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ON PROPERTIES OF THE SINGULAR INTEGRAL OPERATOR FOR THE MAXWELL EQUATIONS

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. For solving boundary value problems for the Maxwell equations, an integral equation method is useful. The Maxwell equation with a piecewise continuous conductivity coefficient in an infinite domain is equivalent to the three-dimensional singular integral equation over a local domain.

In the paper properties of the singular integral operator are analyzed. It has been proved that this operator is a Fredholm type one and is bounded in L_2. Theorems are proved on the existence of the solution of the singular integral equation and on continuous dependence of solution from the right hand side function.

1. Introduction

Physical and other natural science problems are frequently modelled by the vector differential equation like

$$\text{rot rot}\mathbf{E} = k^2\mathbf{E} + \mathbf{j}. \tag{1}$$

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For example the Maxwell equations formulated in frequency variables can be written as vector equation (1), and in this case $\mathbf{j}(\mathbf{R}) \in L_2(\Re^3)$ is a source of an electromagnetic field, the coefficient k^2 is given by

$$k^2 = \mu\omega^2 \left(\varepsilon + \frac{i\sigma}{\omega}\right), \quad \mathrm{Im}\, k \geq 0, \tag{2}$$

where μ, ω and ε are positive constants, $\sigma(\mathbf{R})$ is a nonnegative (and often noncontinuous) conductivity function.

For k (and σ) we assume that $\Re^3$ can be decomposed into a finite number of subdomains G_j, in which k is a continuously differentiable function, and the boundaries of these subdomains are surfaces of Lyapunov type (or they consist of a finite number of such surfaces).

On the surfaces of discontinuity of k boundary conditions have to be given. We suppose on the tangential components of the vectors $\mathbf{E}$ and $\mathbf{H} = \mathrm{rot}\mathbf{E}$ that

$$E_\tau, H_\tau \quad \text{are continuous} \tag{3}$$

(see [1, 9]). For conditions at infinity let a spherical surface S_R be such big that all finite subdomains G_j are inside the S_R. In the infinite domains on the points of S_R, where $\sigma = 0$, the following conditions are supposed to be satisfied:

$$|\mathbf{E}| = o(R^{-1}), \quad |\mathbf{H}| = o(R^{-1}) \tag{4}$$

and if $\sigma \neq 0$ ([1, 9])

$$|\mathbf{E}| = O(R^{-1}), \ |\mathbf{H}| = O(R^{-1}), \ |\mathbf{E} + \sqrt{\mu/\varepsilon}(\mathbf{n} \times \mathbf{H}| = o(R^{-1}). \tag{5}$$

The unicity of solution of this problem is proved in [3].

The vector equation (1) has been formulated in an infinite domain $\Re^3$, and if the coefficients of the equation are not smooth, the compatibility conditions for the solution at the surfaces of discontinuity of the coefficients, like (3), are not easy to satisfy.

Let the coefficient in the equation (1) be

$$k(\mathbf{R}) = \begin{cases} k_c(\mathbf{R}) & \text{if } \mathbf{R} \in \Re^3 \setminus V_T, \\ k_T(\mathbf{R}) & \text{if } \mathbf{R} \in V_T, \end{cases} \tag{6}$$

where V_T is bounded domain, and let the solution of the equation (1) with boundary conditions be known if $k(\mathbf{R}) = k_c$ for $\mathbf{R} \in \Re^3$. Then the *Integral*

Equation Method can be successfully used to the equation (1). By this method the differential problem in an infinite space can be reduced to the solution of the singular integral equation in a finite domain V_T. This reduction is important and useful for numerical solving of the problem.

2. Fundamental solution

Let us suppose that the solution of equation (1) is known if the function $\mathbf{j}$ is the Dirac δ-function. (In the Maxwell equations this source corresponds to electric dipole.) Then the integral representation using a fundamental solution can be used for the solution of the equation (1) with arbitrary $\mathbf{j} \in L_2(V_0)$, where V_0 is a bounded domain in $\Re^3$ ([4]).

