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THE SANDBOX METHOD IN QUADRATIC FIELDS

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This paper is dedicated to our professor Imre Kátai
on the occasion of his 70th birthday

Abstract. We concentrate our attention to the sandbox method and the generalized number systems in real quadratic fields. In the latter area we can find sets generated by a coefficient system whose Hausdorff dimension so far has not been countable exactly. The main goal of this paper is to investigate whether the sandbox method which is efficiently applied in different fields of natural science is suitable for estimating the dimension of the above mentioned sets.

1. Introduction

We focus on a new application of the *sandbox method* which is published by T. Tél, Á. Fülöp and T. Vicsek in [7] in 1989. This process is used for the determination of the generalized dimension (D_q) associated with geometric structure of growing deterministic fractal. The sandbox method was compared with the well-known *boxcounting* process. It is demonstrated that the above-mentioned two methods are equivalent if the sandbox method is applied with an averaging over randomly selected centers. In this case the sandbox approach provides better estimates of the generalized dimensions. At least 50 citations show the importance of sandbox method in several areas of the natural science, for example cell biochemistry, biophysics, physics of the Earth, statistical

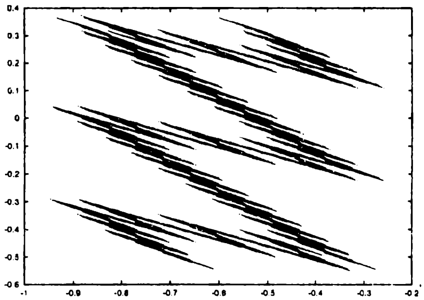

Figure 1: $B_{-\sqrt{3}}$ with S-type digit set ($\alpha = 3 + 3\sqrt{3}$).

mechanics, heterogeneous chemistry, molecular physics, electro analytical chemistry, nuclear physics and so on.

In this paper we investigate the question whether the sandbox method is effectively usable in mathematical research fields. We applied this process for the generalized number systems in real quadratic fields. We computed approximately the Hausdorff dimension of such sets generated by different coefficient systems where this has not been determined in theoretical way yet.

In the mid-seventies I. Kátai started to deal with the problem of generalization of number systems from $\mathbb{N}$ to other algebraic structures.

Let M be an $n \times n$ type regular matrix with integer entries and $\mathcal{L} = M\mathbb{Z}_k$. Then $\mathcal{L}$ is a subgroup in $\mathbb{Z}_k$, and $O(\mathbb{Z}_k/\mathcal{L}) = t = \mid \det M \mid$. Let $t \geq 2$ and $\mathcal{A} = \{a_0 = 0, a_1, \dots, a_{t-1}\}$ be a complete set of representatives of residue classes mod M. Then for every $n \in \mathbb{Z}_k$ there exists a unique $b_0 \in \mathcal{A}$ for which $n - b_0 \in \mathcal{L}$, consequently there exists an unique $n_1 \in \mathbb{Z}_k$ such that

$$n = b_0 + Mn_1.$$

This procedure can be repeated:

$$n_j = b_j + Mn_{j+1}.$$

We say that $(\mathcal{A}, M)$ is a number system if $n_r = 0$ for a large enough r, which is equivalent to the assertion that

$$n = b_0 + Mb_1 + M^2b_2 + \cdots + M^{r-1}b_{r-1}, \quad \text{where} \quad b_j \in \mathcal{A}. \tag{1.1}$$

Thus $(\mathcal{A}, M)$ is a number system if every $n \in \mathbb{Z}_k$ has a finite (1.1) representation.

Let

$$H = \left\{ z \mid z = \sum_{i=1}^{\infty} M^{-i} f_i, \ f_i \in \mathcal{A} \right\}.$$

If $(\mathcal{A}, M)$ is a number system, then

$$\bigcup_{n \in \mathbb{Z}_k} (H+n) = \mathbb{R},$$

furthermore

$$\lambda \left(H + n_1 \bigcap H + n_2 \right) = 0$$

whenever $n_1, n_2 \in \mathbb{Z}_k, \;\; n_1 \neq n_2$. Let S be the set of those $\gamma \in \mathbb{Z}_k, \;\; \gamma \neq 0$ for which

$$H \bigcap H + \gamma \neq 0.$$

Let

$$B_\gamma := H \bigcap H + \gamma.$$

We would like to determine approximately the Hausdorff dimension of B_γ. We shall do it with a probabilistic method for real quadratic fields in the case $k = 2$. The exact definitions and detailed descriptions connected with the generalized number systems in real quadratic fields can be found in the 3 section.

