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ON THE PROBLEM OF THE CHOICE OF THE RADIX

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Dedicated to Professor Imre Kátai
on the occasion of his 70th birthday

Abstract. By the Neumann-principle computers are based on the binary number system, that is they accept as well the data as the instructions as a series of binary digits. Technically this is the best choice and in some point of view this choice is nearly the best also mathematically. Supposing that the cost of the representation of a number is proportional to the product of the number of the digits of its expansion and of the radix of that expansion, and supposing that the number of the digits is a continuous function, the best choice for the radix is the base of the natural logarithm. The nearest integers to that real number are 2 and 3. In the following article we investigate which of these two radices is better and which numbers are, if there is any, exceptional. Instead of the continuous analysis we apply mainly discrete methods.

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1. Introduction

Given a radix r, where r is generally a positive integer greater than 1, all of the real numbers can be represented as a finite or infinite series of digits, that is by a series of nonnegative integers less than r. For the representation an infinite series is theoretical, so we will focus our attention on the finite series. The cost of displaying a number obviously depends on the number of the digits and the cost of representing the digits occuring in the given expansion. The

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simplest but fairly good assumption is that the cost is the product of the before-mentioned two factors. It is an acceptable assumption, too, that the cost of the representation of one position is proportional to the number of the possible characters, that is, to r, if we want to display all of the numbers of a domain containing more elements than r. For the sake of simplicity we will suppose that the numbers are nonnegative integers as the sign and the point separating the integer part and the fractional part of the number are of constant cost not influencing essentially our investigation. In the following we study the cost of the representation of the nonnegative integers less than a given positive integer as the function of the radix of the expansion.

Let N be a given positive integer and let us suppose that we have to display all of the nonnegative integers less than N in the base r number system with the positive integer r greater than 1. There is a uniquely determined integer t with the property of $r^{t-1} \leq N-1 < r^t$. The number of the positions necessary for representing all of the integers of the given domain is equal to t and $t = \lceil \log_r(N) \rceil$ as the values occuring in the previous inequalities are positive integers (supposing that N is at least 2), so $r^{t-1} \leq N-1 < r^t$ is equal to the inequalities of $r^{t-1} < N \leq r^t$. Now the cost of the representation is equal to $rt = r\lceil \log_r(N) \rceil$, so the minimal cost is achieved if r is equal to $\min\left\{\zeta \lceil \log_\zeta(N) \rceil \,|2 \leq \zeta \in \mathbf{N}\right\}$. The analysis is easy if we suppose that $\min\left\{\zeta \lceil \log_\zeta(N) \rceil \,|2 \leq \zeta \in \mathbf{N}\right\} \approx \min\left\{\zeta \log_\zeta(N) \,|1 < \zeta \in \mathbf{R}\right\}$ that is either $r = \lfloor \rho \rfloor$ or $r = \lceil \rho \rceil$ where ρ denotes $\min\left\{\zeta \log_\zeta(N) \,|1 < \zeta \in \mathbf{R}\right\}$. As we shall see this is almost always true but there are exceptions and in the other cases it is a question which of these two integers is the better radix, which gives the lower cost.

Now let us determine $\rho = \min\left\{\zeta \log_\zeta(N) \,|1 < \zeta \in \mathbf{R}\right\}$. As $\log_a(c) = \frac{\log_b(c)}{\log_b(a)}$, where a, b and c are positive real numbers and a and b are different from 1, so $\zeta \log_\zeta(N) = \frac{\zeta \ln(N)}{\ln(\zeta)}$, and ρ is the minimum of the function $f(\zeta) = \frac{\zeta \ln(N)}{\ln(\zeta)}$. This function is differentiable over the positive real numbers greater than 1 and $f'(\zeta) = \ln(N)\frac{\ln(\zeta)-1}{(\ln(\zeta))^2}$. As N and ζ are greater than 1, so neither $\ln(N)$ nor $\ln(\zeta)$ is equal to zero, and then $f'(\zeta) = 0$ if and only if $\ln(\zeta) - 1 = 0$, that is if $\ln(\zeta) = 1$, namely if $\zeta = e$, where e is the base of the natural logarithm. The denominator of the derivative is positive, and the logarithmic function is strictly increasing if the base is greater than 1, so $f(e)$ is the minimal value of f over the given domain. For 2 and 3 are the nearest integers to e, one of these two integers are the best choice as radix.