Let the fundamental solution $\mathcal{E}$ of (1) (an analogy of the Green function for vector equations) be a solution of the following tensor equation

$$\text{rot rot}\mathcal{E} = k^2\mathcal{E} + \mathcal{D}, \quad \mathcal{D} = \delta(\mathbf{R} - \mathbf{R}_0)\mathcal{I}, \tag{7}$$

where $\mathbf{R}_0$ is the pole position and $\mathcal{I}$ is the unit tensor. The tensor $\mathcal{E}(\mathbf{R}, \mathbf{R}_0)$ can be expressed using the tensor potential $\mathcal{A}(\mathbf{R}, \mathbf{R}_0)$ as

$$\mathcal{E} = \mathcal{A} + \nabla_{\mathbf{R}} \left(\frac{1}{k^2} \text{div}_{\mathbf{R}} \mathcal{A} \right) \tag{8}$$

(see [1]). Note, that $\text{div}\,\mathcal{A}$ is a row vector with components $\text{div}\,\mathbf{A}^i$, where the vectors $\mathbf{A}^i$ are the columns of $\mathcal{A}$. The tensor potential $\mathcal{A}$ satisfies the equation

$$\Delta\mathcal{A} + k^2 \nabla \left(\frac{1}{k^2} \right) \cdot \text{div}\mathcal{A} + k^2\mathcal{A} = -\mathcal{D}. \tag{9}$$

Let us analyze the order of the singularity of $\mathcal{E}$. Let

$$\mathcal{E} = \mathcal{E}^0 + \mathcal{E}^1, \tag{10}$$

where $\mathcal{E}^0$ satisfies the equation

$$\text{rot rot}\mathcal{E}^0 = k_0^2\, \mathcal{E}^0 + \mathcal{D}, \quad k_0 = k(\mathbf{R}_0). \tag{11}$$

So, $\mathcal{E}^0$ is a fundamental solution of the equation like (1) in the homogeneous space with constant parameters. $\mathcal{E}^0$ has a simple structure:

$$(12)\qquad \mathcal{E}^0 = \mathcal{A}^0 + \frac{1}{k_0^2}\nabla \mathrm{div}\mathcal{A}^0,\ \mathcal{A}^0 = A_0\mathcal{I},\ A_0 = \frac{e^{ik_0R}}{4\pi R},\ R = |\mathbf{R} - \mathbf{R}_0|.$$

Since the singularity of A_0 in the neighbourhood of the pole is of order R^{-1}, it is easy to see that the singularity of the elements of $\mathcal{E}^0$ is $O(R^{-3})$.

For $\mathcal{E}^1$ we have from (8) and (12)

$$(13)\qquad \mathcal{E}^1 = \mathcal{A}^1 + \nabla\left(\frac{1}{k^2}\,\mathrm{div}\mathcal{A}^1\right) + \nabla\left(\left(\frac{1}{k^2} - \frac{1}{k_0^2}\right)\mathrm{div}\mathcal{A}^0\right),$$

for $\mathcal{A}^1 = \mathcal{A} - \mathcal{A}^0$, because $\Delta\mathcal{A}^0 + k_0^2\mathcal{A}^0 = -\mathcal{D}$. From (9) we have

$$(14)\quad \Delta\mathcal{A}^1 + k^2\nabla\left(\frac{1}{k^2}\right)\cdot\mathrm{div}\mathcal{A}^1 + k^2\mathcal{A}^1 = -k^2\nabla\left(\frac{1}{k^2}\right)\cdot\mathrm{div}\mathcal{A}^0 - (k^2 - k_0^2)\mathcal{A}^0.$$

Since the singularity of the right hand side of the equation (14) is $O(R^{-2})$, the singularity of the second derivatives of the elements of $\mathcal{A}^1$ is of order R^{-2}, and because $1/k^2$ is continuously differentiable function, the elements of $\mathcal{E}^1$ (13) in the neighbourhood of the pole have integrable singularity $O(R^{-2})$.

3. Integral equation method. Separation of a standard singular operator

Let $\mathcal{E}$ be the fundamental solution of (1) if $k = k_c(\mathbf{R})$ from (6) for $\mathbf{R} \in \Re^3$, and let $\mathbf{E}^n$ be the solution of this equation with $k = k_c$ and given $\mathbf{j}$. Then the equation (1) with boundary conditions (3) and conditions at infinity (4) or (5) is equivalent to the singular integral equation

$$(15)\qquad (\mathcal{K}\mathbf{E})(\mathbf{R}) = \mathbf{E}^n(\mathbf{R}),\quad \mathbf{R} \in \Re^3,$$

(see [4, 5]), where the singular operator

$$(16)\qquad (\mathcal{K}\mathbf{E})(\mathbf{R}) = a(\mathbf{R})\mathbf{E}(\mathbf{R}) + \fint_{V_T}(k_c^2 - k^2)(\mathbf{R}_0)\mathcal{E}(\mathbf{R},\mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0,$$

in which

$$a(\mathbf{R}) = 1 - \frac{1}{3}\left(1 - \frac{k^2}{k_c^2}\right).$$

Here by $⨍$ we denote the Cauchy's principle value of singular integrals.

