum_{n \leq x} R\left(\frac{x}{n}\right) v(n) = = c \int_1^x R\left(\frac{x}{t}\right) (\log t)^m dt + O(x(\log x)^m).$$

Put $M(x) = \sum_{n \leq x} (f(n) - An^{ia})$. Then by Theorems 1 and 2, (15) and Lemma 1 with $H(t) = t$ and $m = 1$

$$|M(x)| \log^2 x \leq 2 \int_1^x \left|M\left(\frac{x}{t}\right)\right| \log t \, dt + +O(x \log x) + +O\left(|A| \log x \sum_{p \leq x} \frac{|f(p) - p^{ia}|}{p} \log x\right).$$

Obviously

$$\int_1^x \left|M\left(\frac{x}{t}\right)\right| \log t \, dt \leq \log t \int_1^x \left|M\left(\frac{x}{t}\right)\right| dt = = x \log x \int_1^x \frac{|M(u)|}{u^2} du$$

from which the assertion of Theorem 3 follows.

6. Proof of Theorem 4

By Cauchy's inequality

$$\int_1^x \frac{|M(u)|}{u^2}du \le \left(\int_1^x \frac{|M(u)|^2}{u^3}du\right)^{\frac{1}{2}} \left(\int_1^x \frac{du}{u}\right)^{\frac{1}{2}}$$

Since $1 \le u^{2/\log x} \le e^2$ for $1 \le u \le x$ we get

$$\int_1^x \frac{|M(u)|^2}{u^3}du \ll \int_1^x \frac{|M(u)|^2}{u^{3+2\alpha}}du \le \int_1^\infty \frac{|M(u)|^2}{u^{3+2\alpha}}du,$$

where $\alpha = \frac{1}{\log x}$. Substituting $u = e^\omega$ and using Parseval's Formula (2) gives

$$\tag{23} \begin{aligned} \frac{1}{(\log x)^{\frac{1}{2}}} \int_1^x \frac{|M(u)|}{u^2}du &\ll \left(\int_0^\infty \frac{|M(e^\omega)|^2}{e^{2\omega(1+\alpha)}}dw\right)^{\frac{1}{2}} = \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{F(s) - A\zeta(s-ia)}{s}\right|^2 ds\right)^{\frac{1}{2}}, \end{aligned}$$

where $s = 1 + \frac{1}{\log x} + it$.

Putting $K(u) = \sum_{n \le u} (f(n) - An^{ia}) \log n$ partial summation shows that for $u \ge 2$

$$\tag{24} M(u) = \frac{K(u)}{\log u} + \int_2^u \frac{K(t)}{t(\log t)^2}dt,$$

so that

$$\tag{25} \begin{aligned} \int_2^x \frac{|M(u)|}{u^2}du &\le \int_2^x \frac{|K(u)|}{u^2 \log u}du + \int_2^x \frac{|K(t)|}{t(\log t)^2} \int_t^x \frac{du}{u^2}dt \le \\ &\le \left(1 + \frac{1}{\log 2}\right) \int_2^x \frac{|K(u)|}{u^2 \log u}du. \end{aligned}$$

By Cauchy's inequality

$$\int_2^x \frac{|K(u)|}{u^2 \log u} du \le \left(\int_2^x \frac{|K(u)|^2}{u^3} du \right)^{\frac{1}{2}} \left(\int_2^x \frac{du}{u \log^2 u} \right)^{\frac{1}{2}}$$

and in the same way as above we arrive at

$$\begin{aligned} \int_1^x \frac{|M(u)|}{u^2} du &\ll \left(\int_0^\infty \frac{|K(e^\omega)|^2}{e^{2\omega(1+\alpha)}} d\omega \right)^{1/2} = \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{F'(s) - A\zeta'(s - ia)}{s} \right|^2 ds \right)^{1/2}, \end{aligned} \tag{26}$$

where $s = 1 + \dfrac{1}{\log x} + it.$

7. Some lemmas

First we collect some simple facts about the ζ-function.

Lemma 2. *Let* $s = \sigma + it$. *Then*

$$|\zeta(s)| \ll \frac{1}{|s-1|} \quad \text{if} \quad |t| \le 3$$

and

$$|\zeta(s)| \ll \log|t| \quad \text{if} \quad |t| > 3$$

uniformly in $\sigma > 1$.