In the next part of this paper we point out that apart from finitely many N the optimal choice is $r = 3$, and with the choice of 2 as the radix we get only slightly worse result. In our investigation we apply almost everywhere discrete methods and we shall not refer to the results got from the above-mentioned continuous analysis. We present all of the N's special with the property that they can represented more economically by a radix different from 3 and 2.

2. Development

Theorem 1. *Let u and v be positive real numbers different from 1. If $\frac{u}{\ln(u)} < \frac{v}{\ln(v)}$, then there exists such a real number ζ_0 greater than 1, that $u \cdot \lceil \log_u(\zeta) \rceil < v \cdot \lceil \log_v(\zeta) \rceil$ for any $\zeta_0 \leq \zeta \in \mathbf{R}$.*

Proof. $\vartheta \leq \lceil \vartheta \rceil < \vartheta + 1$ for any real number ϑ, so if $u \cdot (\log_u(\zeta) + 1) \leq \leq v \cdot \log_v(\zeta)$, then

$$
\begin{aligned}
u \cdot \lceil \log_u(\zeta) \rceil &< u \cdot (\log_u(\zeta) + 1) \leq \\
&\leq v \cdot \log_v(\zeta) \leq v \cdot \lceil \log_v(\zeta) \rceil ,
\end{aligned}
\tag{1}
$$

and then also $u \cdot \lceil \log_u(\zeta) \rceil < v \cdot \lceil \log_v(\zeta) \rceil$ is fulfilled. But from $u \cdot (\log_u(\zeta) + 1) \leq \leq v \cdot \log_v(\zeta)$

$$
\begin{aligned}
u &\leq v \cdot \log_v(\zeta) - u \cdot \log_u(\zeta) = \\
&= v \cdot \frac{\ln(\zeta)}{\ln(v)} - u \cdot \frac{\ln(\zeta)}{\ln(u)} = \\
&= \ln(\zeta) \cdot \left(\frac{v}{\ln(v)} - \frac{u}{\ln(u)} \right).
\end{aligned}
\tag{2}
$$

The difference in the parentheses is positive by the condition given in the theorem, thus the inequality can be divided by it, that is

$$
\ln(\zeta) \geq \frac{u}{\frac{v}{\ln(v)} - \frac{u}{\ln(u)}}. \tag{3}
$$

As both the numerator and the denominator of the fraction are positive real numbers, the value of the fraction is positive, too. For $f : x \longmapsto e^x$ is a strictly increasing function, and $e^0 = 1$,

$$
\zeta \geq e^{\frac{u}{\frac{v}{\ln(v)} - \frac{u}{\ln(u)}}} > 1. \tag{4}
$$

Now it can be seen that

$$\zeta_0 = e^{\frac{u}{\frac{v}{\ln(v)} - \frac{u}{\ln(u)}}} \tag{5}$$

satisfies the condition stated in the theorem.

Theorem 2. *Let* $3 \leq s \in \mathbf{N}$. *Then* $\frac{s}{\ln(s)} < \frac{s+1}{\ln(s+1)}$.

Proof. By the condition given in the theorem s, and then $s+1$, too, is greater than 1, therefor $\ln(s) \cdot \ln(s+1)$ is positive, so multiplying the inequality of the theorem by this product we get a correct inequality. This means that instead of the original inequality we can study the truth of the relation of $s \cdot \ln(s+1) < (s+1) \cdot \ln(s)$. Applying the well-known identities of the logarithmic function

$$(s+1) \cdot \ln(s) - s \cdot \ln(s+1) = \ln\left(\frac{s^{s+1}}{(s+1)^s}\right), \tag{6}$$

and this value is positive if and only if the argument of the right hand side is greater than 1 that is if the nominator of the argument is greater than the denominator of the argument. Now we have to point out only that if s is an integer not less than 3 then $s^{s+1} > (s+1)^s$. By the binomial theorem