2. The sandbox method

The sandbox method has been introduced for the determination of the generalized dimension $D^{sb}(q)$ associated with the geometric structure of growing deterministic fractals. In this paper we present only the most important concepts, the details are found in [7].

The definition of sandbox dimension is based on the *generalized dimension* which phenomena is well-known in the natural science. It has been introduced in [8]. Although the generalized dimension is an important quantity in the research field mentioned in the introduction, to determine its value is too complicated. The sandbox dimension can be counted in an easy way and suitable for estimating the dimension of infinite sets using computations carried out in finite sets.

2.1. Generalized dimensions

We introduce the phenomena of generalized dimension based on the definitions published by R. H. Riedi in [10]. Let K be a compact set $K \subset \mathbb{R}^d$. Let us consider a lattice C with linear size δ ($\delta > 0,\ \delta \in \mathbb{R}$) in the following way:

$$C_j = \prod_{k=1}^{d} [l_k\delta, (l_k+1)\delta[\quad l_k \in \mathbb{N}, \tag{2.1}$$

$$C = \bigcup_j C_j, \quad \text{where} \quad j \in \{1 \ldots N^d\}, \quad C_j \cap K \neq \emptyset. \tag{2.2}$$

Let us denote $T_j = C_j \cap K \quad j \in \{1 \ldots N^d\}$ and

$$T = \bigcup_{j=1}^{N^d} T_j, \quad \text{where} \quad T_j \cap T_i = \emptyset, \quad j \neq i.$$

Let μ be a measure and G_δ means the set of δ-box, where ($\mu(T_j) \neq 0$). Then the *singularity exponents* are equal to

$$\tau(q) = \limsup_{\delta \to 0} \frac{\lg(S_\delta(q))}{-\lg(\delta)}, \quad \text{where} \tag{2.3}$$
$$S_\delta(q) = \sum_{T_i \in G_\delta} \mu(T_i)^q, \quad q \in \mathbb{R}.$$

The *generalized dimensions* [8] are defined by this expression

$$D(q) = \frac{1}{q-1}\tau(q) = \frac{1}{q-1} \limsup_{\delta \to 0} \frac{\lg(S_\delta(q))}{-\lg(\delta)}, \quad q \neq 1, \tag{2.4}$$

$$D(1) = \limsup_{\delta \to 0} \sum_{T \in G_\delta} \mu(T_i) \frac{\lg(\mu(T))}{\lg(\delta)},$$

where μ is a appropriate measure.

2.2. Application of sandbox method for finite sets

Since we can only study finite sets, therefore we apply the sandbox method for them and we extrapolate from the counted values to properties of infinite sets. There is a similar situation in the case of the generalized number systems in real quadratic fields, since nevertheless the B_γ sets investigated in this paper are infinite compact sets, we can produce just finite self-affine subsets instead of them.

Since in our case we study finite sets we can not use the definition of sandbox dimension introduced in [7], therefore we give an "empirical" definition of this value. We carry out the sandbox method and the given value is called "sandbox dimension". Thus we can not prove the correctness of the sandbox method in a mathematical way, so we present numerical results.

Let us consider a finite nonempty set $K \subset \mathbb{R}^d$. Further we assume that $d = 2$. As a matter of fact we count the elements of K with a lattice C with linear size δ and a closed ball covering. We get an approximate value because we consider only such T_i where each element of T_i belongs to $K \cap C$. We divide the given values by $|T|$. We note that for a finite set A $|A|$ means the number of elements of A. We define a closed ball with center $\underline{x} \in K$ and radius R $(R > 0,\ R \in \mathbb{R})$ in the following way:

$$B(\underline{x}, R) = \{\underline{y} : \mid \underline{x} - \underline{y} \mid \leq R, \underline{y} \in K\}. \tag{2.5}$$

Let us introduce two constant values $a = \min\{\mid \underline{x} - \underline{y} \mid : \underline{x} \neq \underline{y}\ \ \underline{x}, \underline{y} \in K\}$, $L = \max\{\mid \underline{x} - \underline{y} \mid : \underline{x}, \underline{y} \in K\}$, i.e. a means the minimal distance which is distinguishable between two points and L is the diameter of set K.