So the differential equation with complicated conditions, defined in an infinite domain, has been reduced to the singular integral equation in a bounded domain. It is *the Integral Equation Method.*

The tensor $\mathcal{E}^0$ (12) can be rewritten as

$$\mathcal{E}^0 = \frac{e^{ik_0R}}{4\pi R}\mathcal{I} + \frac{1}{4\pi k_0^2}\nabla\mathrm{div}\left(\frac{e^{ik_0R}-1}{R}\mathcal{I}\right) + \tag{17}$$
$$+\left(\frac{1}{4\pi k_0^2} - \frac{1}{4\pi k_c^2(\mathbf{R})}\right)\nabla\mathrm{div}\left(\frac{1}{R}\mathcal{I}\right) + \frac{1}{4\pi k_c^2(\mathbf{R})}\nabla\mathrm{div}\left(\frac{1}{R}\mathcal{I}\right),$$

where $k_0 = k_c(\mathbf{R}_0)$. Now from $\mathcal{E}^0$ the tensor $\mathcal{E}^S$ can be separated as

$$\mathcal{E}^S = \frac{1}{4\pi k_c^2(\mathbf{R})}\nabla\mathrm{div}\left(\frac{1}{R}\mathcal{I}\right). \tag{18}$$

In the remaining part $\mathcal{E}^0 - \mathcal{E}^S$ the second term in right hand side of (17) is analytical function, further

$$\left|\frac{1}{k_c^2(\mathbf{R}_0)} - \frac{1}{k_c^2(\mathbf{R})}\right| = R\cdot\left|\nabla\left(\frac{1}{k_c^2}\right)(\widetilde{\mathbf{R}})\right|, \tag{19}$$

where $\widetilde{\mathbf{R}} = \lambda\mathbf{R}_0 + (1-\lambda)\mathbf{R}$, $\quad 0 \le \lambda \le 1$, and so the singularity of $\mathcal{E}^0 - \mathcal{E}^S$, as well as of $\mathcal{E} - \mathcal{E}^S$, is an integrable singularity $O(R^{-2})$.

Since $k^2 - k_c^2 = 0$ for $\mathbf{R} \in \Re^3 \setminus V_T$, the domain of integration in (16) can be enlarged to $\Re^3$.

The singular integral in (16) can be rewritten as

$$\int\limits_{\Re^3}((k_c^2 - k^2)(\mathbf{R}_0)(\mathcal{E} - \mathcal{E}^S)\mathbf{E}d\mathbf{R}_0 + \int\limits_{\Re^3}((k_c^2 - k^2)(\mathbf{R}_0) - (k_c^2 - k^2)(\mathbf{R}))\mathcal{E}^S\mathbf{E}d\mathbf{R}_0 +$$

$$+ ⨍\limits_{\Re^3}(k_c^2 - k^2)(\mathbf{R})\mathcal{E}^S\mathbf{E}d\mathbf{R}_0. \tag{20}$$

The first two integrals have integrable integrands. In the third integral the integrand has the singularity order R^{-3}. Now the operator $\mathcal{K}$ from (16) can be written as

$$\mathcal{K} = \mathcal{R} + \mathcal{B}, \tag{21}$$

where $\mathcal{R}$ is *a standard singular operator*

$$(\mathcal{R}\mathbf{E})(\mathbf{R}) = a(\mathbf{R})\mathbf{E}(\mathbf{R}) + \int_{\Re^3} \frac{\mathcal{F}(\mathbf{R}, \vec{\alpha})}{R^3} \mathbf{E}(\mathbf{R}_0) d\mathbf{R}_0 \tag{22}$$

with matrix $\mathcal{F}$:

$$\frac{\mathcal{F}(\mathbf{R}, \vec{\alpha})}{R^3} = \left(\frac{k_c^2 - k^2}{4\pi k_c^2}\right)(\mathbf{R}) \bigtriangledown \mathrm{div}\left(\frac{1}{R}\mathcal{I}\right), \quad \vec{\alpha} = \frac{1}{R}(\mathbf{R}_0 - \mathbf{R}), \tag{23}$$