Proof. Partial summation shows

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^\infty n^{-s} = s \int_1^\infty \frac{[u]}{u^{s+1}} du = \\ &= \frac{1}{s-1} + 1 + s \int_1^\infty \frac{[u] - u}{u^{s+1}} du \end{aligned}$$

which obviously implies $\zeta(s) = O(|s-1|^{-1})$ for $|t| \le 3$.

In the same manner we conclude for every $\sigma > 1$ and positive integer N

$$\zeta(s) - \sum_{n=1}^{N} n^{-s} = \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}du.$$

Hence

$$\begin{aligned}|\zeta(s)| &\le \sum_{n=1}^{N} n^{-1} + \frac{1}{|s-1|} + |s|\int_N^\infty \frac{du}{u^{1+\sigma}} \le \\ &\le \log N + \frac{1}{|s-1|} + \frac{|s|}{\sigma}N^{-\sigma} + \text{constant}\end{aligned}$$

and the desired result is obtained by choosing N suitably.

Without loss of generality we may assume that $f(p) = p^{ia}$ if $x < p$ since these values do not influence the sum $M(x)$. Then the following holds.

Lemma 3. *Let* $\sigma = 1 + \dfrac{1}{\log x}$ *and let* $\delta_0(x)$ *be given as in Theorem 5. Then, as* $x \to \infty$

$$\sum_{y(x)<p\le x} \frac{|f(p)p^{-ia}-1|}{p} \ll \left(\delta_0(x)\log\frac{1}{\delta_0(x)}\right)^{\frac{1}{2}} \tag{27}$$

and

$$\sum_{p\le y(x)} |f(p)p^{-ia}-1|\left|\frac{1}{p^s} - \frac{1}{p}\right| \ll \delta_0(x)|s-1|\log x. \tag{28}$$

Proof. By Cauchy's inequality

$$\begin{aligned}\sum_{y(x)<p\le x} \frac{|f(p)p^{-ia}-1|}{p} &\le (\delta_o(x))^{\frac{1}{2}}\left(\sum_{y(x)<p\le x}\frac{1}{p}\right)^{1/2} \ll \\ &\ll \left(\delta_0(x)\log\frac{1}{\delta_0(x))}\right)^{1/2}\end{aligned}$$

and thus (27) holds. Further we observe

$$\begin{aligned}&\sum_{p\le y(x)} |f(p)p^{-ia}-1|\left|\frac{1}{p^s} - \frac{1}{p}\right| \le \\ &\le 2\sum_{p\le y(x)}\frac{1}{p}|1-\exp((1-s)\log p)| \le \\ &\le 2|s-1|\sum_{p\le y(x)}\frac{\log p}{p} \ll \delta_0(x)|s-1|\log x\end{aligned}$$

which ends the proof of Lemma 3.

From Lemma 3 we conclude

Lemma 4. *Let $K > 0$ and $\sigma = 1 + \frac{1}{\log x}$. Then*

$$F_a(s) - A\zeta(s) \ll |\zeta(s)| \left\{ \left(\delta_0(x) \log \frac{1}{\delta_0(x)}\right)^{1/2} + K\delta_0(x) \right\}$$

uniformly for $|t| \le K(\sigma - 1)$ and every $K > 0$.

Proof. By Lemma 3 we have

$$\frac{F_a(s)}{A\zeta(s)} =$$

$$= \exp\left\{ \sum_p \frac{f(p)p^{-ia} - 1}{p^s} + h(s) - \sum_{p \le x} \frac{f(p)p^{-ia} - 1}{p} - h(1) \right\} =$$

$$= \exp\left(\sum_{p \le x} (f(p)p^{-ia} - 1) \left(\frac{1}{p^s} - \frac{1}{p}\right) + O(|s-1|) \right) =$$

$$= \exp\left(O\left(\left(\delta_0(x) \log \frac{1}{\delta_0(x)}\right)^{1/2} \right) + O(\delta_0(x)K) \right)$$

uniformly in $|t| \le K(\sigma - 1)$ which proves Lemma 4.

