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EXPONENTIAL SUM ON ALGEBRAIC VARIETY

$$\frac{\alpha}{f_1(x,y)}+\frac{\beta}{f_2(X,Y)}=1$$

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. Let $f_i(x,y)\in\mathbb{F}_q[x,y]$, $i=1,2$, be quadratic polynomials. We obtain nontrivial estimates for exponential sums on the algebraic variety $\alpha f_1^{-1}(x,y)+\beta f_2^{-1}(X,Y)=1$, where $\alpha,\beta\in\mathbb{F}_q$.

1. Introduction

Let $\mathbb{F}_q$, $q=p^r$ be a finite field and let $f(x,y)$, $f_1(x,y)$, $f_2(x,y)$ be quadratic polynomials over $\mathbb{F}_q$. For α, $\beta\in\mathbb{F}_q$ we define the algebraic variety

$$V(\alpha,\beta)=\left\{(x,y,X,Y)\in\mathbb{F}_q^4 \;\Big|\; \frac{\alpha}{f_1(x,y)}+\frac{\beta}{f_2(X,Y)}=1\right\}.$$

Let χ be an additive character of field $\mathbb{F}_q$, $r,s\in\mathbb{F}_q$ with the condition $r\neq 0$ or $s\neq 0$. Consider the exponential sum

$$S(\alpha,\beta)=\sum_{(x,y,X,Y)\in V(\alpha,\beta)}\chi(rx+sy+rX+sY).$$

Let $\mathbb{F}_{q^n}$ be an extension of the field $\mathbb{F}_q$ of degree n. For $x\in\mathbb{F}_{q^n}$ we put

$$Tr(x)=x+x^q+\cdots+x^{q^{n-1}},\quad Tr(x)\in\mathbb{F}_q.$$

We denote by χ_n an extension of a character χ in the field $\mathbb{F}_{q^n}$, i.e. for every $x \in \mathbb{F}_{q^n}$ $\chi_n(x) := \chi(Tr(x))$.

We define the algebraic variety

$$V_n(\alpha,\beta) = \left\{ (x,y,X,Y) \in \mathbb{F}_{q^n}^4 \;\middle|\; \frac{\alpha}{f_1(x,y)} + \frac{\beta}{f_2(X,Y)} = 1 \right\},$$

$$S_n(\alpha,\beta) = \sum_{(x,y,X,Y)\in V_n(\alpha,\beta)} \chi_n(rx + sy + rX + sY).$$

Consider the function

$$\zeta(V(\alpha,\beta),t) = \exp\left(\sum_{n=1}^{\infty} s_n(\alpha,\beta)\frac{t^n}{n} \right).$$

From the paper of B. Dwork [1] it follows that $\zeta(V(\alpha,\beta),t)$ is a rational function $\frac{h(t)}{g(t)}$, where $h(t)$, $g(t)$ relatively prime polynomials are with complex coefficients. We denote by $\omega_1^{-1},\dots,\omega_\ell^{-1}$ and $\omega_{\ell+1}^{-1},\dots,\omega_k^{-1}$ the roots of $g(t)$ and $h(t)$ (respectively). Moreover, the following equality

$$S_n(\alpha,\beta) = \omega_1^n + \dots + \omega_\ell^n - \omega_{\ell+1}^n - \dots - \omega_k^n, \qquad n = 1,2,\dots$$

holds. The complex numbers $\omega_1,\dots,\omega_k$ are called the characteristic roots of the sum $S(\alpha,\beta)$.

The aim of this paper to construct an estimate for $S(\alpha,\beta)$.

B. Birch and E. Bombieri [1] obtained the estimate $S(\alpha,\beta) \ll q^{\frac{3}{2}}$ in the case $f_1(x,y) = f_2(x,y) = xy$. This permitted to obtain the asymptotic formulae for the summatory function for $\tau_3(n)$ in an arithmetic progression (see [7], [9]). Gunyavy [8] investigated the distribution of values of the function

$$\widetilde{\tau}_3(n) = \sum_{n=(u^2+v^2)\omega} 1$$

in an arithmetic progression using the estimate $S(\alpha,\beta) \ll q^{\frac{3}{2}}$ in the case $f_1(x,y) = f_2(x,y) = x^2 + y^2$.

In the sequel we shall use the following notation:

$$f(x,y) = c_{11}x^2 + 2c_{12}xy + c_{22}y^2 + 2c_{13}x - 2c_{23}y + c_{33}, \;\; c_{ij} \in \mathbb{F}_q;$$

$$\omega = c_{11}s^2 - 2c_{12}sr + c_{22}r^2, \quad \delta = \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix}, \quad \Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{vmatrix}.$$

The polynomials $f_1(x,y)$, $f_2(x,y)$ are defined similarly, their coefficients denoted by a_{ij}, b_{ij} and parameters by ω_1, δ_1, Δ_1 and ω_2, δ_2, Δ_2, respectively. We shall suppose that even one of coefficients c_{11}, c_{12} or c_{22} differs from 0.

We define for every $c \in \mathbb{F}_q$

$$K_f(c) := \sum_{\substack{x,y\in\mathbb{F}_q \\ f(x,y)=c}} \chi(rx+sy).$$

We shall distinguish seven cases:

1. $\delta \neq 0,\ \omega \neq 0,\ \Delta = 0;$
2. $\delta \neq 0,\ \omega \neq 0,\ \Delta \neq 0;$
3. $\delta \neq 0,\ \omega = 0;$
4. $\delta = 0,\ \omega \neq 0,\ \Delta = 0;$
5. $\delta = \omega = \Delta = 0;$
6. $\delta = 0,\ \omega = 0,\ \Delta \neq 0;$
7. $\delta = 0,\ \omega \neq 0,\ \Delta \neq 0.$

Further if in the variety $V(\alpha,\beta)$ the polynomial $f_1(x,y)$ belongs to a case i) and the polynomial $f_2(x,y)$ belongs to a case j), then we denote this case by (i,j). Furthermore, $(i,*)$ is the union of the cases (i,j), $j = 1,\dots,7$.

