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AVERAGE ORDERS OF DIVISOR FUNCTION OF INTEGER MATRICES IN $M_3(\mathbb{Z})$

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Dedicated to Professor Imre Kátai on his seventieth birthday

Abstract. Construct the asymptotic formula of summatory function for the number of divisors of matrices from $M_3(\mathbb{Z})$ with a determinant $\leq x$. Derive the estimate of the second moment of error term of this asymptotic formula.

1. Introduction

The interest to study the arithmetic functions over the ring of integer matrices connects with applications in theory of groups, theory of rings, in cryptography (see [1]-[4], [8]). The investigation of structure of abelian groups, in particular, the problems on the member of subgroups of certain types of groups, the average number of their formal direct factors and their formal unitary factors can be realized by the results on the distribution of the divisor function of integer matrices.

It is easy to determine a natural bijection between the set of subgroups of a finite abelian group G of rank r represented by the Smith canonical form

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \otimes ... \otimes \mathbb{Z}/n_r\mathbb{Z}, \quad n_j|n_{j+1}, \quad j = 1, ..., r-1,$$

and the set of divisors of an $r \times r$ matrix C for which a diagonal matrix $S(C) = = diag(n_1, ..., n_r)$ is a Smith Normal Form (see [8]). Let $\tau(G)$ denote the number subgroups of an abelian group G of rank r and let

$$t_r(n) = \sum_{|G|=n} \tau(G).$$

In the works G. Bhowmik [2], G. Bhowmik and I. Wu [3] obtained the asymptotic formula

$$T(x) = \sum_{n \le x} t_2(n) = xP_2(\log x) + \Delta(x), \tag{1}$$

where $P_2(u)$ is quadratic polynomial, $\Delta(x)$ is an error term for which in [3] the estimate $\Delta(x) \ll x^{5/8} \ln^4 x$ has been proved.

A. Ivič [6] obtained the "O" and "Ω" estimates for $\Delta(x)$ in mean square

$$\Omega(x^2 \ln^4 x) = \int_1^x \Delta^2(x) dx = O(x^2 (\log x)^{31/3}). \tag{2}$$

In our paper we investigate the divisor function $t_3(n)$. We proved the following theorems:

Theorem 1. *For $x \to \infty$ we have*

$$\sum_{n \le x} t_3(n) = xP_4(\log x) + O(x^{5/6}), \tag{3}$$

where $P_4(u)$ is a polynomial of degree 4.

Theorem 2. *Let $\Delta(x) = \sum_{n \le x} t_3(n) - xP_4(\log x)$, then for $x \to \infty$ the following estimate*

$$\int_1^x \Delta^2(x) dx \ll x^{51/20} \tag{4}$$

holds.

2. Notation and auxiliary lemmas

The aim of this section is to introduce some notations and to recall some known results, which we shall use later. The notations $f = O(g)$, $f \ll g$ mean that $|f| \leq cg$ with some positive constant c. We write $|C|$ for the determinant of the matrix C; $S(C)$ for Smith Normal Form of C, $H(C)$ for its Hermite Normal Form. $M_3(\mathbb{Z})$ denotes the ring of matrices over $\mathbb{Z}$ and $U(\mathbb{Z}) := \{U \in \in M_3(\mathbb{Z}) | |U| = \pm 1\}$. The letter s denotes a complex number, $s = \sigma + it$.

Let A_1, A_2 be two matrices from $M_3(\mathbb{Z})$. We say that the matrices A_1 and A_2 are associated on the left (accordingly, on the right) if $A_1 = UA_2$ (accordingly, $A_1 = A_2V$), $U, V \in U(\mathbb{Z})$. Through $\tau(C)$, $C \in M_3(\mathbb{Z})$ we denote the number of representations of the C in the form $C = AB$, $A, B \in M_3(\mathbb{Z})$, moreover, two representations $C = A_1B_1 = A_2B_2$ are considered as identical if A_1 and A_2 are associated on the right (B_1, B_2 are associated on the left).