Lemma 5. *If $K(\sigma - 1) < |t| < K$ then*

$$F_a(\sigma + it) \ll \frac{1}{K^{1/2}} \cdot \frac{1}{\sigma - 1}.$$

Proof. Since

$$|1 - p^{it}|^2 \le 2|1 - f(p)p^{-ia}|^2 + 2|f(p)p^{-ia} - p^{it}|^2$$

we have

$$2\textstyle\sum_p p^{-\sigma}(1 - \operatorname{Re} p^{it}) = \sum_p p^{-\sigma}|1 - p^{it}|^2 \le$$

$$\le 2\textstyle\sum_p p^{-\sigma}|1 - f(p)p^{-ia}|^2 + 4\sum_p p^{\sigma}(1 - \operatorname{Re} f(p)p^{-ia}p^{-it})$$

which implies together with (3)

$$\left| \frac{\zeta(\sigma)}{\zeta(\sigma + it)} \right|^2 \ll \left| \frac{\zeta(\sigma)}{F_a(\sigma - it)} \right|^4.$$

This proves Lemma 5 since by Lemma 2

$$|\zeta(\sigma+it)| \ll \frac{1}{|\sigma+it-1|} + \log(3+|t|) \ll \frac{1}{K(\sigma-1)}.$$

Lemma 6. *Let f be a nonnegative multiplicative function, $f(p^\alpha) = O(1)$ for all prime powers p^α. Then*

$$x^{-1}\sum_{n\le x} f(n) \ll \exp\left(\sum_{p\le x}\frac{f(p)-1}{p}\right) \text{ for all } x \ge 2. \tag{29}$$

Proof. Put $M = \mathbf{1} * f$. Then

$$LM = \mathbf{1} * L_0 f + L * f$$

which leads to

$$\begin{aligned}(\log x)M(x) &= \sum_{n\le x} f(n)\log n + O\left(\sum_{n\le x}|f(n)|\log\frac{x}{n}\right) = \\ &= \sum_{n\le x} f(n)\sum_{p^\alpha\|n}\log p^\alpha + O\left(x\sum_{n\le x}\frac{|f(n)|}{n}\right) = \\ &= \sum_{p^\alpha\le x}\log p^\alpha \sum_{\substack{n\le \frac{x}{p^\alpha}\\ p\nmid n}} f(n)f(p^\alpha) + O\left(x\sum_{n\le x}\frac{|f(n)|}{n}\right) \le \\ &\le \sum_{n\le x} f(n)\sum_{p^\alpha\le\frac{x}{n}}\log p^\alpha + O\left(x\sum_{n\le x}\frac{|f(n)|}{n}\right) = \\ &= O\left(x\sum_{n\le x}\frac{f(n)}{n}\right).\end{aligned}$$

Since

$$\sum_{n\le x}\frac{f(n)}{n} \le \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^\alpha)}{p^2}+\ldots\right) \ll \exp\left(\sum_{p\le x}\frac{f(p)}{p}\right)$$

the assertation of Lemma 6 follows immediately.

8. Proof of Theorem 5

By changing the variable t into $t + ia$ we conclude

$$\int_{-\infty}^{\infty} \frac{|F(s) - A\zeta(s - ia)|^2}{|s|^2} dt = \int_{-\infty}^{\infty} \frac{|F_a(s) - A\zeta(s)|^2}{|s + ia|^2} dt \ll$$
$$\ll \int_{-\infty}^{\infty} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt.$$

Therefore, it is enough to estimate the integral

$$\int_{-\infty}^{\infty} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt,$$

where $s = \sigma + it$ with $\sigma = 1 + \dfrac{1}{\log x}$. For this purpose we divide the range of integration into the three parts

$$I_1 := \{t \in \mathbb{R} : |t| \le K(\sigma - 1)\},$$
$$I_2 := \{t \in \mathbb{R} : K(\sigma - 1) < |t| \le K\},$$
$$I_3 := \{t \in \mathbb{R} : K < |t|\}$$

and choose $K = (\delta_0(x))^{-4/5}$. For the interval I_1 we use Lemma 4 and obtain

$$\int_{I_1} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt \ll$$
$$\ll \left\{ \left(\delta_0(x) \log \frac{1}{\delta_0(x)} \right)^{1/2} + K\delta_0(x) \right\} \int_{I_1} \frac{|\zeta(s)|^2}{|s|^2} dt \ll \tag{30}$$
$$\ll (\delta_0(x))^{1/5} \log x.$$

Concerning the intervals of I_2 we have, by Lemma 2 and Lemma 5,

$$\int_{I_2} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt \ll$$
$$\ll \max_{t \in I_2} |F_a(s)|^{1/2} \int_{I_2} \frac{|F_a(s)|^{3/2}}{|s|^2} dt + \max_{t \in I_2} |\zeta(s)|^{1/2} \int_{I_2} \frac{|\zeta(s)|^{3/2}}{|s|^2} dt \ll$$
$$\ll \frac{1}{K^{1/4}(\sigma - 1)^{1/2}} \int_{I_2} \frac{|F_a(s)|^{3/2}}{|s|^2} dt + \frac{1}{K^{1/2}(\sigma - 1)^{1/2}} \int_{I_2} \frac{|\zeta(s)|^{3/2}}{|s|^2} dt.$$

It remains to estimate the two integrals on the right hand side. We shall proceed as in [1] and [8].