The following statement is the main result of this paper.

Theorem. *There exist absolute constants c_0 and c_1 such that for $p > c_0$ the following estimates*

$$S(\alpha,\beta) \ll \begin{cases} q\sqrt{q} & \text{for the cases } (7,1),(7,2),(1,1),(2,2),(7,7); \\ q^2 & \text{for the cases } (7,5),(5,1),(5,2); \\ q^2\sqrt{q} & \text{for the case } (5,5) \end{cases}$$

hold. Moreover,

$$S(\alpha,\beta) = 0 \quad \textit{for the cases } (4,*),(6,*);$$

and for the cases $(3,*)$

$$S(\alpha,\beta) = \begin{cases} \chi(ra_1+sa_2)\cdot\left(K_{f_2}(0)+K_{f_2}(\beta)\right) & \textit{if}\ \alpha = \dfrac{\Delta_1}{\delta_1}; \\ \chi(ra_1+sa_2)\left(K_{f_2}(0)+K_{f_2}(\beta)+qK_{f_2}\left(\frac{\beta\Delta_1}{\Delta_1-\alpha\delta_1}\right)\right) & \textit{if}\ \alpha \neq \dfrac{\Delta_1}{\delta_1}. \end{cases}$$

2. Some lemmas

Lemma 1. (Deligne [4], [5]) *For the characteristic roots ω_j we have the equality*

$$|\omega_j| = q^{\frac{m_j}{2}}, \quad m_j \in \mathbb{N}\cup\{0\}, \quad j=1,\ldots,k.$$

Moreover, all conjugates with ω_j over $\mathbb{Q}$ have equal modules (number m_j is called the weight of root ω_j).

Lemma 2. (Bombieri [2]) *Let $f(x,y)$ be an absolutely irreducible polynomial over $\mathbb{F}_p$. Then*

$$\sum_{\substack{x,y\in\mathbb{F}_{q^n} \\ f(x,y)=0}} \chi_n(rx+sy) \ll q^{\frac{n}{n}}.$$

Lemma 3. *Let $f(x,y,z)\in\mathbb{F}_q[x,y,z]$ and let V be an algebraic variety, defined by the polynomial f. Suppose that for $\alpha,\beta,\gamma\in\mathbb{F}_p$ and for all $\tau\in\overline{\mathbb{F}}_p$ except for $O(1)$ values of them, the polynomial*

$$\varphi_\tau(x,y) = f(x,y,\tau\gamma^{-1}-\alpha\gamma^{-1}x-\beta\gamma^{-1}y)$$

is absolutely irreducible over $\mathbb{F}_p$. Then

$$\sum_{(x,y,z)\in V\cap\mathbb{F}_p^3} e^{2\pi i\frac{Tr(\alpha x+\beta y+\gamma z)}{p}} \ll q.$$

Proof. We shall follow the scheme of C. Hooley [10].

Let us consider

$$M(\alpha,\beta,\gamma) = \sum_{\mu\in\mathbb{F}_q^*} |S(\mu\alpha,\mu\beta,\mu\gamma)|^2 = \sum_{\mu\in\mathbb{F}_q^*}\left|\sum_{(x,y,z)\in V} e^{2\pi i\frac{Tr(\mu\alpha x+\mu\beta y+\mu\gamma z)}{p}}\right|^2$$

Let $N(\tau)$ be the number of solutions of the system of the equations

$$f(\xi,\eta,\zeta)=0,\ \ \alpha\xi+\beta\eta+\gamma\zeta=\tau\ \ (\tau\in\mathbb{F}_q).$$

Put

$$\overline{N}=\frac{1}{q}\sum_{\tau\in\mathbb{F}_q}N(\tau).$$

Then, we have (since $\sum_{\tau\in\mathbb{F}_q}\chi(\tau)=0$)

$$S(\mu\alpha,\mu\beta,\mu\gamma)=\sum_{\tau\in\mathbb{F}_q}N(\tau)\chi(\mu\tau)=\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})\chi(\mu\tau).$$

Hence,

$$M=M(\alpha,\beta,\gamma)=\sum_{\mu\in\mathbb{F}_q^*}\sum_{\tau_1,\tau_2}(N(\tau_1)-\overline{N})(N(\tau_2)-\overline{N})e^{2\pi i\frac{Tr(\mu(\tau_1-\tau_2))}{p}}=$$

$$=(q-1)\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})^2+\sum_{\tau_1\neq\tau_2}(N(\tau_1)-\overline{N})(N(\tau_2)-\overline{N})\sum_{\mu\in\mathbb{F}_q^*}e^{2\pi i\frac{Tr(\mu)}{p}}=$$

$$=q\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})^2-\sum_{\tau_1,\tau_2\in\mathbb{F}_q}(N(\tau_1)-\overline{N})(N(\tau_2)-\overline{N})=$$

$$=q\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})^2-\left(\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})\right)^2=q\sum_{\tau\in\mathbb{F}_q}(N(\tau)-\overline{N})^2=$$

$$=q\sum_{\tau\in\mathbb{F}_q}(N(\tau)-q)^2-q^2(\overline{N}-q)^2\leq q\sum_{\tau\in\mathbb{F}_q}(N(\tau)-q)^2.$$