Let $C = US(C)V$ and let $A|C$, i.e. $\exists B \in M_3(\mathbb{Z})$, $C = AB = US(C)V$, and, hence, $S(C) = U^{-1}ABV^{-1}$, i.e. $U^{-1}A|S(C)$. Clearly, the correspondence $A \leftrightarrow U^{-1}A$ realizes bijection between the set of left divisors of the matrix $S(C)$. Thus, $\tau(C) = \tau(S(C))$. Also clear, that if $A|S(C)$, then for any $U \in U(\mathbb{Z})$ follows $AU|S(C)$. Hence, $AU(\mathbb{Z})$ is the associated left divisor of matrix $S(C)$ generated by the divisor A. But $AU(\mathbb{Z})$ has single matrix $\hat{A}$ of type

$$\hat{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ a_{11}, \ a_{22}, \ a_{33} \in \mathbb{N}, \ 0 \leq a_{21} < a_{22}, \ 0 \leq a_{31}, a_{32} < a_{33}.$$

The matrix $\hat{A}$ is called the Hermite Normal Form of matrix A (denoted by $H(A)$). So, $\tau(S(C))$ is the number of matrices of type $\hat{A}$ such that $\hat{A}|S(C)$, i.e. $\hat{A}^{-1}S(C) \in M_3(\mathbb{Z})$. We have

$$\hat{A}^{-1}S(C) = \begin{pmatrix} \dfrac{c_1}{a_{11}} & 0 & 0 \\ -\dfrac{a_{21}c_1}{a_{11}a_{22}} & \dfrac{c_{22}}{a_{22}} & 0 \\ \dfrac{a_{21}a_{32} - a_{22}a_{31}}{a_{11}a_{22}a_{33}} & -\dfrac{a_{32}}{a_{22}a_{33}} & \dfrac{c_3}{a_{33}} \end{pmatrix}.$$

Now, we set $c_1 = a_1$, $c_2 = a_1a_2$, $c_3 = a_1a_2a_3$. Hence, $a_{11}|c_1$, $a_{22}|c_2$, $a_{33}|c_3$.

Denote $d_1 = \frac{a_1}{a_{11}}$, $d_2 = a_{22}$, $d_3 = a_{33}$. We infer that the matrix $\hat{A}^{-1}S(C) \in$ $\in M_3(\mathbb{Z})$ if and only if

$$a_{21} = \frac{d_2}{(d_1, d_2)}k,\ k = 0, \dots, (d_1, d_2)-1;\ a_{32} = \frac{d_3}{d}l,\ \ l = 0, \dots, d-1;\ d = \left(\frac{a_1 a_2}{d_3}\right),$$

the congruence $d_1(a_{21}a_{32} - d_2 x) \equiv 0 \bmod d_2 d_3$ resolves.

It is easy to see that for

$$\hat{A} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2 & 0 \\ 0 & 0 & a_1 a_2 a_3 \end{pmatrix}$$

we have

$$\tau(S(C)) =$$

$$= \sum_{d_1 | a_1} \sum_{d_2 | a_1 a_2} \sum_{d_3 | a_1 a_2 a_3} (d_1, d_2) \left((d_1, d_2) \cdot \left((d_1, d_2), \left(d_3, \frac{a_1 a_2}{d_2} \right) \right), \frac{d_1 d_3}{(d_1, d_3)} \right) \cdot$$

$$\sum_{t \left| \left(\left(d_3, \frac{a_1 a_2}{d_2} \right), \frac{d_1 d_3}{(d_1, d_2)(d_1, d_3)} \right) \right.} \frac{\varphi(t)}{t}. \tag{5}$$

Moreover, for $(a_1 a_2 a_3, b_1 b_2 b_3) = 1$

$$\tau(S(C_1 C_2)) = \tau(S(C_1))\tau(S(C_2)),$$

where

$$S(C_1) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2 & 0 \\ 0 & 0 & a_1 a_2 a_3 \end{pmatrix}, \quad S(C_2) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_1 b_2 & 0 \\ 0 & 0 & b_1 b_2 b_3 \end{pmatrix}.$$