In the halfplane $\sigma > 1$ we have

$$|F_a(s)|^{3/4} \asymp |\exp\left(\sum_p \frac{3}{4} f(p)p^{-ia}p^{-s}\right) \asymp \left|\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{\Omega(n)} \frac{f(n)}{n^{ia}} n^{-s}\right|$$

and thus, by Parseval's equality and Lemma 6,

$$\int_{I_2} \frac{|F_a(s)|^{3/2}}{|s|^2} dt \ll \int_0^{\infty} \exp\left(-2\sum_{p\le e^{\omega}} \frac{1}{4}p^{-1}\right) e^{2\omega(\sigma-1)} d\omega \ll$$
$$\ll \int_0^{\infty} \omega^{-1/2} e^{-2\omega(\sigma-1)} d\omega \ll (\sigma-1)^{-1/2}.$$

In the same way we conclude

$$\int_{I_2} \frac{|\zeta(s)|^{3/2}}{|s|^2} dt \ll (\sigma-1)^{-1/2}.$$

Collecting the estimates we arrive at

$$\int_{I_2} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt \ll (\delta_0(x))^{1/5} \log x. \tag{31}$$

Last of all we deal with the intervals of I_3. Again using Parseval's formula (2) we get

$$\begin{aligned}\int_{I_3} \frac{|F_a(s) - A\zeta(s)|^2}{|s|^2} dt &\ll \\ &\ll \sum_{m\ge K} \frac{1}{m^2} \int_{|t-m|\le 1} (|F_a(s)|^2 + |\zeta(s)|^2) dt = \\ &= \sum_{m\ge K} \frac{1}{m^2} \int_{-1}^{+1} \frac{\left|\sum_{n=1}^{\infty} \frac{f(n)}{n^{ia}} n^{imt} n^{-s}\right|^2 + \left|\sum_{n=1}^{\infty} n^{imt} n^{-s}\right|^2}{|s|^2} dt \ll \\ &\ll \frac{1}{K(\sigma-1)} = (\delta_0(x))^{4/5} \log x.\end{aligned} \tag{32}$$

Since

$$\frac{1}{\log x} \sum_{p\le x} \frac{|f(p)p^{-ia} - 1|}{p} \log p \ll \left(\frac{1}{\log x} \sum_{p\le x} \frac{|f(p)p^{-ia} - 1|^2}{p} \log p\right)^{1/2}$$

and

$$\frac{1}{\log x}\sum_{p\le x}\frac{|f(p)p^{-ia}-1|^2}{p}\log p \ll \delta_0(x)$$

and by (30), (31) and (32) the proof of Theorem 5 is completed.

References

[1] **Daboussi H. and Indlekofer K.-H.,** Two elementary proofs of Halász's theorem, *Math. Zeitschrift,* **209** (1992), 43-52.

[2] **Halász G.,** Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta. Math. Acad. Sci. Hung.,* **19** (1968), 365-403.

[3] **Hardy G.H. and Wright E.M.,** *Introduction to the theory of numbers,* Oxford University Press, Oxford, 1960.

[4] **Indlekofer K.-H.,** A mean value theorem for multiplicative functions, *Math. Zeitschrift,* **172** (1980), 255-271.

[5] **Indlekofer K.-H., Kátai I. and Wagner R.,** A comparative result for multiplicative functions, *Liet. Matem. Rink.,* **41** (2) (2001), 183-201.

[6] **Indlekofer K.-H.,** On an elementary proof of the prime number theorem (submitted)

[7] **Indlekofer K.-H.,** On the prime number theorem, *Annales Univ. Sci. Budapest. Sect. Comp.,* **27** (2007), 167-185.

[8] **Indlekofer K.-H.,** Remarks on an elementary proof of Halász's theorem, *Liet. Matem. Rink.,* **33** (4) (1993), 417-423.

[9] **Selberg A.,** An elementary proof of the prime number theorem, *Annals of Mathematics,* **50** (1949), 305-313.

[10] **Shapiro H.N.,** *Introduction to the theory of numbers,* J. Wiley and Sons, New York, 1983.

K.-H. Indlekofer
Institute of Mathematics
University of Paderborn
Warburger Str. 100
D-33098 Paderborn, Germany
k-heinz@math.uni-paderborn.de