Let $\mu(T)$ be a map which is used instead of the measure in expression (2.4)

$$\mu(T_i) = \frac{\mid T_i \mid}{\mid T \mid}, \tag{2.6}$$

where $i \in \{1...N^d\}$. We note that

$$\sum_{i=1}^{N^d} \frac{\mid T_i \mid}{\mid T \mid} = 1.$$

>From μ we can derive the really used quantity $\nu(B(\underline{x}, R))$ on the set T in the following way

$$\nu(B(\underline{x}, R)) = \sum_{\forall i \in I} \frac{\mid T_i \mid}{\mid T \mid} = \sum_{\forall i \in I} \mu(T_i), \tag{2.7}$$

where $I = \{1...n'\} \subseteq \{1, ...N^d\}$ and the index set I contains each integer i $(1 \leq i \leq N^d,\ i \in \mathbb{N})$ for which $K \cap T_i = T_i$. We apply the expression

$$\left\langle \nu(B(\underline{x}, R))^{(q-1)} \right\rangle = \sum_{i=1}^{n} \frac{\nu(B(\underline{x}_i, R))^{(q-1)}}{n}$$

instead of $S_\delta(q)$ (in 2.4), where $\underline{x}_i \in K \cap C$. The centers of the balls $\underline{x}_i$ are chosen by random number generator with uniform probability distribution from the set $K \cap C$.

For the next step of the sandbox method let us consider the following expression introduced in [7]

$$D^{sb}(q) = \frac{\lg \langle \nu(B(\underline{x}, R))^{(q-1)} \rangle}{\lg(R/L)} \frac{1}{q-1} \quad \text{if } q \neq 1 \tag{2.8}$$

and

$$D^{sb}(1) = \frac{\lg \langle \nu(B(\underline{x}, R)) \rangle}{\lg(R/L)} \quad \text{if } q = 1,$$

where $a \leq R \leq L$. The original definition of sandbox dimension is the limit of the expression (2.8), if $R/L \to 0$ and n, L are constant. For the finite set K this limit is not defined. Thus we carry out the following steps to estimate the sandbox dimension.

Let us determine the value of the expression

$$\lg \left\langle \nu(B(\underline{x}, R))^{(q-1)} \right\rangle \frac{1}{q-1}$$

for different R and constant n. Then we graph the given values with respect to $\lg(L/R)$. Let us find those subintervals of $[a, L]$ where the points show linear dependence. Then we fit a straight line on these points. **The slope of the given line is considered the sandbox dimension.** The precision of the approximations can be increased by changing of δ and n.

The sandbox method includes numerical and statistical error. The numerical error depends on the implementation and the line fitting. The randomly chosen centers of closed ball covering can cause statistical error. The error can be estimated at every calculation. The error of the calculated sandbox dimension values, published in this paper, is less than 0.05.

3. Generalized number systems in real quadratic field

3.1. Quadratic fields

Let $\beta \in \mathbb{C}$ be a root of a polynomial $x^2 + ax + b \in \mathbb{Q}[x]$. Let $\mathbb{Q}(\beta)$ be the smallest field containing β. If $\beta \notin \mathbb{Q}$, then $\mathbb{Q}(\beta)$ is called *quadratic*, or *quadratic* *extension field.* It is known that every quadratic field is of form $\mathbb{Q}(\sqrt{D})$, where D is a positive or negative square-free integer. We speak about *real quadratic field* if $D > 0$. The set I of algebraic integers in $\mathbb{Q}(\sqrt{D})$ is given by