Thus after short calculations we easily obtain

$$t_3(p^n) = \begin{cases} (k+1)p^{2k}(1 + O(\frac{k}{p})), & \text{if } n = 3k; \\ 2(k+1)p^{2k}(1 + O(\frac{k}{p})), & \text{if } n = 3k+1; \\ (k+1)p^{2k+1}(1 + O(\frac{k}{p})), & \text{if } n = 3k+2. \end{cases} \tag{6}$$

Consequently,

$$\sum_1^{\infty} \frac{t_3(n)}{n^s} = \sum_1^{\infty} \frac{1}{n^s} \sum_{\substack{S(C) \\ |S(C)|=n}} \tau(S(C)) =$$

$$= \sum_{(a_1,a_2,a_3)=1} \frac{1}{a_1^{3s} a_2^{2s} a_3^{s}} \tau\left(\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2 & 0 \\ 0 & 0 & a_1 a_2 a_3 \end{pmatrix}\right) =$$

$$= \prod_p \left(1 + \frac{2 + O(\frac{1}{p})}{p^s} + \frac{p(1+O(\frac{1}{p}))}{p^{2s}} + 2\frac{p^2(1+O(\frac{1}{p}))}{p^{3s}} + 2\frac{2p^2(1+O(\frac{1}{p}))}{p^{4s}} + \right.$$

$$+2\frac{p^3(1+O(\frac{1}{p}))}{p^{5s}} + \ldots + (k+1)\frac{p^{2k}(1+O(\frac{c(k)}{p}))}{p^{3ks}} + (k+1)\frac{2p^{2k}(1+O(\frac{c(k)}{p}))}{p^{(3k+1)s}} +$$

$$\left. +(k+1)\frac{p^{2k+1}(1+O(\frac{c(k)}{p}))}{p^{(3k+2)s}} + \ldots\right),$$

where $c(k) = \sum_{\substack{k_1,k_2,k_3 \geq 0 \\ k_1+k_2+k_3=k}} 1$. It can be easily shown that

$$F(s) = \prod_{p \leq p_0} \left(1 + \frac{2+O(\frac{1}{p})}{p^s} + \frac{p(1+O(\frac{1}{p}))}{p^{2s}} + \ldots + (k+1)\frac{p^{2k}(1+O(\frac{1}{p}))}{p^{3ks}} + \right.$$

$$\left. +(k+1)\frac{2p^{2k}(1+O(\frac{1}{p}))}{p^{(3k+1)s}} + (k+1)\frac{p^{2k+1}(1+O(\frac{1}{p}))}{p^{(3k+2)s}} + \ldots\right) \times$$

$$\times \prod_{p > p_0} \left(\sum_{k=0}^{\infty} \left((k+1)\frac{p^{2k}}{p^{3ks}} + (k+1)\frac{2p^{2k}}{p^{(3k+1)s}} + (k+1)\frac{p^{2k+1}}{p^{(3k+2)s}}\right) + \right.$$

$$\left. + \sum_{k=0}^{\infty} \left(O\left(\frac{k^3}{p^{3ks-2k+1}}\right) + O\left(\frac{k^3}{p^{(3k+1)s-2k+1}}\right) + +O\left(\frac{k^3}{p^{(3k+2)s-2k}}\right)\right)\right),$$

for some p_0.

Thus we have

$$F(s) = G_1(s) \cdot \tag{7}$$

$$\cdot \prod_{p > p_0} \left(\sum_{k=0}^{\infty} \left((k+1)\frac{p^{2k}}{p^{3ks}} + (k+1)\frac{2p^{2k}}{p^{(3k+1)s}} + (k+1)\frac{p^{2k+1}}{p^{(3k+2)s}} + \ldots\right) g_p(s)\right),$$

where

$$G_1(s) =$$

$$= \prod_{p \le p_0} \left(\sum_{k=0}^{\infty} (k+1) \frac{p^{2k}(1+O(\frac{1}{p}))}{p^{3ks}} + (k+1) \frac{2p^{2k}(1+O(\frac{1}{p}))}{p^{(3k+1)s}} + \right.$$