It is clear that $N(\tau)$ is the number of solutions of the equation

$$f(\xi,\eta,(\tau-\alpha\xi-\beta\eta)\gamma^{-1})=0$$

or

$$\varphi_\tau(x,y)=0.$$

Now, using the Weil's estimate for the number of points on an algebraic curve over $\mathbb{F}_q$ defined by an absolutely irreducible polynomial, we obtain

$$(1)\qquad M=q\sum_{\substack{\tau\in\mathbb{F}_q\\ \varphi_\tau(x,y)\text{ is abs. irreducibility}}}\left(O(q^{\frac{1}{2}})\right)^2+qO(1)\cdot O(q^2)=O(q^3).$$

Further, by Lemma 1

$$S(\mu\alpha,\mu\beta,\mu\gamma)=\omega_{1,\mu}^{r}+\cdots-\omega_{\ell+1,\mu}^{r}-\cdots-\omega_{k,\mu}^{r},\quad (\mu\in\mathbb{F}_q^*)$$

and $|\omega_{j,\mu}|$ does not depend on μ. Let

$$p^{\frac{N}{2}}=\max_{1\le j\le k}|\omega_{j,\mu}|,\quad (N\ge 0).$$

If $N\le 2$ then we have $S(\alpha,\beta,\gamma)\ll q$. Thus we suppose that $N\ge 3$. Let k_0 be the number of ω_j, $j=1,\dots,k$, for which $|\omega_j|=q^{\frac{N}{2}}$. Then we have

$$S(\mu\alpha,\mu\beta,\mu\gamma)=e_1\omega_{1,\mu}^{r}+\cdots+e_{k_0}\omega_{k_0,\mu}^{r}+O\left(q^{\frac{N-1}{2}}\right),$$

where $|\omega_{j,\mu}^r|=q^{\frac{N}{2}}$, $\omega_{j_1,\mu}\ne\pm\omega_{j_2,\mu}$ for $j_1\ne j_2$, and $e_1,\dots,e_{k_0}$ are integers, $|e_0|+\cdots+|e_{k_0}|>0$. Hence

$$S(\mu\alpha,\mu\beta,\mu\gamma)=q^{\frac{N}{2}}(e_1z_{1,\mu}+\cdots+e_{k_0}z_{k_0,\mu})+O\left(q^{\frac{N-1}{2}}\right),\tag{2}$$

where $z_{j,\mu}$ are complex numbers, $|z_{j,\mu}|=1$, $z_{j_1,\mu}\ne\pm z_{j_2,\mu}$ for $j_1\ne j_2$.

Now, from (2) we obtain

$$q^{-N}M(\alpha,\beta,\gamma)=q^{-N}\sum_{\mu\in\mathbb{F}_q^*}|S(\mu\alpha,\mu\beta,\mu\gamma)|^2\ge q^{-N}\sum_{\mu\in\mathbb{F}_p^*}|S(\mu\alpha,\mu\beta,\mu\gamma)|^2=$$

$$=\sum_{\mu\in\mathbb{F}_p^*}\left|e_1z_{1,\mu}^r+\cdots+e_{k_0}z_{k_0,\mu}^r\right|+O(p^{1-\frac{r}{2}}).$$

Applying the Bombieri-Davenport lemma [3], we obtain

$$\sum_{\mu\in\mathbb{F}_p^*}\frac{1}{R}\sum_{\substack{r<2R\\ r\equiv 1 (\mathrm{mod}\ 2)}}\left|e_1z_{1,\mu}^r+\cdots+e_{k_0}z_{k_0,\mu}^r\right|^2=O(1)+O\left(\frac{\sqrt{p}}{R}\right),$$

and, hence,

$$(p-1)(e_1^2+\cdots+e_{k_0}^2)=\sum_{\mu\in\mathbb{F}_p^*}\lim_{\substack{r\to\infty\\ r\text{ is odd}}}\left|e_1z_{1,\mu}^r+\cdots+e_{k_0}z_{k_0,\mu}^r\right|^2=$$

$$=\sum_{\mu\in\mathbb{F}_p^*}\overline{\lim_{R\to\infty}}\frac{1}{R}\sum_{\substack{r\le 2R\\ r\text{ is odd}}}\left|e_1z_{1,\mu}^r+\cdots+e_{k_0}z_{k_0,\mu}^r\right|^2=O(1).$$

But the equality $(p-1)(e_1^2+\cdots+e_{k_0}^2)=O(1)$ means that $p=O(1)$. Hence, there exist $c_0>0$ such that for $p\geq c_0$ a greatest module of characteristic roots $|\omega_j|\leq p$. Consequently,

$$S(\alpha,\beta,\gamma)\ll q.$$

Suppose that $r,s,c\in\mathbb{F}_q$, $f(x,y)\in\mathbb{F}_q[x,y]$, $\deg f(x,y)=2$, and that χ is a character of the field $\mathbb{F}_q$. We define

$$K_f(c)=K_f(r,s;c):=\sum_{\substack{x,y\in\mathbb{F}_q\\ f(x,y)=c}}\chi(rx+sy).$$

Lemma 4. *If $(r,s)\in\mathbb{F}_q^2\backslash\{0,0\}$, then we have*

$$K_f(c)=\begin{cases}\ll\sqrt{q} & \textit{in the cases 1),2) or 7);}\\ \ll q & \textit{in the case 5);}\\ 0 & \textit{in the cases 4),6);}\\ q-1 & \textit{in the case } \delta\neq0,\ \omega=0,\ c=\dfrac{\Delta}{\delta};\\ -1 & \textit{in the case } \delta\neq0,\ \omega=0,\ c\neq\dfrac{\Delta}{\delta}.\end{cases}$$