$$I = \mathbb{Z} + \mathbb{Z}\sqrt{D}$$

if $D \equiv 2, 3 \pmod 4$,

$$I = \mathbb{Z} + \mathbb{Z}\omega$$

if $D \equiv 1 \pmod 4$, where $\omega = \frac{1+\sqrt{D}}{2}$. The conjugate of an algebraic integer $\alpha \in \mathbb{Q}(\sqrt{D})$ is denoted by $\overline{\alpha}$. It is clear that $\overline{\alpha} = a - b\sqrt{D}$ if $\alpha = a + b\sqrt{D}$ and $\overline{\alpha} = a + b\overline{\omega}$ if $\alpha = a + b\omega$, where $\overline{\omega} = \frac{1-\sqrt{D}}{2}$. The norm of $\alpha \in I$ is defined by $N(\alpha) = \alpha\overline{\alpha}$.

3.2. S-type digit sets

Here we investigated such quadratic fields where $D \equiv 2, 3 \pmod 4$, therefore $\{1, \sqrt{D}\}$ is an integral bases in I. We assumed further that each considered base number α is greater than 2. Thus we can construct an S-type digit set in the analogues way as for the set of Gaussian integers has been found by G. Steidl in [11], and by which α is a base of a number system. We shall define the corresponding digit sets $E_\alpha^{(\varepsilon,\delta)}$ exactly as follows. Let $\varepsilon = \pm 1$, $\delta = \pm 1$ and for some $\alpha \in \mathbb{Q}(\sqrt{D})$ let $d = N(\alpha) = \alpha\overline{\alpha}$.

Let $\alpha = a + b\sqrt{D}$ and $E_\alpha^{(\varepsilon,\delta)}$ be the sets of those $f = k + l\sqrt{D}$, $k, l \in \mathbb{Z}$ for which $f\overline{\alpha} = (k + l\sqrt{D})(a - b\sqrt{D}) = (ka - bl2) + (la - kb)\sqrt{D} = r + s\sqrt{D}$ satisfy the following conditions:

- if $(\varepsilon, \delta) = (1, 1)$, then $r, s \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$,
- if $(\varepsilon, \delta) = (-1, -1)$, then $r, s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$,
- if $(\varepsilon, \delta) = (-1, 1)$, then $r \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$, $s \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$,
- if $(\varepsilon, \delta) = (1, -1)$, then $r \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$, $s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$.

It is proved in [4] that for each value of ε and δ the pair $\left(\alpha, E_\alpha^{(\varepsilon,\delta)}\right)$ is a number system.

3.3. Construction of $\underline{B}_\gamma$

We used a method suggested by I. Kátai in [2] to construct the set $\underline{B}_\gamma$. Since $\mathbb{Q}(\sqrt{D})$ is a two dimensional vector space over $\mathbb{Q}$ we can consider the elements of $\mathbb{Z} + \mathbb{Z}\sqrt{D}$ as the elements of $\mathbb{Z}^2$. In order to distinguish the one and two dimensional sets we underline the two dimensional ones. Let E_α

be one of $E_\alpha^{(\varepsilon,\delta)}$. We assume that (α, E_α) is a number system in I, where $\alpha = a + b\sqrt{D}$ and E_α is the suitable S-type digit set. Let α correspond to the matrix $M = \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$ and E_α to $\mathcal{A}$ with the mapping $f \mapsto \underline{f}$ where $f = k + l\sqrt{D} \in E_\alpha, \underline{f} = \begin{pmatrix} k \\ l \end{pmatrix} \in \mathcal{A}$. It is clear that both eigenvalues of M are greater than 1 and the order of the factor group $\mathbb{Z}^2/M\mathbb{Z}^2$ is $|\det M| = |d|$, thus $\mathcal{A}$ is a complete residue system mod M in $\mathbb{Z}^2$. According to the proof published in [4] $(\mathcal{A}, M)$ is a number system in $\mathbb{Z}^2$.