$$\left. +(k+1) \frac{p^{2k+1}(1+O(\frac{1}{p}))}{p^{(3k+2)s}} \right),$$

$$g_p(s) = 1 + O\left(\frac{1}{p^{\frac{15}{11}s}} \right).$$

Now from (6)-(7) we obtain for $Re\, s > 1$

$$F(s) = \zeta^2(s)\zeta(2s-1)\zeta^2(3s-2)\zeta^{-1}(4s-2)G_0(s), \tag{8}$$

where $\zeta(s)$ is the Riemann zeta-function and $G_0(s)$ is a Dirichlet series absolutely convergent in $Re\, s > \frac{11}{15}$,

$$G_0(s) = \sum_1^{\infty} \frac{g_n}{n^s}. \tag{9}$$

We will need the following facts about $\zeta(s)$:

$$\begin{cases} |\zeta(1/2+it)| \ll t^{\alpha}, \quad (\alpha \le 89/570 + \epsilon < 9/56 \text{ (see [5])}), \\ |\zeta(\sigma+it)| \ll t^{2\alpha(1-\sigma)}(\log|t|)^{2\sigma-1}, \quad \text{if } 1/2 \le \sigma \le 1,\ |t| \ge 3, \\ \int_1^T |\zeta(1/2+it)|^2 dt \ll T\log T, \\ \int_1^T |\zeta(1/2+it)|^4 dt \ll T\ln^4 T, \\ |\zeta'(1+it)| \ll \ln(|t|+3). \end{cases} \tag{10}$$

3. Proof of the upper bound estimate $\Delta(x)$

By using Perron's summation formula we have for $c > 1,\ T > 1$:

$$\sum_{n\le x} t_3(n) = xP_4(\log x) + \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{T(c-1)^5}\right). \tag{11}$$

Shifting the line of integration on the line $Re\, s = b$ ($b = 3/4$), and taking into account that the function $F(s)$ has pole of 5-th order at the point $s = 1$, we obtain

$$\sum_{n\le x} t_3(n) = xP_4(\log x) + \frac{1}{2\pi}\int_{b-iT}^{b+iT} \zeta^2(b+it)\zeta(2b-1+2it)\cdot \tag{12}$$

$$\cdot\zeta^2(3b-2+3it)\frac{\zeta^{-1}(4b-2+4it)}{b+it}\cdot x^{b+it}\cdot G_0(s)dt + O\left(\frac{x^c}{T(c-1)^5}\right)$$

(here $P_4(u)$ be a polynomial of degree 4).

Applying the Cauchy inequality, from (12) we obtain

$$\sum_{n\le x} t_3(n) = xP_4(\log x) + O\left(\left(\int_1^T |\zeta(3/4+it)|^2|\zeta(1/2+2it)|\frac{dt}{t}\right)^{1/2}\right.$$

$$\left.\left(T\int_1^T |\zeta(1/4+3it)|^2|\zeta(1/2+2it)|\frac{dt}{t}\right)^{1/2} x^b \max_{1\le t\le T}|\zeta^{-1}(1+it)|\right) +$$

$$+O\left(\frac{x^c}{T(c-1)^5}\right).$$

Note that $|\zeta^{-1}(1+it)| \ll \log T$ for $1 \le |t| \le T$ and

$$\int_1^T |\zeta(\sigma+it)|^8 dt \ll T^{1+\varepsilon}, \tag{13}$$

for $\sigma \geq \frac{5}{7}$ (see Ch.8 of [7]). Then taking into account that $|\zeta(1/4+it)| \ll \ll |t|^{1/2-1/4}|\zeta(3/4-it)|$ and using the Cauchy inequality, we have

$$\sum_{n\leq x} t_3(n) =$$

$$= xP_4(\ln x) + O\left(x^b \left(\int_1^T |\zeta(3/4+it)|^8 \frac{dt}{t} \right)^{1/4} \left(\int_1^T |\zeta(1/2+2it)|^2 \frac{dt}{t} \right)^{1/4} \right.$$