Proof. We can suppose that $s\neq0$. Then

$$K_f(c)=\sum_{\tau\in\mathbb{F}_q}\ \sum_{f\left(x,\frac{\tau-rx}{s}\right)=c}\chi(\tau)=\sum_{\tau\in\mathbb{F}_q}N(\tau)\chi(\tau),$$

where $N(\tau)$ is the number of solutions of the equation $f\left(x,\frac{\tau-rx}{s}\right)=0$ over $\mathbb{F}_q$:

$$f\left(x,\frac{\tau-rx}{s}\right)=$$

$$\frac{1}{s^2}\left[\omega x^2+2(\tau(a_{12}s-a_{22}r)+s(a_{13}s-a_{23}r))x+(a_{22}\tau^2+2a_{23}s\tau+a_{33}s^2)\right]=$$

$$=c. \tag{3}$$

For the discriminant D of the equation (3) we have

$$\frac{S^4D}{4}=\tau^2\left[(a_{12}s-a_{22}r)^2-\omega a_{22}\right]+2\tau s\left[(a_{12}s-a_{22}r)(a_{13}s-a_{23}r)-\omega a_{23}\right]+$$

$$+s^2\left[(a_{13}s-a_{23}r)^2-\omega a_{33}\right]+\omega cs^2.$$

If the curve $f(x,y)=0$ has center (a,b) then

$$K_f(c)=\chi(ra+sb)\sum_{f(x+a,y+b)=c}\chi(rx+sy).$$

Denote

$$F(x,y)=f(x+a,y+b)=a_{11}x^2+2a_{12}xy+a_{22}y^2+a'_{33},\quad a'_{33}=f(a,b).$$

Then we have

$$K_f(c)=\chi(ra+sb)\sum_{\tau\in\mathbb{F}_q}N_1(\tau)\chi(\tau),$$

where $N_1(\tau)$ is the number of solutions of the equation

$$(4)\qquad F\left(x,\frac{\tau-rx}{s}\right)=\frac{1}{s^2}\left[\omega x^2+2\tau(a_{12}s-a_{22}r)x+(a_{22}\tau^2+a'_{33}s^2)\right]=c.$$

If $\omega\neq 0$, for a discriminant D of the quadratic equation (4) we have

$$\frac{s^2D}{4}=\omega(c-a'_{33})-\delta\tau^2,$$

and hence,

$$(5)\qquad N(\tau)=1+\eta(D)=1+\eta(\omega(c-a'_{33})-\delta\tau^2).$$

Now we consider various cases:

1. $\delta\neq 0,\ \omega\neq 0,\ \Delta=0$.
   In this case

$$a=\frac{1}{\delta}(a_{13}a_{22}-a_{12}a_{23}),\quad b=\frac{1}{\delta}(a_{11}a_{23}-a_{13}a_{12}),\quad a'_{33}=f(a,b)=\frac{\Delta}{\delta}=0.$$

Thus by (1), (5)

$$K_f(c)=\chi(ra+sb)\sum_{\tau\in\mathbb{F}_q}\eta(\omega c-\delta\tau^2)\chi(\tau)=$$
$$=\eta(-\delta)\chi(ra+sb)\sum_{xy=c}\chi\left(x+\frac{\omega}{4\delta}y\right)\ll\sqrt{q}.$$

2. $\delta\neq 0,\ \omega\neq 0,\ \Delta\neq 0$.

From (1), (5) we obtain

$$K_f(c) = \chi(ar+bs)\sum_{\tau\in\mathbb{F}_q}\eta\left(\omega\left(c-\frac{\Delta}{\delta}\right)-\delta\tau^2\right)\chi(\tau) =$$
$$= \eta(-\delta)\chi(ar+bs)\sum_{xy=c-\frac{\Delta}{\delta}}\chi\left(x+\frac{\omega}{4\delta}y\right) \ll \sqrt{q}.$$

3. $\delta \neq 0$, $\omega = 0$.
In this case the equation (4) has form

$$s^2F\left(x, \frac{\tau - rx}{s}\right) = 2\tau(a_{12}s - a_{22}r) + (a_{22}\tau^2 + a'_{33}s^2) = cs^2.$$

Hence,

$$N_1(\tau) = \begin{cases} 1 & \text{if } \tau \neq 0, \\ q & \text{if } \tau = 0,\ c = \dfrac{\Delta}{\delta}, \\ 0 & \text{if } \tau = 0,\ c \neq \dfrac{\Delta}{\delta}, \end{cases}$$

$$K_f(c) = \chi(ra+sb)\cdot\begin{cases} q-1, & \text{if } c = \dfrac{\Delta}{\delta}, \\ -1, & \text{if } c \neq \dfrac{\Delta}{\delta}. \end{cases}$$

4. $\delta = 0$, $\omega \neq 0$, $\Delta = 0$.
Then $\frac{s^2D}{4} = \omega(c - a'_{33})$, i.e. D is independent on τ. Hence,

$$K_f(c) = \chi(ra+sb)\sum_{\tau\in\mathbb{F}_q}(1+\eta(D))\chi(\tau) = 0.$$

5. $\delta = \omega = \Delta = 0$.
The equation (3) has the form

$$\frac{1}{s^2}(a_{22}\tau^2 + 2a_{23}s\tau + a_{33}s^2) = c.$$

Hence,

$$N_1(\tau) = \begin{cases} q & \text{if } a_{22}\tau^2 + 2a_{23}s\tau + a'_{33}s^2 = cs^2, \\ 0 & \text{else.} \end{cases}$$

$$K_f(c) = q \sum_{\substack{\tau\in\mathbb{F}_q \\ a_{22}\tau^2+2a_{23}s\tau+a'_{33}s^2=cs^2}} \chi(\tau) = q\chi\left(-\frac{a_{23}}{a_{22}}s\right) \sum_{\substack{\tau\in\mathbb{F}_q \\ a_{22}\tau^2=s^2(c-a'_{33})}} \chi(\tau) \ll q$$

(here $a'_{33} = a_{33} - \frac{a_{23}^2}{a_{22}}$).