Thus the fundamental region can be given in the following way:

$$\underline{H} = \{\underline{z} \mid \underline{z} = \sum_{i=1}^{\infty} M^{-i} \underline{f_i},\ \underline{f_i} \in \mathcal{A}\}. \tag{3.1}$$

Now we can get a possible expansion of each elements of $\underline{B_\gamma} \subset \underline{H}$. Since $\underline{B_\gamma} = \underline{H} \cap \underline{H} + \gamma$, therefore

$$\underline{B_\gamma} = \{\underline{x} \mid \underline{x} \in \underline{H}, \underline{x} - \gamma \in \underline{H}\}.$$

We can write all expansions of γ as

$$\gamma = M^{-1}\underline{e_1} + M^{-2}\underline{e_2} + \dots,$$

where $\underline{e}_1, \underline{e}_2, \dots \in \mathcal{B} = \mathcal{A} - \mathcal{A}$. Then $\underline{e}_i = \underline{f}_i - \underline{f}'_i$, where $\underline{f}_i, \underline{f}'_i \in \mathcal{A}$ $(i = 1, 2, \dots)$ and let

$$\underline{x} = M^{-1}\underline{f}_1 + M^{-2}\underline{f}_2 + \dots . \tag{3.2}$$

Then we construct a directed graph $G(S)$. The nodes of $G(S)$ are the elements of S defined in the first section. From each node $\gamma \in S$ goes an edge to $\eta \in S$ if

$$M\gamma - \underline{\delta} = \eta$$

holds for some $\underline{\delta} \in \mathcal{B} = \mathcal{A} - \mathcal{A}$. This edge is labeled by $\underline{\delta}$. In order to get the elements of $\underline{B_\gamma}$ let us start from γ, walk on the graph $G(S)$ and write down the sequence of labels $\underline{\delta}_1, \underline{\delta}_2, \dots$. Since for every $\underline{\delta}_i = \underline{f}_i - \underline{f}'_i$ $(i = 1, 2, \dots)$, we can compute all possible values of the digits $\underline{f}_i$ which are substituted into (3.2).

The Figure 1 shows an example of $\underline{B_\gamma}$ with S-type digit set and Figure 2 another one with canonical digit set.

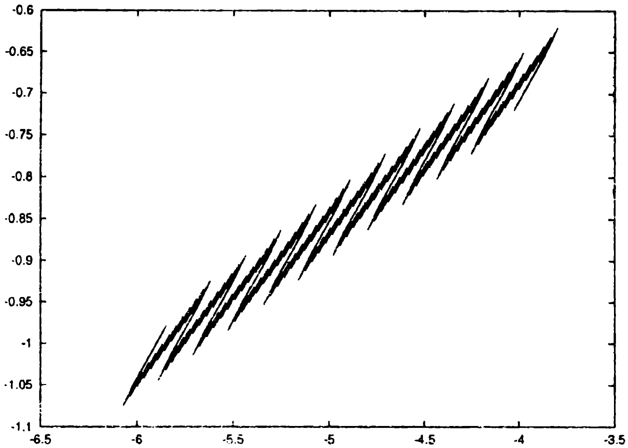

Figure 2: $B_{-5-\sqrt{3}}$ with canonical digit set ($\alpha = 6 + \sqrt{3}$).

3.4. Application of sandbox method for $\widetilde{B_\gamma}$

As it can be seen in previous subsection the set $\underline{B_\gamma}$ is an infinite compact set. We generate a finite set $\widetilde{B_\gamma}$ so that we substitute the infinite sums (3.2) with finite sums for some fixed k and γ. Thus we get that each element of $\widetilde{B_\gamma}$ forms

$$\overline{x} = \sum_{i=1}^{k} M^{-i} \underline{f}_i. \tag{3.3}$$

We have applied the sandbox method for $\widetilde{B_\gamma}$ whose every element corresponds to a subset of $\underline{B_\gamma}$. We think that the self-affine behavior of $\widetilde{B_\gamma}$ gives a good estimate of the dimension of the $\underline{B_\gamma}$. As a matter of fact we applied the sandbox method for $\widetilde{B_\gamma}$ in the following way. Let us use the notations which are introduced in subsections 2.1 and 2.2 with substitution $K = \widetilde{B_\gamma}$. As it can be seen in the subsection 4.1, our computations support that the sandbox dimension counted in finite set $\widetilde{B_\gamma}$ is a good approximation of the theoretical value of the boxcounting dimension of $\underline{B_\gamma}$.