$$(s+1)^s = \sum_{k=0}^{s} \binom{s}{k} s^{s-k}. \tag{7}$$

If $s \geq 3$ and $2 \leq k \leq s$ then

$$\begin{aligned} \binom{s}{k} &= \frac{s \cdot (s-1) \cdot \dots \cdot (s-k+1)}{1 \cdot 2 \cdot \dots \cdot k} < \\ &< \frac{\overbrace{s \cdot s \cdot \dots \cdot s}^{k}}{2 \cdot \underbrace{1 \cdot \dots \cdot 1}_{k-1}} = \frac{s^k}{2}, \end{aligned} \tag{8}$$

so

$$(s+1)^s = \sum_{k=0}^{s} \binom{s}{k} s^{s-k} =$$

$$(9)\qquad \begin{aligned} &= s^s + s\cdot s^{s-1} + \sum_{k=2}^{s}\binom{s}{k}s^{s-k} < \\ &< 2s^s + \sum_{k=2}^{s}\frac{s^k}{2}s^{s-k} = 2s^s + \frac{1}{2}\sum_{k=2}^{s}s^s = \\ &= 2s^s + \frac{s-1}{2}s^s = \frac{s+3}{2}s^s \le \\ &\le \frac{s+s}{2}s^s = s\cdot s^s = s^{s+1}. \end{aligned}$$

Corollary 1. *The minimum of the function $\frac{s}{\ln(s)}$ restricted to the integers greater than 1 is at $s=3$. If $s\neq 3$ then $\frac{2}{\ln(2)}\le\frac{s}{\ln(s)}$, and the equality is held only at $s=4$.*

Proof. By Theorem 2 $\frac{3}{\ln(3)}<\frac{4}{\ln(4)}$, and again by the previous theorem if $\frac{3}{\ln(3)}<\frac{s}{\ln(s)}$ for $s\ge 4$ then

$$(10)\qquad \frac{3}{\ln(3)}<\frac{s}{\ln(s)}<\frac{s+1}{\ln(s+1)},$$

so the function takes its minimal value in the set of the integers not less than 3 at $s=3$. At the same time

$$(11)\qquad \begin{aligned} \frac{3}{\ln(3)} < \frac{4}{\ln(4)} &= \frac{2\cdot 2}{\ln(2^2)} = \\ &= \frac{2\cdot 2}{2\cdot\ln(2)} = \frac{2}{\ln(2)} \end{aligned}$$

that proves the first statement. It can be got similarly that $\frac{4}{\ln(4)}<\frac{s}{\ln(s)}$ for an arbitrary integers $4<s$, and this shows that the second statement is true, too.

Now let $u=3$, and let us determine the appropriate values of ζ_0 for the integers $v>3$. On the base of the results we got till now $\zeta_0^{(v)}$ is a strictly decreasing function of v, and the function is bounded from below, thus if $\zeta_0^{(v_0)}\le$ ≤ 3 for a v_0, then $\zeta_0^{(v)}<3$ for any integer v greater than v_0. But if $n+1\le 3$ then $n\le 2$, and this n can be written with only one digit both in its radix 3 expansion and in any number system where the base v is greater than 3, so the cost of displaying this number is surely less in the ternary number system, thus it is enough to expand our investigation for the integers less than or equal to the maximum of the v's with the property of $\zeta_0^{(v)}>3$. On the basis of these results we get the following values:

v	$=$	2	$\zeta_0 \approx$	265156957, 85
		4		265156957, 85
		5		2920, 87
		6		128, 35
		7		31, 88
		8		14, 69
		9		9, 00
		10		6, 43
		11		5, 03
		12		4, 18
		13		3, 61
		14		3, 21
		15		2, 91

For every radix less than 15 all of the numbers less than the ones indicated in the table above can be checked, whether the ternary or the radix-v expansion is cheaper. For the small values of the given v's it takes much time to execute this investigation, so we examine this problem in another way.

