

## ON A FAMILY OF FUNCTIONAL EQUATIONS WITH ONE PARAMETER

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*Dedicated to my friend Imre Kátai on his 70th birthday*

**Abstract.** Let  $I \subset \mathbb{R}$  be a non-void open interval and let  $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{2}$  be a given parameter. The functions  $f, g : I \rightarrow \mathbb{R}_+$  are solutions of the functional equation

$$f\left(\frac{x+y}{2}\right) \left(2\alpha g(y) - g(x)\right) = \alpha f(x)g(y) - (1-\alpha)f(y)g(x)$$

( $x, y \in I$ ), if and only if  $f$  and  $g$  are constant functions on  $I$ .

### 1. Introduction

We consider a one-parameter family of functional equations which plays important role in the solution of Matkowski-Sutô type problem ([1], [2], [3], [4], [8], [5], [6], [7]). These equations are interesting themselves independently of the original problem.

Let  $I \subset \mathbb{R}$  be a non-void open interval and let  $0 < \alpha < 1$  be a given parameter. We investigate the functional equation

$$(1) \quad f\left(\frac{x+y}{2}\right) \left(2\alpha g(y) - g(x)\right) = \alpha f(x)g(y) - (1-\alpha)f(y)g(x)$$

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for all  $x, y \in I$ , where  $f, g : I \rightarrow \mathbb{R}_+$  ( $\mathbb{R}_+$  is the set of positive real numbers) are unknown functions.

For the case  $\alpha = \frac{1}{2}$  we have the following proposition ([1]).

**Theorem 1.** *If  $\alpha = \frac{1}{2}$  and the functions  $f, g : I \rightarrow \mathbb{R}_+$  are continuous solutions of the functional equation (1), then there exist constants  $a, b \in \mathbb{R}$  and  $c \in \mathbb{R}_+$ , such that*

$$f(x) = ax + b > 0 \quad \text{and} \quad g(x) = \frac{c}{ax + b}$$

for all  $x \in I$ .

We remark, that in the case  $I = \mathbb{R}$  the constant  $a$  is 0 necessarily (and  $b \in \mathbb{R}_+$ ).

In this paper we discuss the following problems:

- (1) Is the continuity in Theorem 1 necessary, i.e. there exist solutions  $f, g : I \rightarrow \mathbb{R}_+$  of (1) in the case  $\alpha = \frac{1}{2}$  which are non continuous at the interval  $I$  or not.
- (2) What is the situation in the case  $\alpha \neq \frac{1}{2}$ .

## 2. On the non continuous solutions in the case $\alpha = \frac{1}{2}$

**Theorem 2.** *There exist non continuous solutions  $f, g : I \rightarrow \mathbb{R}_+$  of the functional equation (1) in the case  $\alpha = \frac{1}{2}$ .*

**Proof.** Let  $t \in I$  be fix. We define

$$f(x) := \begin{cases} a_1 & \text{if } x = t \\ a & \text{if } x \neq t \end{cases} \quad (x \in I),$$

and

$$g(x) := \begin{cases} b_1 & \text{if } x = t \\ b & \text{if } x \neq t \end{cases} \quad (x \in I),$$

where  $a_1, a, b_1, b$  are positive constants and  $a_1 \neq a$ ,  $b_1 \neq b$ . These functions  $f, g : I \rightarrow \mathbb{R}_+$  are non continuous at  $I$  and solutions of (1) in the case  $\alpha = \frac{1}{2}$ , if

$$(2) \quad 2 = \frac{a_1 b - a b_1}{a(b - b_1)}.$$

This assertion is trivial if  $x \neq t$ ,  $y \neq t$ .

If  $x = t$  and  $y \neq t$ , then (1) yields

$$a(b - b_1) = \frac{1}{2}(a_1b - ab_1),$$

i.e. by (2)  $f$  and  $g$  are solutions of (1). By the symmetry of (1) we have the proof of our assertion. For the equation (2) is an example the following one:  $a = b = 3$ ,  $a_1 = 4$  and  $b_1 = 2$ .