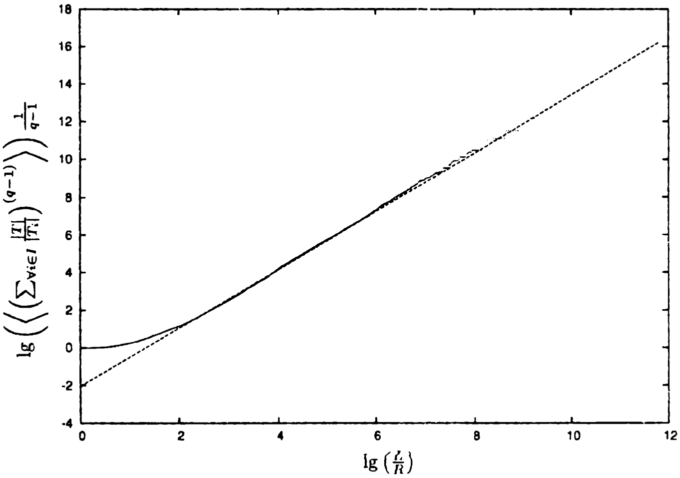

Figure 3: Sandbox dimension of $B_{-\sqrt{3}}$ with S-type digit set ($\alpha = 3 + 3\sqrt{3}$ N=10000, number of centum: 5000).

4. Analysis of the numerical results

In this section we present the numerical results of our investigations. In the Figure 3 we show the estimation of the sandbox dimension of B_γ for a given (α, E_α) number system. The slope of the line fitting on the curve displays the approximated value of the sandbox dimension (defined in (2.2).

After the fitting we get the value of $D_\delta^{sb}(q)$ which depends on δ. Since $D_\delta^{sb}(q)$ has the property so called *scaling nature* (described in [13]), therefore we can calculate the lattice independent value of the sandbox dimension. The given values of $D_\delta^{sb}(q)$ are displayed with respect to δ (Figure 4). The fitting line of the intersection on y axis causes a lattice independent value ($\delta \to 0$) which yields the numerical estimation of the sandbox dimension.

4.1. Comparison of the boxcounting and sandbox method

In this subsection we compare the well-known boxcounting method with the sandbox method. We study such canonical digit sets where the theoretical value of the boxcounting dimension is known. Remark, the sandbox method contradicting to the boxcounting method is used on the lattice size is a constant δ. In the Figure 4 we displayed the theoretical value of the boxcounting

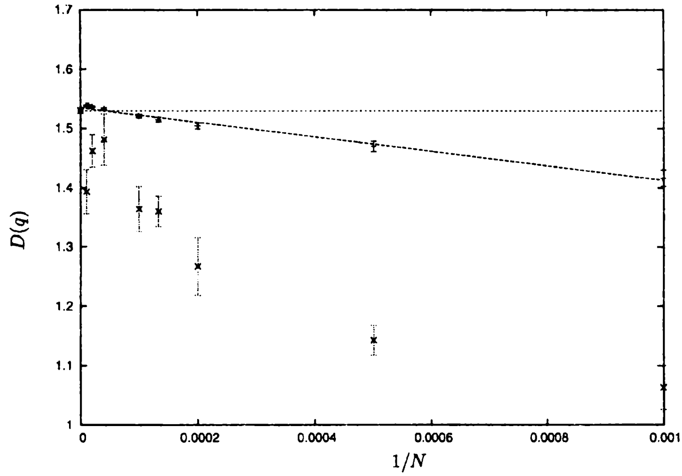

Figure 4: The convergence of sandbox and boxcounting dimension to the theoretical value in case $\alpha = 6 + \sqrt{3}$ with canonical digit set.