Now we consider the cases, when the curve $f(x,y) = 0$ is a noncentrical curve.

6. $\delta = 0$, $\omega = 0$, $\Delta \neq 0$.
   The equation (3) accepts the form

$$\frac{1}{s^2}\left[2s(a_{13}s - a_{23}r)x + (a_{22}\tau^2 + 2a_{23}s\tau + a'_{33}s^2)\right] = c.$$

Then $N_1(\tau) = 1$. Hence,

$$K_f(c) = \sum_{\tau\in\mathbb{F}_q} \chi(\tau) = 0 \qquad \text{for any} \quad c \in \mathbb{F}_q.$$

7. $\delta = 0$, $\omega \neq 0$, $\Delta \neq 0$.
   Denote

$$a = \frac{1}{s}\left[(a_{12}s - a_{22}r)(a_{13}s - a_{23}r) - \omega a_{23}\right], \quad b = (a_{13}s - a_{23}r)^2 - \omega a_{33}.$$

Then for a discriminant D of the equation (3) we have

$$\frac{s^2 D}{4} = 2\tau a + b + \omega c.$$

From $\delta = 0$, $\Delta = 0$ we easily infer that $a \neq 0$. Thus

$$K_f(c) = \sum_{\tau\in\mathbb{F}_q} (1 + \eta(D))\chi(\tau) = \sum_{\tau\in\mathbb{F}_q} \eta(2\tau a + b + \omega c)\chi(\tau) =$$
$$= \eta(2a)\chi\left(-\frac{b+\omega c}{2a}\right) \sum_{\tau\in\mathbb{F}_q} \eta(\tau)\chi(\tau) \ll \sqrt{q}.$$

3. Auxiliary sum

We consider the auxiliary sum

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(uc)}, \qquad u\in\mathbb{F}_q^*$$

(here $\overline{a}$ is the complex conjugate of a). We have

$$\sum_{c\in\mathbb{F}_q} K_f(c) = \sum_{x,y\in\mathbb{F}_q} \chi(rx+sy) = 0.$$

Let $\delta\neq 0$, $\omega\neq 0$, i.e. the conditions 1) or 2) from the Section 2 are carried out. We have

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(cu)} =$$

$$= \sum_{c\in\mathbb{F}_q}\sum_{\tau_1,\tau_2\in\mathbb{F}_q} \eta\left(c\omega-\frac{\omega\Delta}{\delta}-\delta\tau_1^2\right)\eta\left(uc\omega-\frac{\omega\Delta}{\delta}-\delta\tau_2^2\right)\chi(\tau_1)\chi(\tau_2) =$$

$$= \eta(-u)\sum_{\tau_1,\tau_2\in\mathbb{F}_q}\chi(\tau_1+\tau_2)\sum_{c\in\mathbb{F}_q}\eta\left(c-\left(\frac{\Delta}{\delta}+\frac{\delta\tau_1^2}{\omega}\right)\right)\eta\left(\frac{1}{u}\left(\frac{\Delta}{\delta}+\frac{\delta\tau_2^2}{\omega}\right)-c\right)$$

$$= \eta(-u)\sum_{\tau_1,\tau_2\in\mathbb{F}_q}\chi(\tau_1+\tau_2)\mathfrak{J}_{\frac{1}{u}(\frac{\Delta}{\delta}+\frac{\delta\tau_2^2}{\omega})-(\frac{\Delta}{\delta}+\frac{\delta\tau_1^2}{\omega})}(\eta,\eta) =$$

$$= \eta(-u)\sum_{\tau_1,\tau_2\in\mathbb{F}_q}\chi(\tau_1+\tau_2)+q\eta(u)\sum_{\substack{\tau_1,\tau_2\in\mathbb{F}_q\\ \delta^2\tau_2^2+\omega\Delta=u(\delta^2+\tau_1^2+\omega\Delta)}}\chi(\tau_1+\tau_2) =$$

$$= q\eta(u)\sum_{\substack{\tau_1,\tau_2\in\mathbb{F}_q\\ \delta^2\tau_2^2+\omega\Delta=u(\delta^2+\tau_1^2+\omega\Delta)}}\chi(\tau_1+\tau_2).$$

Hence, we make use the relations

$$\mathfrak{J}_a(\eta,\eta)=\mathfrak{J}_1(\eta,\eta)=-\eta(-1), \text{ if } a\in\mathbb{F}_q^*,$$

$$\mathfrak{J}_0(\eta,\eta)=\eta(-1)\cdot(q-1),$$

where $\mathfrak{J}_a(\eta,\eta)$ is the Jacobi sum

$$\mathfrak{J}_a(\eta,\eta)=\sum_{\substack{x,y\in\mathbb{F}_q\\ x+y=a}}\eta(x)\eta(y).$$