## 2. Main result about the equation (1) in the case $\alpha \neq \frac{1}{2}$

Our main result of this paper is the following surprising

**Theorem 3.** *Let  $\alpha \in ]0, 1[$  and  $\alpha \neq \frac{1}{2}$ . The functions  $f, g : I \rightarrow \mathbb{R}_+$  are solutions of the functional equation (1) if and only if there exist constants  $a, b \in \mathbb{R}_+$  such that*

$$f(x) = a \quad \text{and} \quad g(x) = b$$

for all  $x \in I$ .

To prove this theorem we need the following lemmas.

**Lemma 1.** *Let  $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{2}$ . If the pair  $(f, g)$  ( $f, g : I \rightarrow \mathbb{R}_+$ ) satisfies the functional equation (1) then the following equations*

$$(3) \quad f\left(\frac{x+y}{2}\right) = \frac{f(x)g(y) + f(y)g(x)}{g(x) + g(y)},$$

$$(4) \quad \alpha g(y) - (\alpha + 1)g(x) \neq 0,$$

$$(5) \quad \frac{f(x)g(y)}{f(y)g(x)} = \frac{\alpha g(x) - (\alpha + 1)g(y)}{\alpha g(y) - (\alpha + 1)g(x)}$$

for all  $x, y \in I$  are true.

**Proof.** In equation (1) we interchange  $x$  and  $y$ , then

$$(6) \quad f\left(\frac{x+y}{2}\right) [2\alpha g(x) - g(y)] = \alpha f(y)g(x) - (1 - \alpha)f(x)g(y).$$

We add equations (1) and (6), then we have

$$f\left(\frac{x+y}{2}\right)(2\alpha-1)[g(x)+g(y)] = (2\alpha-1)[f(x)g(y)+f(y)g(x)].$$

From this equation by  $2\alpha-1 \neq 0$  it follows (3). From (3) by (1) we obtain

$$\frac{f(x)g(y)+f(y)g(x)}{g(x)+g(y)}[2\alpha g(y)-g(x)] = \alpha f(x)g(y) - (1-\alpha)f(y)g(x)$$

for all  $x, y \in I$ . From the above equation by short computation we have

$$(7) \quad f(x)g(y)[\alpha g(y) - (\alpha+1)g(x)] = f(y)g(x)[\alpha g(x) - (\alpha+1)g(y)]$$

for all  $x, y \in I$ . If  $x = y$ , then  $\alpha g(x) - (\alpha+1)g(x) = -g(x) < 0$ , therefore assertion (4) is true. If  $x \neq y$  in (7) and  $\alpha g(y) - (\alpha+1)g(x) = 0$ , then from (7) we have  $\alpha g(x) - (\alpha+1)g(y) = 0$ , i.e.

$$\frac{g(x)}{g(y)} = \frac{\alpha}{1+\alpha} \quad \text{and} \quad \frac{g(y)}{g(x)} = \frac{\alpha}{1+\alpha},$$

which is impossible. Hence (4) is true for all  $x, y \in I$ . From (7) by (4) we have (5) for all  $x, y \in I$ .

**Lemma 2.** *Let  $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{2}$  be a fixed number. If the functions  $f, g : I \rightarrow \mathbb{R}_+$  with the property  $f(y_0) = g(y_0) = 1$  ( $y_0 \in I$ ) satisfy functional equation (1), then*

$$(8) \quad [g(x) - g(y)][1 - g(x)][1 - g(y)] = 0$$

for all  $x, y \in I$ .

**Proof.** By Lemma 1 we know that (4) and (5) are true. From (5) with  $y = y_0 \in I$  we have

$$f(x) = g(x) \frac{\alpha g(x) - (\alpha+1)}{\alpha - (\alpha+1)g(x)}$$

for all  $x \in I$ . We substitute this form of  $f$  in equation (5), then we obtain

$$\frac{g(x) \frac{\alpha g(x) - (\alpha+1)}{\alpha - (\alpha+1)g(x)} g(y)}{g(y) \frac{\alpha g(y) - (\alpha+1)}{\alpha - (\alpha+1)g(y)} g(x)} = \frac{\alpha g(x) - (\alpha+1)g(y)}{\alpha g(y) - (\alpha+1)g(x)}$$