$$\left. \cdot \left(T^2 \int_1^T |\zeta(3/4+3it)|^8 \frac{dt}{t} \right)^{1/4} \left(\int_1^T |\zeta(1/2+2it)|^2 \frac{dt}{t} \right)^{1/4} x^{3/4} \right) +$$

$$+ O\left(\frac{x^c}{T(c-1)^5} \right) =$$

$$= xP_4(\log x) + O(x^{3/4}T^{1/2+\varepsilon}) + O\left(\frac{x^c}{T(c-1)^5} \right).$$

Hence, setting $c = 1 + \frac{1}{\log x}$, $T = x^{1/6}$ we obtain the assertion of Theorem 1.

4. The second moment for $\Delta(x)$

From the relation

$$\Delta(x) = \sum_{n\leq x} t_3(n) - P_4(\log x) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s) \frac{x^s}{s} ds \quad (b > 3/4),$$

follows that $\Delta(\frac{1}{x})$ and $F(s)$ generate the Mellin pair, and, hence, by Parseval's identity for Mellin transform we may immediately deduce that

$$(15) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F(b+it)|^2}{|b+it|^2} dt = \int_0^{\infty} \Delta^2\left(\frac{1}{x}\right) x^{2b-1} dx = \int_0^{\infty} \Delta^2(x) x^{-2b-1} dx.$$

Take $b = 3/4 + \delta$, where δ will be chosen later. By (14) we obtain for $x \geq x_0 > 0$

$$\int_x^{2x} \Delta^2(x)dx \ll$$

$$\ll x^{2b+1} \int_{-\infty}^{\infty} |\zeta^2(b+it)\zeta(2b-1+2it)\zeta^2(3b-2+3it)\zeta^{-1}(4b-2+4it)|^2 \frac{dt}{|b+it|^2} \ll$$

$$\ll x^{2b+1} \left(\int_0^1 + \int_1^{T_0} + \int_{T_0}^{\infty} \right) = x^{2b+1}(I_1 + I_2 + I_3),$$

say.

The estimate $I_1 \ll 1$ is clear. Further applying the estimates from (10) and the functional equation for $\zeta(s)$ we have

$$(17)\quad I_2 \ll \int_1^{T_0} |\zeta(b+it)|^4 |\zeta(3-3b-3it)|^4 \left(t^{1/2-(3b-2)}\right)^4 \left(t^{2\alpha(1-2b+1)}\right)^2 \frac{dt}{t^2} \ll$$

$$\ll \max_{1 \leq T_1 \leq T_0} \max \left(T_1^{-12\delta-8\alpha\delta+2\alpha} \log T_1 \left(\int_{T_1}^{2T_1} |\zeta(b+it)|^4 |\zeta(3-3b+3it)|^4 \frac{dt}{t} \right) \right).$$

Setting $\delta = \frac{1}{40}$, $\alpha = \frac{9}{56}$, $\varepsilon = \frac{1}{28}$, using (13) and the Cauchy inequality, after short calculations we obtain

$$(18)\qquad I_2 \ll 1.$$

Moreover, for $Q \geq T_0$

$$(19)\qquad \int_{-\infty}^{\infty} \frac{|F(b+it)|^2}{|b+it|^2} dt \ll Q^{-5\varepsilon}.$$

Setting $Q = 2^j T_0$ we have

$$(20)\qquad I_3 = \int_T^{\infty} = \sum_{\infty}^{j=1} \int_{2^{j-1}T_0}^{2^j T_0} \ll \sum_{\infty}^{j=1} 2^{-\frac{20\varepsilon}{9} j} T_0^{-5\varepsilon} \ll 1.$$

Thus $I_1 + I_2 + I_3 \ll 1$, and consequently we have

$$\int_{x}^{2x} \Delta^2(x)dx \ll x^{51/20}.$$

Corollary. For all but $o(x)$ of $x < X$,

$$|\Delta(x)| \ll x^{31/40} \tag{21}$$

holds.

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