Let $N_2(\tau)$ be the number of solutions of the system

$$\tau_1 + \tau_2 = \tau, \qquad \delta_2\tau_2^2 + \omega\Delta = u(\delta^2\tau_1^2 + \omega\Delta).$$

Then

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(uc)} = q\eta(u)\sum_{c\in K} N_2(\tau)\chi(\tau).$$

If put $\tau_2 = \tau - \tau_1$, then we obtain that $N_2(\tau)$ equals the number of the solutions of the equation

$$\delta^2(u-1)\tau_1^2 + 2\delta^2\tau\tau_1 + \omega\Delta(u-1) - \delta^2\tau^2 = 0. \tag{6}$$

If $u = 1$, we have

$$N_2(\tau) = \begin{cases} 1 & \text{if } \tau \in \mathbb{F}_q, \\ q & \text{if } \tau = 0. \end{cases}$$

Hence, for $u = 1$ we obtain

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(uc)} = q(q-1).$$

If $u \neq 1$, we have for a discriminant of quadratic equation (6)

$$\frac{D_1}{4\delta^2} = u\delta^2\tau^2 - \omega\Delta(u-1)^2.$$

Hence,

$$N_2(\tau) = 1 + \eta(u\delta^2\tau^2 - \omega\Delta(u-1)^2),$$

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(uc)} = q\eta(u)\sum_{\tau\in\mathbb{F}_q} \eta(u\delta^2\tau^2 - \omega\Delta(u-1)^2)\chi(\tau).$$

Thus we infer

$$\sum_{c\in\mathbb{F}_q} K_f(c)\overline{K_f(cu)} = \begin{cases} q(q-1) & \text{if } u = 1, \\ -q & \text{if } u \neq 1,\ \Delta = 0, \\ q\widetilde{K}(u) \ll q^{\frac{3}{2}} & \text{if } u \neq 1,\ \Delta \neq 0, \end{cases}$$

where $\widetilde{K}(u) = \eta(u) \sum_{\tau\in\mathbb{F}_q} \eta(u\delta^2\tau^2 - \omega\Delta(u-1)^2)\chi(\tau)$.

Let $\delta = \omega = \Delta = 0$ (i.e. the conditions of (5) are carried out). As in the previous case we obtain

$$\sum_{c \in \mathbb{F}_q} K_f(c)\overline{K_f(cu)} = \begin{cases} q^2(q-1) & \text{if } u = 1, \\ -\eta(u)q^2 & \text{if } u \neq 1,\ a'_{33} = 0, \\ q^2 \widetilde{K}(u) & \text{if } u \neq 1,\ a'_{33} \neq 0, \end{cases}$$

where

$$a'_{33} = a_{33} - \frac{a_{23}^2}{a_{22}}, \qquad \widetilde{K}(u) = \sum_{\tau \in \mathbb{F}_q} \eta(ua_{22}^2\tau^2 - s^2 a_{22} a'_{33}(u-1)^2)\chi(\tau) \ll \sqrt{q}.$$

At last, let $\delta = 0$, $\omega \neq 0$, $\Delta \neq 0$. Then

$$K_f(c) = \eta(2a)\chi\left(-\frac{b + \omega c}{2a}\right) G(\eta, \chi),$$

where

$$a \neq 0, \quad b = (a_{13}s - a_{23}r)^2 - \omega a_{33}, \quad G(\eta, \chi) = \sum_{\tau \in \mathbb{F}_q} \eta(\tau)\chi(\tau)$$

is the Gauss sum. Hence,

$$\sum_{c \in \mathbb{F}_q} K_f(c)\overline{K_f(uc)} = \begin{cases} q^2 & \text{if } u = 1, \\ 0 & \text{if } u \neq 1. \end{cases} \tag{7}$$

4. Proof of Theorem

Let $\alpha, \beta \in \mathbb{F}_q^*$. Define the algebraic variety

$$V(\alpha, \beta) = \left\{ (x, y, X, Y) \in \mathbb{F}_q^4 \;\middle|\; \frac{\alpha}{f_1(x,y)} + \frac{\beta}{f_2(X,Y)} = 1 \right\},$$

where $f_1(x,y)$, $f_2(x,y) \in \mathbb{F}_q[x,y]$, $\deg f_i = 2$, $i = 1,2$. Let χ be an additive character of the field $\mathbb{F}_q$, $(r,s) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$. Define

$$S(\alpha,\beta) = \sum_{x,y,X,Y \in V(\alpha,\beta)} \chi(rx + sy + rX + sY).$$

Let

$$L_j(c) = K_{f_j}(c) = \sum_{\substack{x,y \in \mathbb{F}_q \\ f_j(x,y)=c}} \chi(rx + sy) \qquad (j = 1,2).$$

Then

$$S(\alpha,\beta) = \sum_{u \in \mathbb{F}_q^* \setminus \{1\}} L_1\left(\frac{\alpha}{u}\right) L_2\left(\frac{\beta}{1-u}\right). \tag{8}$$

In order to prove the Theorem we consider the various pairs (i,j), $i,j = 1,\dots,7$, which correspond the various sets of the polynomials $f_1(x,y)$ and $f_2(x,y)$. Let, for example, $f_1(x,y)$ belong to the case 3), i.e. $\delta_1 \neq 0$, $\omega = 0$. Let (a_1,a_2) be the center of the curve $f_1(x,y) = 0$. Then

$$L_1(c) = \begin{cases} \chi(ra_1 + sa_2)(q-1) & \text{if } c = \dfrac{\Delta_1}{\delta_1}, \\ -\chi(ra_1 + sa_2) & \text{if } c \neq \dfrac{\Delta_1}{\delta_1}. \end{cases}$$

Hence, in view of (8)

$$S\left(\frac{\Delta_1}{\delta_1},\beta\right) = \chi(ra_1 + sa_2)(L_2(0) + L_2(\beta)), \tag{9}$$

and for $\alpha \neq \dfrac{\Delta_1}{\delta_1}$

$$S(\alpha,\beta) = \chi(ra_1 + sa_2)\left(L_2(0) + L_2(\beta) + qL_2\left(\frac{\beta\Delta_1}{\Delta_1 - \alpha\delta_1}\right)\right). \tag{10}$$

Hence, in this case an estimate of the sum $S(\alpha,\beta)$ depends from an estimate of $L_2(c)$, $c \in K$.