and the operator $\mathcal{B}$ is

$$(\mathcal{B}\mathbf{E})(\mathbf{R}) = \int_{\Re^3} ((k_c^2 - k^2)(\mathbf{R}_0)(\mathcal{E} - \mathcal{E}^S)(\mathbf{R}, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0) d\mathbf{R}_0 + \tag{24}$$

$$+ \int_{\Re^3} ((k_c^2 - k^2)(\mathbf{R}_0) - (k_c^2 - k^2)(\mathbf{R}))\mathcal{E}^S(\mathbf{R}, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0) d\mathbf{R}_0,$$

the operator with a weak integrable singularity.

4. Properties of the singular operator

Let us give some known results and definitions.

If $D(\mathbf{R}, \mathbf{R}_0)$ is bounded function on $G \subset \Re^m$, the operator with integrable singularity

$$(\mathcal{B}u)(\mathbf{R}) = \int_G \frac{D(\mathbf{R}, \mathbf{R}_0)}{R^\lambda} u(\mathbf{R}_0) d\mathbf{R}_0), \quad 0 < \lambda < m, \tag{25}$$

is compact operator in $L_p(G)$, $1 \le p$ ([8]).

If in the singular operator is defined in $\Re^m$, like (25) with $\lambda = m$, the function $D(\mathbf{R}, \mathbf{R}_0)$ can be represented as $f(\mathbf{R}, \vec{\alpha})$

$$\fint_{\Re^m} \frac{f(\mathbf{R}, \vec{\alpha})}{R^m} u(\mathbf{R}_0) d\mathbf{R}_0), \quad \vec{\alpha} = \frac{1}{R} (\mathbf{R}_0 - \mathbf{R}), \tag{26}$$

then the function f is called *the characteristic function* of the singular operator ([8]).

The necessary and sufficient condition for existence of a singular integral like (26) is

$$\oint_S f(\mathbf{R}, \vec{\alpha})\, dS = 0, \tag{27}$$

where S is the unit sphere ([8]).

The operator $\mathcal{R}$

$$(\mathcal{R}u)(\mathbf{R}) = a(\mathbf{R})u(\mathbf{R}) + \fint_{\Re^m} \frac{f(\mathbf{R}, \vec{\alpha})}{R^m} u(\mathbf{R}_0) d\mathbf{R}_0)$$

is a singular operator if the following conditions are satisfied:

$$|a(\mathbf{R}_0) - a(\mathbf{R})| \le A \cdot R^\lambda \left[(1 + |\mathbf{R}|^2)(1 + |\mathbf{R}_0|^2) \right]^{-\lambda/2}, \quad A, \lambda > 0, \tag{28}$$

$$|f(\mathbf{R}_0, \vec{\alpha}) - f(\mathbf{R}, \vec{\alpha})| \le B \cdot R^\mu \left[(1 + |\mathbf{R}|^2)(1 + |\mathbf{R}_0|^2) \right]^{-\mu/2}, \quad B, \mu > 0. \tag{29}$$

If the characteristic $f(\mathbf{R}, \vec{\alpha})$ has been expanded in a series with m-dimensional spherical functions $Y^l_{n(m)}$

$$f(\mathbf{R}, \vec{\alpha}) = \sum_{n=1}^{\infty} \sum_{l=-n}^{n} a^l_n(\mathbf{R}) Y^l_{n(m)}(\vec{\alpha}), \tag{30}$$

then the *symbol* $G(\mathbf{R}, \vec{\alpha})$ of the singular operator $\mathcal{R}$ is a series

$$G(\mathbf{R}, \vec{\alpha}) = a(\mathbf{R}) + \sum_{n=1}^{\infty} \sum_{l=-n}^{n} \gamma_{n(m)} a^l_n(\mathbf{R}) Y^l_{n(m)}(\vec{\alpha}), \quad \gamma_{n(m)} = \frac{i^n \pi^{m/2} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)} \tag{31}$$

(see [8]). The symbol of a singular operator does not change if a compact operator is added.

The *index* of singular operator $\mathcal{R}$ is the difference of the numbers of nulls of operators $\mathcal{R}$ and complex-conjugate $\mathcal{R}^*$. If the index of an integral operator equals zero, this operator is the Fredholm type one ([6]).