Unfortunately it does not follows from $u \cdot \log_u(n) < v \cdot \log_v(n)$ that $u \cdot \lceil \log_u(n) \rceil < v \cdot \lceil \log_v(n) \rceil$ or even the weaker form of $u \cdot \lceil \log_u(n) \rceil \leq v \cdot \lceil \log_v(n) \rceil$. For instance let $u = 3$, $v = 4$ and $n = 244$, then $\log_3(244) \approx 5,004$ and $\log_4(244) \approx 3,966$, and from these data we get that $3 \cdot \log_3(244) \approx 15,012 < 15,864 \approx 4 \cdot \log_4(244)$, while $3 \cdot \lceil \log_3(244) \rceil = 3 \cdot \lceil 5,004 \rceil = 3 \cdot 6 = 18 > 16 = 4 \cdot 4 = 4 \cdot \lceil 3,966 \rceil = 4 \cdot \lceil \log_4(244) \rceil$, that is, the expansion of 243 needs six digits in the ternary number system and four digits in the radix-4 system, and the cost of the presentation of the latter case is lower. However, we show that this is possible only in the cases when $v = 2$, $v = 4$ and $v = 5$ (and also in these cases only for a finite set of integers, as we saw this earlier).

Theorem 3. *Let u and $v > u$ be such positive integers, that $u \cdot \lceil \log_u(v) \rceil \leq v$. Then $u \cdot \lceil \log_u(n+1) \rceil \leq kv$ for any $k \in \mathbf{N}$ and for any integer $v^{k-1} \leq n < v^k$, and if furthermore $u \cdot \log_u(v)$ is less than v, then there is such an integer $m \in \mathbf{N}$, that $u \cdot \lceil \log_u(n+1) \rceil < kv$, if $k \geq m$.*

Proof. $\lceil a + b \rceil \leq \lceil a \rceil + \lceil b \rceil$ for any real numbers a and b. Applying this relationship we get that

$$
\begin{aligned}
& u \cdot \lceil \log_u(n+1) \rceil \leq u \cdot \lceil \log_u(v^k) \rceil = \\
(12) \qquad & = u \cdot \lceil k \cdot \log_u(v) \rceil \leq u \cdot (k \cdot \lceil \log_u(v) \rceil) = \\
& = k \cdot (u \cdot \lceil \log_u(v) \rceil) \leq k \cdot v.
\end{aligned}
$$

If both $u \cdot \log_u(v) < v$ and $u \cdot \lceil \log_u(v) \rceil \leq v$ are true then either $u \cdot \lceil \log_u(v) \rceil < v$, or $\log_u(v)$ is not an integer. In the first case we can exchange in (12) the last

$\leq$ by $<$, and $m = 1$ meets the requirements. But if $\log_u(v) < \lceil \log_u(v) \rceil$, then there exists such an $s \in \mathbf{N}$, that $s \cdot (\lceil \log_u(v) \rceil - \log_u(v)) \geq 1$, and from here

$$\begin{aligned} \lceil s \cdot \log_u(v) \rceil &= \lceil s \cdot \lceil \log_u(v) \rceil - s \cdot (\lceil \log_u(v) \rceil - \log_u(v)) \rceil \leq \\ &\leq \lceil s \cdot \lceil \log_u(v) \rceil - 1 \rceil = s \cdot \lceil \log_u(v) \rceil - 1 < s \cdot \lceil \log_u(v) \rceil \end{aligned} \tag{13}$$

thus if m is the least value of the former integers s, then in (13) instead of $\leq$ separating the two rows we can write $<$.

The significance of the theorem above is the following with respect to our investigation. An integer $v^{k-1} \leq n < v^k$ needs k digits in its radix-v expansion, so the cost of the representation of this integer is equal to kv in this expansion. The same number can be given by $\lceil \log_u(n+1) \rceil$ digits if the radix is equal to u, and then the cost is equal to $u \cdot \lceil \log_u(n+1) \rceil$. From this follows that if the inequality given in the theorem is fulfilled then the cost of displaying an arbitrary number with u different symbols is not greater than the cost when we can apply v different digits, that is the radix-v expansion is at least so expensive as the radix-u expansion.