for all  $x, y \in I$ . From this equation with the notation

$$F(x, y) = [\alpha g(x) - (\alpha+1)][\alpha - (\alpha+1)g(y)][\alpha g(y) - (\alpha+1)g(x)]$$

we have

$$F(x, y) = F(y, x)$$

for all  $x, y \in I$ . From this equation with an easy computation and with the notation  $A := \alpha^2(\alpha + 1) + \alpha(\alpha + 1)^2 > 0$  it follows

$$Ag(x) - Ag(y) + Ag(y)g^2(x) - Ag(x)g^2(y) + Ag^2(y) - Ag^2(x) = 0,$$

i.e.

$$[g(x) - g(y)][1 + g(x)g(y) - g(x) - g(y)] = 0.$$

But this is (8) for all  $x, y \in I$ .

#### 4. Proof of the Theorem 3

In this section we give a complete proof of Theorem 3.

- (i) First we suppose that the functions  $f, g : I \rightarrow \mathbb{R}_+$  are solutions of the functional equation (1) (where  $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{2}$ ) and  $f(y_0) = g(y_0) = 1$  for  $y_0 \in I$ . We assert, that in this case  $f(x) = g(x) = 1$  for all  $x \in I$ . Contrary, we suppose that there exists  $y_1 \in I$  ( $y_1 \neq y_0$ ), such that

$$g(y_1) = c \neq 1 \quad \text{and} \quad c > 0.$$

With the substitution  $y = y_1$  in (8) we have

$$(9) \quad [g(x) - c][1 - g(x)] = 0$$

for all  $x \in I$ . We define

$$E := \{ x \mid x \in I, g(x) = 1 \} \neq \emptyset$$

and

$$E^* := \{ x \mid x \in I, g(x) = c \} \neq \emptyset.$$

By equation (9) any  $x \in I$  is in  $E$  or in  $E^*$ , i.e.  $E \cap E^* = \emptyset$  and  $I = E \cup E^*$ . By Lemma 2

$$(10) \quad f(x) = g(x) \frac{\alpha g(x) - (\alpha + 1)}{\alpha - (\alpha + 1)g(x)} = \begin{cases} 1 & \text{if } x \in E, \\ c \frac{\alpha c - (\alpha + 1)}{\alpha - (\alpha + 1)c} & \text{if } x \in E^*. \end{cases}$$

If  $x \in E$  and  $y \in E^*$  then by equation (3) we have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)c + f(y)}{c+1} = \frac{c + c\frac{\alpha c - (\alpha+1)}{\alpha - (\alpha+1)c}}{c+1}.$$

Now,  $\frac{x+y}{2} \in E$  or  $\frac{x+y}{2} \in E^*$ . In the first case we have

$$\frac{c + c\frac{\alpha c - (\alpha+1)}{\alpha - (\alpha+1)c}}{c+1} = 1,$$

or in the second case

$$\frac{c + c\frac{\alpha c - (\alpha+1)}{\alpha - (\alpha+1)c}}{c+1} = c\frac{\alpha c - (\alpha+1)}{\alpha - (\alpha+1)c}.$$

In both cases we obtain  $c^2 = c$ , i.e.  $c = 1$ , which is a contradiction.

Then  $g(x) = 1$  for all  $x \in I$  and by (10) it follows  $f(x) = 1$  for all  $x \in I$ .

- (ii) If the pair  $(f, g)$  ( $f, g : I \rightarrow \mathbb{R}_+$ ) is a solution of (1) ( $0 < \alpha < 1$ ,  $\alpha \neq \frac{1}{2}$ ) then the pair  $\left(\frac{f}{f(y_0)}, \frac{g}{g(y_0)}\right)$  ( $y_0 \in I$ ) is a solution of (1), too, and

$$\frac{f(y_0)}{f(y_0)} = 1, \quad \frac{g(y_0)}{g(y_0)} = 1.$$

By (i) we have

$$\frac{f(x)}{f(y_0)} = 1, \quad \frac{g(x)}{g(y_0)} = 1$$

for all  $x \in I$ . With  $f(y_0) := a > 0$  and  $g(y_0) := b > 0$  we obtain the assertion of Theorem 3.

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