These results and definitions can be easy generalized to vector integral operators like (16).

The operator $\mathcal{B}$ (24) with weak singularity ($\lambda = 2$, $m = 3$) is compact operator.

The function $a(\mathbf{R})$ from (16), because $a(\mathbf{R}) = 1$ if $\mathbf{R} \in \Re^3 \setminus V_T$, satisfies condition (28) with $\lambda = 1$ and

$$A = \max\ (1 + |\mathbf{R}|^2) \cdot \max \left| \left(\frac{k^2}{k_c^2} \right)' \right| (\mathbf{R}), \quad \mathbf{R} \in V_T.$$

The components of matrix $\mathcal{F}$ (23) can be written as

$$f_{pq} = \left(\frac{k_c^2 - k^2}{4\pi k_c^2} \right) (\mathbf{R}) \cdot (3\alpha_p \alpha_q - \delta_{pq}), \quad p, q = 1, 2, 3, \tag{32}$$

where $\alpha_1 = \sin\theta\cos\phi$, $\alpha_2 = \sin\theta\sin\phi$, $\alpha_3 = \cos\theta$ are co-ordinates of unit vector $\vec{\alpha}$ in the spherical system of co-ordinates $\{R, \theta, \phi\}$. For these components condition (27) satisfies obviously. So the singular integral in (22) exists.

The condition (29) satisfies, because $k_c^2 - k^2 = 0$ if $\mathbf{R} \in \Re^3 \setminus V_T$. So $\mathcal{R}$ (22) is a singular integral operator, and $\mathcal{F}$ is a characteristic matrix function of this singular operator (see (26)).

Theorem 1. *The index of the standard singular operator $\mathcal{R}$ (22) is equal to zero.*

Proof. It is seen, that equality (32) is an expansion of f_{pq} in a series with three-dimensional spherical functions (see (30)), and in our case only the terms with $n = 2$ are present in the series. Since $\gamma_{2\,(3)} = -4\pi/3$, for elements of the symbol matrix $\mathcal{G} = \{G_{pq}\}$ of singular operator $\mathcal{R}$ (see (31)) we obtain

$$G_{pq} = a\delta_{pq} - \frac{1}{3}\left(1 - \frac{k^2}{k_c^2}\right)(3\alpha_p\alpha_q - \delta_{pq}) = \delta_{pq} - \left(1 - \frac{k^2}{k_c^2}\right)\alpha_p\alpha_q.$$

It is a known result (see [8]), that the fulfilment of the following conditions is sufficient for an index of singular operator $\mathcal{R}$ be equal to zero:

$$\int\limits_{\Re^3} \frac{\overline{\mathcal{F}(\mathbf{R}_0, -\vec{\alpha})} - \overline{\mathcal{F}(\mathbf{R}, -\vec{\alpha})}}{R^3}\ \mathbf{E}(\mathbf{R}_0) d\mathbf{R}_0 \quad \text{is a compact operator,} \tag{33}$$

and lower bounds of modules of

$$\Delta_1 = G_{11}, \quad \Delta_2 = \begin{vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{vmatrix}, \quad \Delta_3 = \det \{\mathcal{G}\}$$

are positive.

Condition (33) is satisfied because of an equality like (19) for f_{pq} (32). For Δ_i, $i = 1, 2, 3$, we obtain that

$$\Delta_1 = 1 - \left(1 - \frac{k^2}{k_c^2}\right) \sin^2\theta \cos^2\phi, \quad \Delta_2 = 1 - \left(1 - \frac{k^2}{k_c^2}\right) \sin^2\theta, \quad \Delta_3 = \frac{k^2}{k_c^2}.$$

The lower bounds of modules are

$$\text{for } \Delta_1 \text{ and } \Delta_2: \quad \min\left(1, \left|\frac{k^2}{k_c^2}\right|\right), \quad \text{for } \Delta_3: \quad \left|\frac{k^2}{k_c^2}\right|.$$

It is obvious that these numbers not equal to zero. Therefore the index of singular operator $\mathcal{R}$ is equal to zero. So operator $\mathcal{R}$, as well as $\mathcal{K}$, are of Fredholm type ones.

Note that the separation of standard singular operator $\mathcal{R}$ is useful not only for the analysis of properties of operator $\mathcal{K}$, but because of the simple structure of the integrand, this form of the singular operator facilitates a numerical solution of the integral equation (15) (see for example [2]).