Let $u \geq 2$ be an integer, and $\tau = \frac{u}{u-1} \leq 2$. Then

$$\begin{aligned} 1 = \log_u(u) &= \log_u(1 + (u-1)) = \\ &= \log_u\left(1 + \frac{u}{\frac{u}{u-1}}\right) = \log_u\left(1 + \frac{u}{\tau}\right) = \\ &= \log_u\left(\frac{\tau + u}{\tau}\right) = \log_u(\tau + u) - \log_u(\tau), \end{aligned} \tag{14}$$

and if $\rho > \tau$, then $0 < \log_u(\rho + u) - \log_u(\rho) < 1$, as the logarithmic function is strictly increasing and $\frac{u}{\rho} < \frac{u}{\tau}$. Now let $v > u \geq 2$ integer and let's suppose that $u \cdot \lceil \log_u(v) \rceil \leq v$. Then

$$\begin{aligned} u \cdot \lceil \log_u(v+u) \rceil &\leq u \cdot \lceil \log_u(v) + 1 \rceil = \\ &= u \cdot \lceil \log_u(v) \rceil + u \leq v + u, \end{aligned} \tag{15}$$

that is if $u \cdot \lceil \log_u(v) \rceil \leq v$ is true for a series of u consecutive integers $r \leq v < r + u$, where $r > u \geq 2$, then the inequality is true for all of the integers not less than r (and similar is true, if instead of the relation less than or equal the stricter relation less is held everywhere).

Looking up in tables or another appropriate places we get the following values:

v	$=$	4	$\log_3(v)$	$\approx$	1,262	$3 \cdot \lceil \log_3(v) \rceil$	$=$	6
		5			1,465			6
		6			1,631			6
		7			1,771			6
		8			1,893			6
		9		$=$	2			6

This table shows that $3 \cdot \lceil \log_3(v) \rceil \leq v$ for the three consecutive integers $v = 6$, $v = 7$ and $v = 8$, that is, the use of any radix greater than 5 is at least as expensive as the use of the ternary expansion, and if the radix is greater than or equal to 7 then the ternary system is definitely cheaper. $6 = 3 \cdot \lceil \log_3(6) \rceil$, but $3 \cdot \log_3(6) < 6$, the minimal value of m is

$$\left\lceil \frac{1}{\lceil \log_3(6) \rceil - \log_3(6)} \right\rceil = \left\lceil \frac{1}{2 - 1,631} \right\rceil =$$

$$= \left\lceil \frac{1}{0,369} \right\rceil = 3, \tag{16}$$

so at most among the integers less than 36 are those integers which can be represented with the same cost also in the base-6 number system as in the ternary number system. Representing the numbers from 0 to 35 in these number systems we can see that only the numbers $3 \leq n \leq 5$ and $27 \leq n \leq 35$ are of the same cost.

Now we investigate the cases of radix 5 and radix 6. For an arbitrary positive integer r greater than 2 and for every positive integer k it is true that the cost of the representation of each integer $r^{k-1} \leq n < r^k$ is equal to $k \cdot r$, so if the representation of r^{k-1} is more expensive or at least of the same cost in the base-r' number system then this statement is true for all of the aforementioned integers n. This means that we have only to check whether in the case of $r = 3$ there exists such a t that $3t > r' \left\lceil \log_{r'}\left(r^{t-1} + 1\right) \right\rceil$, where r' is equal to 4 or 5. If $t = 1$ then this is surely not possible, as in this case also in the base-r' number system is 1 digit required, and so the cost is $r' > r$. Now let us consider the case $t \geq 2$. $\lceil u + \varepsilon \rceil \geq u + \varepsilon > u$ for an arbitrary real number u and a positive real number ε, so

$$\left\lceil \log_{r'}\left(r^{t-1} + 1\right) \right\rceil > \log_{r'}\left(r^{t-1}\right) = \frac{t-1}{\log_3(r')}$$