Further, for the cases 4) and 6) we have $L(c) = 0$, $c \in \mathbb{F}_q$, and, hence, in view of (8) we infer $S(\alpha,\beta) = 0$ for the pairs $(4,*)$ and $(6,*)$.

We now shall consider the combinations of the cases 1), 2), 5) and 7).

Case (7,7). Let $a, b \in \mathbb{F}_q^*$. We have

$$\sum_{u\in\mathbb{F}_q^*\setminus\{1\}} \chi\left(\frac{a}{u}\right)\chi\left(\frac{b}{1-u}\right) = \sum_{\tau\in\mathbb{F}_q} N(\tau)\chi(\tau),$$

where $N(\tau)$ is the number of the solutions of the equation

$$\frac{a}{u} + \frac{b}{1-u} = \tau.$$

But we have

$$\frac{a}{u} + \frac{b}{1-u} = \tau \Leftrightarrow \tau u^2 + (b - a - \tau)u + a = 0.$$

The last equation has the discriminant $D = (b - a - \tau)^2 - 4a\tau$.

a) $a = b$. Then $N(0) = 0$, $N(\tau) = 1 - \eta(\tau^2 - 4a\tau)$ for $\tau \neq 0$. Hence,

$$\sum_{u\in\mathbb{F}_q^*\setminus\{1\}} \chi\left(\frac{a}{u}\right)\chi\left(\frac{b}{1-u}\right) = -1 + \sum_{\tau\in\mathbb{F}_q} \eta(\tau^2 - 4a\tau)\chi(\tau). \tag{11}$$

b) $a \neq b$. Then $N(0) = 1$, $N(\tau) = 1 + \eta((b-a-\tau)^2 - 4a\tau)$ **for** $\tau \neq 0$, and then

$$\sum_{u\in\mathbb{F}_q^*\setminus\{1\}} \chi\left(\frac{a}{u}\right)\chi\left(\frac{b}{1-u}\right) = -1 + \sum_{\tau\in\mathbb{F}_q} \eta((b-a-\tau)^2 - 4a\tau)\chi(\tau). \tag{12}$$

The sums on the right hand side of the relations (11), (12) are the Kloosterman sums and, hence,

$$S(\alpha, \beta) \ll q\sqrt{q}.$$

Case (7,5). We have

$$\sum_{\alpha\in\mathbb{F}_q} |S(\alpha,\beta)|^2 =$$

$$= \sum_{\alpha\in\mathbb{F}_q} \sum_{u\in\mathbb{F}_q^*\setminus\{1\}} L_1\left(\frac{\alpha}{u}\right) L_2\left(\frac{\beta}{1-u}\right) \sum_{\nu\in\mathbb{F}_q^*\setminus\{1\}} \overline{L_1\left(\frac{\alpha}{\nu}\right) L_2\left(\frac{\beta}{1-\nu}\right)} =$$

(13)

$$= \sum_{u,\nu\in\mathbb{F}_q^*\setminus\{1\}} \overline{L_2\left(\frac{\beta}{1-\nu}\right)} L_2\left(\frac{\beta}{1-\nu}\right) \sum_{\alpha\in\mathbb{F}_q} L_1\left(\frac{\alpha}{u}\right) \overline{L_1\left(\frac{\alpha}{\nu}\right)} = \text{[in view of (8)]} =$$

$$= q^2 \sum_{u\neq 0,1} \left| L_2\left(\frac{\beta}{1-u}\right)\right|^2 = \text{[by (7)]} = q^2\left[q^2(q-1) - |L_2(0)|^2 - |L_2(\beta)|^2\right] \leq$$

$$\leq q^5 - q^4.$$

Hence, $|S(\alpha,\beta)| < q^{\frac{5}{2}}$.

For the case 5)

$$L_2(c) = q\chi\left(-\frac{b_{23}}{b_{22}}s\right) \sum_{\substack{\tau\in\mathbb{F}_q \\ b_{22}\tau^2 = s^2(c-b_{33}')}} \chi(\tau),$$

moreover,

$$\sum_{\substack{\tau\in\mathbb{F}_q \\ b_{22}\tau^2 = s^2(c-b_{33}')}} \chi(\tau) \in \mathbb{R}.$$

Hence, $\overline{L_2(c)} = \varepsilon L_2(c)$, where ε is a fixed number for any $c \in K$, $|\varepsilon| = 1$, $\chi(-a) = \overline{\chi(a)}$. Thus, from the representation of $L_1(c)$ (for the case 7)) and the relation (8) we infer

$$S(-\alpha,\beta) = \varepsilon'\overline{S(\alpha,\beta)}, \qquad |\varepsilon'| = 1. \tag{14}$$