5. Existence of the solution of the Singular Integral Equation

Theorem 2. *Singular integral operator* K *is bounded in* L_2, *continuous and closed.*

Proof. The elements of characteristic matrix $\mathcal{F}$ under fixed $\mathbf{R}$ are twice continuously differentiable functions over components of vector $\vec{\alpha}$. Further, they are bounded independently from $\mathbf{R}$. The satisfaction of these conditions is sufficient for the singular operator to be bounded in L_2 ([8]).

If an operator is bounded from the Banach space to the Banach space, then it is continuous and closed ([6]). Therefore, the operator $\mathcal{K} : L_2 \Rightarrow L_2$ is continuous and closed.

Theorem 3. *The solution of the singular integral equation* (15) *exists in* L_2 *and is unique.*

Proof. In [3] it has been proved that the the solution of the equation (1) with boundary conditions (3) and conditions at infinity (4), (5) is unique. Because the equation (1) with referred conditions is equivalent to the singular integral equation (15), the solution of this integral equation is unique as well.

Since operator $\mathcal{K}$ is a Fredholm type operator, the Fredholm's alternatives are satisfied for them. Because of the Fredholm's alternatives ([7, 10, 11]) if the solution of the Fredholm type integral equation is unique, it exists as well in L_2 with an arbitrary right hand side function from L_2.

Theorem 4. *The solution of the integral equation* (15) *continously depends on the right hand side function.*

Proof. Since the solution of the integral equation (15) exists and is unique with an arbitrary right side function from L_2, there exists inverse operator $\mathcal{K}^{-1} : L_2 \Rightarrow L_2$, further, this operator is bounded, and therefore (because of the Banach theorem) is continuous and closed. So the solution of the integral equation

$$\mathbf{E} = \mathcal{K}^{-1}\mathbf{E}^n$$

continuously depends from $\mathbf{E}^n$.

References

[1] **Дмитриев В.И. и Захаров Е.В.,** *Интегральные уравнения для краевых задач электродинамики,* Изд. Московского университета, Москва, 1987. (Dmitriev V.I. and Zakharov E.V., *Integral equations for the boundary value problems in electrodynamics,* Moscow Univ. Publ., Moscow, 1987. (in Russian))

[2] **Farzan R.H.,** Numerical solution of a three-dimensional integral equation, *Numerical analysis and mathematical modelling,* Banach Center Publ. **24**, Warsaw, 1990, 409-414.

[3] **Farzan R.H.,** Unicity of solution of the Maxwell equations with discontinuous coefficients in an infinite domain, *Annales Univ. Sci. Budapest. Sect. Math.,* **36** (1993), 247-252.

[4] **Farzan R.H.,** Integral equation method for three-dimensional partial differential equations, *Colloquia Math. Soc. János Bolyai* **62**. *Differential equations, Budapest'91,* North-Holland Publ., Amsterdam, 1993, 119-132.

[5] **Farzan R.H.,** Integral equation method for Maxwell equations, *Annales Univ. Sci. Budapest. Sect. Comp.,* **16** (1996), 103-127.

[6] **Крейн С.Г. ред.**, *Функциональный анализ,* СМБ, Наука, Москва, 1972. (Krein S.G. ed., *Functional analysis,* Nauka, Moscow, 1972. (in Russian))

[7] **Михлин С.Г.**, *Интегральные уравнения и их применения,* ГИТТЛ, Москва-Ленинград, 1949. (Mikhlin S.G., *Integral equations and their applications,* GITTL, Moscow-Leningrad, 1949. (in Russian))

[8] **Михлин С.Г.**, *Многомерные сингулярные интегралы и интегральные уравнения,* Физматгиз, Москва, 1962. (Mikhlin S.G., *Multidimensional singular integrals and integral equations,* Fizmatgiz, Moscow, 1962. (in Russian))

[9] **Müller C.**, *Foundations of the mathematical theory of electromagnetic waves,* Springer Verlag, 1969.

[10] **Петровский И.Г.**, *Лекции по теории интегральных уравнений,* Наука, Москва, 1965. (Petrovskij I.G., *Lectures on the theory of integral equations,* Nauka, Moscow, 1965. (in Russian))

[11] **Забрейко П.П. и др.**, *Интегральные уравнения,* **СМБ,** Наука, Москва, 1965. (Zabreiko et al., *Integral equations,* Nauka, Moscow, 1965. (in Russian))

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