and then the ternary number system is surely cheaper if $3t \leq r' \frac{t-1}{\log_3(r')}$, that is if $t \geq \frac{r'}{r' - 3\log_3(r')}$. We get that the potentially wrong values of t are less than 19 if $r' = 4$ and less than 9 if $r' = 5$. The given values of t are only possibly wrong so we have to check every pair (r', t) whether the condition $3t > r' \left\lceil \log_{r'}\left(3^{t-1} + 1\right) \right\rceil$ is held. By this examination only the cases of $r' = 4$ and $t = 2$, 3, 6, 7, 11, as well as $r' = 5$ and $t = 2$ are cheaper than the ternary number system for the same values of t. Now let us take into consideration that if $a_{t'-1} \dots a_0$ is the representation of 3^{t-1} in the base-r' number system and this representation is cheaper than the ternary one then this is true for all of the numbers $3^{t-1} \le n < r'^{t'} - 1$, as the costs of the representations of these numbers are the same in the radix-r' number system (the number of the digits does not change). Finally, if $r'^{t'} < 3^t$, then we have to investigate wether the cost of the representation of the number on the left hand side is lower than the cost of the representation of the right hand side, namely, in this case the costs of the representations of the numbers of the domain determined by the former inequality are lower, too, in the radix-r expansion. On the base of these results we get with some calculation that in the case of $r' = 4$ the greatest number which can be represented with lower cost in this number system than in the ternary one is 65535, and there are altogether 6803 such numbers, but if $r' = 5$, then the numbers 3 and 4 representable with only one digit in the quinary number system are of this property. The numbers exceptional in the base-4 number system can be seen in the following table:

t								
2					3_4	$=$	3	1
3	21_4	$=$	9	$-$	33_4	$=$	15	7
6	3303_4	$=$	243	$-$	3333_4	$=$	255	13
7	23121_4	$=$	729	$-$	33333_4	$=$	1023	295
11	32122221_4	$=$	59049	$-$	33333333_4	$=$	65535	6487

Let us see the binary number system. If a number is cheaper in the quaternary number system than in the ternary one then it is cheaper also in the binary one as the binary expansion can be got from the quaternary one exchanging every digit by a diagram of bits and omit the possible (only) leading zero, that is, the number of the binary digits is at most the double of the number of the digits of the quaternary representation and the cost of a digit with the greater radix is the double of the cost with the smaller one. At the same time it can be that there are such numbers represented by odd numbers of bits in their binary expansions which are cheaper in the base-2 number system than in the ternary number system, but more expensive in the radix-4 number system than in the ternary one (as in the quaternary case the binary equivalent contains a leading zero bit, too). The investigation is similar to the former one: if $3t > 2 \cdot \lceil \log_2 (3^{t-1} + 1) \rceil$ then the binary system is better than the ternary one (t is the number of the digits in the letter system, that is in the ternary case). Applying the well-known properties of the functions $y = \log_a(x)$ and $y = \lceil x \rceil$ we get that

$$
\begin{aligned}
& 2 \cdot \log_2 \left(3^{t-1}\right) < 2 \cdot \log_2 \left(3^{t-1} + 1\right) \le \\
& \le 2 \cdot \lceil \log_2 \left(3^{t-1} + 1\right) \rceil ,
\end{aligned}
\tag{17}
$$

so if $3t \leq 2 \cdot \log_2(3^{t-1})$, then it is surely true that $3t \leq 2 \cdot \lceil \log_2(3^{t-1}+1) \rceil$. From the previous expression we get that if $t \geq \frac{2}{2-3 \cdot \log_3(2)}$, that is, if $t \geq 19$, as t is an integer, then the representation of a number is more economic in the ternary system. Among the numbers of the interval $1 \leq t \leq 18$ the inequality $3t > 2 \lceil \log_2(3^{t-1}+1) \rceil$ is fulfilled in the case of $t = 1, \ldots, 7, 9$ and 11.