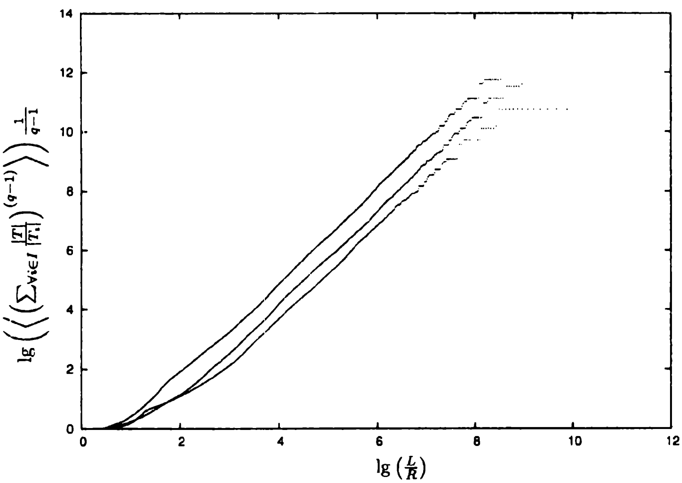

Figure 5: Dimension of B_γ for different γ with S-type digit set in case $\alpha = 3 + 3\sqrt{3}$. $\gamma_1 = -1$, $\gamma_2 = -1 + \sqrt{3}$, $\gamma_3 = 0 - \sqrt{3}$

a	b	D	γ	$D^{sb}(q)$	digit
6	1	3	$-5-\sqrt{3}$	1.53	canon.
6	1	3	$-5-\sqrt{3}$	1.24	S-type
3	3	3	$0-\sqrt{3}$	1.52	S-type
5	1	6	1	1.37	S-type
3	3	2	$-2-\sqrt{2}$	1.69	S-type

Table 1: Approximations of sandbox dimensions for different B_γ.

dimension of the set $B_{-5-\sqrt{3}}$ for $\alpha = 6+\sqrt{3}$. This is published in [5] by J.M. Thuswaldner. Also it can be seen in this figure the approximated values of the boxcounting and sandbox dimension of the same set B_γ with respect to δ. It is clear that the sandbox method arises faster convergence to the right value than the boxcounting using the same parameters and precision.

The Table 1 summarizes the lattice independent sandbox dimension values of the boundary set of fundamental region for coefficient systems and number systems for different a, b, D.

4.2. Dimension of B_γ for different γ

It is conjectured that if the graph $G(S)$ is strongly connected, the Hausdorff dimension of B_γ is the same for different values of γ. The numerical estimation confirms this conjecture as can be seen in Figure 5.

4.3. Modified S-type digit sets

The fundamental region is investigated even if (α, E_α) is not a number system. The Figure 6 shows an example for this case.

We conjecture that an S-type digit set for a given α can be modified to a digit set which consists of a number system with this α. This is proved in case $D = 2$, where $\alpha, 1-\alpha$ is not a unit and $\mid \alpha \mid, \mid \overline{\alpha} \mid > \sqrt{2}$.

4.4. Conclusion and further goals

The presented numerical results prove the effectiveness of sandbox method for estimating the Hausdorff dimension of the boundary set of H in real quadratic fields. Thus we strongly suppose that this method will be proved to be a useful tool in the investigation of generalized number systems and connected research fields.

Our prospective purpose is to study the dynamic systems by generalized entropy which has been introduced in the articles [12], [14].

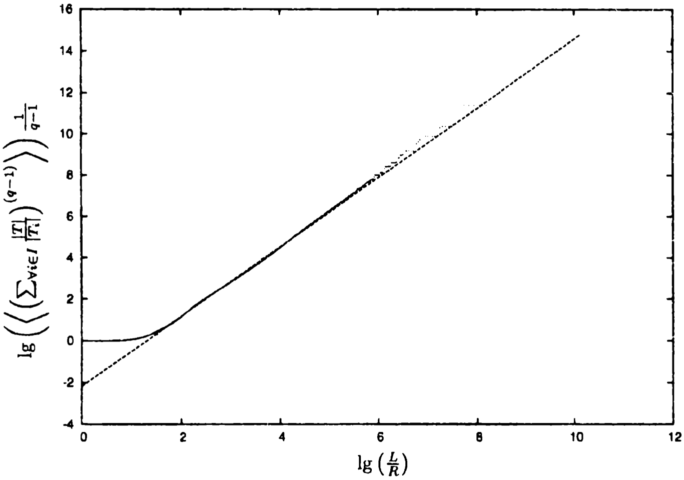

Figure 6: Dimension of $B_{-2+\sqrt{2}}$ with S-type digit set in case $\alpha = 3 + 3\sqrt{2}$, where (α, E_α) is not a number system.

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