Suppose that there exists $\alpha^0 \in \mathbb{F}_q$ such that $S(\alpha^0,\beta)$ has a characteristic root of weight 5. Then from Lemma 2 and the relation (13) it follows that for $\alpha \neq \alpha^0$ the sum $S(\alpha,\beta)$ has characteristic roots of weight < 5. But from (14) the sum $S(-\alpha^0,\beta)$ has such a root. Hence, for any $\alpha \in \mathbb{F}_q$ the sum $S(\alpha,\beta)$ has some characteristic roots of weight ≤ 4. Thus for any $\alpha,\beta \in \mathbb{F}_q^*$

$$S(\alpha,\beta) \ll q^2.$$

Cases (7,1) and (7,2). For the cases 1) and 2) (i.e. $\delta\omega \neq 0$) we have

$$L(c) = \chi(ra+sb)\sum_{\tau\in\mathbb{F}_q} \eta\left(\omega\left(c - \frac{\Delta}{\delta}\right) - \delta\tau^2\right)\chi(\tau),$$

$$\sum_{\tau\in\mathbb{F}_q}\eta\left(\omega\left(c-\frac{\Delta}{\delta}\right)-\delta\tau^2\right)\chi(\tau)\in\mathbb{R}.$$

Hence, $\overline{L_2(c)}=\varepsilon L_1(c)$, where ε is a fixed number for any $c\in\mathbb{F}_q$, $|\varepsilon|=1$. Thus, using 6) and 8) we similarly have

$$S(\alpha,\beta)\ll q\sqrt{q}\quad\text{for any}\quad \alpha,\beta\in\mathbb{F}_q^*.$$

Cases (1,1) and (2,2). The algebraic variety from the paper Birch, Bombieri [1] belongs to this case. Moreover, repeating the proof in [1] almost word for word, we obtain

$$S(\alpha,\beta)\ll q\sqrt{q}.$$

Case (5,5). In this case we can take the polynomials $f_1(x,y)$ and $f_2(x,y)$ in the form

$$f_1(x,y)=(rx+sy)^2+a,\qquad f_2(x,y)=(rx+sy)^2+b.$$

Then we have

$$S(\alpha,\beta)=\sum_{(x,y)\in V(\alpha,\beta)}\chi(rx+sy+rX+sY)=q^2\sum_{\substack{x,y\in\mathbb{F}_q\\ \frac{\alpha}{x^2+a}+\frac{\beta}{y^2+b}=1}}\chi(x+y)=$$

$$=q^2\sum_{\alpha(y^2+b)+\beta(x^2+a)=(x^2+a)(y^2+b)}\chi(x+y)-q^2\sum_{x^2+a=0}\chi(x)\sum_{y^2+b=0}\chi(y)=$$

$$=q^2\left(\Sigma_1-\Sigma_2\right),$$

say. By Lemma 3 we have $\sum_1\ll\sqrt{q}$. The second sum is $O(1)$. Thus $S(\alpha,\beta)\ll$ $\ll q^2\sqrt{q}$.

Cases (1,5) and (2,5). In this case we can take

$$f_1(x,y)=f(x,y)=a_{11}x^2+2a_{12}xy+a_{22}y^2+a,$$

where $\delta=a_{11}a_{22}-a_{12}^2\neq 0$, $\omega=a_{11}s^2-2a_{12}sr+a_{22}r^2\neq 0$. Moreover, in the case (1,5) $a=0$, and in the case (2,5) $a\neq 0$, $f_2(x,y)=(rx+sy)^2+b$. Then

$$S(\alpha,\beta)=\sum_{V(\alpha,\beta)}\chi(rx+sy+rX+sY)=q\sum_{\frac{\alpha}{f(x,y)}+\frac{\beta}{g(z)}=1}\chi(rx+sy+z),$$

where $g(z) = z^2 + b$. Hence,

$$S(\alpha,\beta) = q \sum_{f(x,y)g(z)-\alpha g(z)-\beta f(x,y)=0} \chi(rx+sy+z) - L_1(0)L_2(0). \tag{15}$$

The last summand is $4q\sqrt{q}$. The first summand can be estimated by Lemma 3 with $f(x,y,z) = f(x,y)g(z) - \alpha g(z) - \beta f(x,y)$. In this case for any $\tau \in \mathbb{F}_q$ the polynomial $f_\tau(x,y)f(x,y,\tau - rx - sy)$ is absolutely irreducible over $\mathbb{F}_p$. In view of the relation (13) and Lemma 3 we conclude

$$S(\alpha,\beta) \ll q^2 \text{ for } \alpha,\beta \in \mathbb{F}_q^*.$$

Collecting together the estimates of $S(\alpha,\beta)$ we obtain the assertion of Theorem.

Remark 1. The case (1,2) remained without consideration.

Remark 2. Let $\varphi(x,y)$ be a quadratic form over $\mathbb{Z}$ and let

$$d(\varphi;n) = \#\left\{\varphi(u,v)\omega = n \mid u,v,\omega \in \mathbb{Z}\right\}.$$

Let $M(x,q;\varphi)$ (respectively, $\Delta(x,q;\varphi)$) be the main term (respectively, the error term) in an asymptotic formula

$$\sum_{\substack{n\equiv a \pmod q\\ n\le x}} d(\varphi;n) = M(x,q;\varphi) + \Delta(x,q;\varphi).$$

Then, applying the method of Heath-Brown [9] one can prove that

$$\Delta(x,q;\varphi) \ll \begin{cases} x^{\frac{86}{107}+\varepsilon} + q^{-\frac{66}{107}} & \text{if } \varphi(x,y) \text{ is a hyperbole,} \\[2ex] x^{\frac{26}{33}+\varepsilon} + q^{-\frac{58}{99}} & \text{if } \varphi(x,y) \text{ is an ellipse.} \end{cases}$$

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