We give again the numbers more economic in the binary number system:

t			
1	$0_2 = 0$ –	$1_2 = 1$	2
2		$11_2 = 3$	1
3	$1001_2 = 9$ –	$1111_2 = 15$	7
4	$11011_2 = 27$ –	$11111_2 = 31$	5
5	$1010001_2 = 81$ –	$1111111_2 = 127$	47
6	$11110011_2 = 243$ –	$11111111_2 = 255$	13
7	$1011011001_2 = 729$ –	$1111111111_2 = 1023$	295
9	$1100110100001_2 = 6561$ –	$1111111111111_2 = 8191$	1631
11	$1110011010101001_2 = 59049$ –	$1111111111111111_2 = 65535$	6487
			8488

In the light of the previous results it is understandable why the computer apply the binary number system: economically it is almost ideal, only slightly worse than the cheapest ternary system (and in the domain of the most frequently occuring numbers there are quite a lot such numbers where the radix-2 number system is more economic), at the same time this is technically the simplest and most reliable radix, for in this case there are only two values to distinguish.

We know that $s \cdot \frac{\ln(N)}{\ln(s)}$ takes its minimal value at 3 over the whole domain of $2 \leq s \in \mathbf{N}$, but the deviation from this minimal value is not considerable at 2 and 3, as $\frac{3}{\ln(3)} / \frac{2}{\ln(2)} = \frac{\ln(8)}{\ln(9)} = 0,946$. It is worth examining which numbers can be represented more economically with radices different from 3 than in the binary system (the numbers more economically representable in the binary system than in the ternary one were already given earlier). We do not have to examine the quaternary system, too, as we saw that the numbers having an event numbers of digits in their binary expansion are of same cost in the binary and in the quaternary number systems, while the others are of lower cost in the case of using the binary system. All of the other cases are enumerated in the following table.

v	$\log_2(v)$						
5	2,322	$2 \cdot \log_2(v)$	$\approx$	4,644	$2 \cdot \lceil \log_2(v) \rceil$	$=$	6
6	2,585			5,170			6
7	2,807			5,615			6

From this table we can read out that we have to study only the case of $v = 5$. $\frac{5}{5-2\cdot\log_2(5)} \approx 14,040$, so we have to check the relation $2 \cdot t > 5 \cdot \lceil \log_5 \left(2^{t-1}+1\right) \rceil$ only over the interval $2 \leq t \leq 14$. The relation is fulfilled for the integers belonging to the given interval only in one case namely if $t = 3$. Really, $2^{3-1} = 4$, the binary expansion of 4 is 100_2, that is, the cost is now 6, but on the other hand $4 = 4_5$, and the cost of this expansion is only 5 (but from 5 on returns egain the normal case, for the binary cost of 5 is 6, and the cost of the quinary representation of the same number is 10).

3. Conclusion

Technically the binary number system is the most preferable for the computers, surely, and with the exception of a really insignificant number of machines, computers apply the binary principle. But the question is grounded, too, what integer is theoretically the optimal choice as the radix of a number system. For answering this question it is necessary to define, in what sense is optimal an actual choice. Such consideration can be for instance the minimum of the cost of the representation of the numbers. In this case we have to define a cost-function. A very simple but fairly reasonable cost-function is $f(n) = u \log_u(n)$. Now the cost of the representation of a number depends not only on the given number but on the base of the logarithm, too. By the help of a continuous analysis it is easy to point out that $u = e$ is the optimal choice as the base of the logarithm (where e is the base of the natural logarithm), but normally the radices are integers greater than 1, so we have to find the optimum among the integer bases. It is not surprising that in the majority of the cases 2 and 3 are the two best choices. By discrete methods we pointed out that if $u > 6$ then the ternary system is definitely better for every number then the base-u number system, if $u = 6$ then the base-3 system is at least as good as the base-u system and there are only 12 numbers of the same cost, two numbers can be represented more economically in the quinary number system than in the ternary one, and if the radix is equal to 2 or 4, then there are altogether 8488 exceptional cases, that is, when the ternary system is less economic. The analysis shows that with a few exceptional cases the ternary system is the best choice with respect to the given condition on the optimum. However, the technically best choice is hardly worse than the ternary one, and apart from the ternary system, the binary one is the best for all of the positive integers with only one exception.

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