Annales Univ. Sci. Budapest., Sect. Comp. 28 (2008) 107-139

ASYMPTOTICS OF RENEWAL PROCESSES: SOME RECENT DEVELOPMENTS

V.V. Buldygin (Kiev, Ukraine)
K.-H. Indlekofer (Paderborn, Germany)
O.I. Klesov (Kiev, Ukraine)
J.G. Steinebach (Cologne, Germany)

Dedicated to Academician Imre Kátai on the occasion of his 70th birthday

Abstract. We give a survey of recent asymptotic results in renewal theory. Earlier investigations of renewal sequences, constructed from random walks with multidimensional time, lead to an equivalent definition of the renewal process which, in turn, resulted in other (generalized) renewal functionals. The techniques for dealing with these generalized renewal functionals are purely analytical, and they provide a link between renewal theory and some new developments in the theory of regular variation (of functions).

1. Introduction

Academician Imre Kátai wrote three papers on renewal theory, i.e. [12-13], joint with J. Galambos, and [11], joint with J. Galambos and K.-H. Indlekofer. The papers deal with the asymptotic behavior of the renewal sequence constructed from a random walk with multidimensional time (precise definitions are given below). Kátai, as a well-known number theorist, has been attracted by this probabilistic problem, since, in the first place, there

Partially supported by DFG grants
Mathematics Subject Classification: Primary 60F05; Secondary 60F05, 60F15, 60E07

is a remarkable relationship between renewal theory in multidimensional time and the problem of Dirichlet divisor functions in number theory. Moreover, the precise rate of convergence in certain renewal asymptotics depends on a corresponding one for Dirichlet divisors which, in turn, depends on the Riemann conjecture on the zeta function. This has clearly been understood in those papers, where a "... lack of knowledge in number theory imposes limitations on our (probabilistic) results ..." [11]. Nevertheless a new estimate for Dirichlet divisors in a small growing interval and a deep local central limit theorem allow I. Kátai and his coauthors to obtain in [11] a final (in a certain sense) result on the renewal function.

Below we give a short account of Kátai's results for renewal sequences constructed from random walks with multidimensional time. Then we show how further generalizations of his results give rise to the definition of a new class of (so-called) *pseudo-regularly varying* (PRV) functions, generalizing the notion of regular variation. Another way, leading to similar classes of functions, is to study the asymptotic behavior of generalized renewal processes. We describe both approaches leading to the PRV functions mentioned before. A comprehensive study of PRV functions and other similar classes is given in [2-5]. A recent survey of analytic properties of these functions can be found in [6]. Some other applications of PRV functions can also be found in [2-6].

The paper is organized as follows. Section 2 contains a brief list of properties of renewal processes in the classical setting. Sections 3 and 4 deal with earlier results on renewal sequences and functions constructed from random walks with multidimensional time. The equivalence between strong limit theorems for random walks and renewal processes is described in Section 5. Generalized renewal processes are treated in Section 6. A link between renewal theory and (extended) regular variation is discussed in Section 7.

2. Some definitions and results of the classical theory for renewal processes

Let us first discuss some classical results for random walks in one-dimensional time and their corresponding renewal processes, renewal functions and renewal sequences (see the definitions below).

Assume X, $\{X_n,\ n \geq 1\}$ to be independent, identically distributed (i.i.d.) random variables, and put $S_n = \sum_{k=1}^{n} X_k$. The sequence $\{S_n, n \geq 1\}$ is called a *random walk*. The *renewal (counting) process* N is defined via the random walk as follows:

$$N(t) = \max\{n : \ S_n < t\}, \qquad t \geq 0. \tag{2.1}$$

The process $\{N_t, \ t \geq 0\}$ is well defined if, for example, the random variables X_n are nonnegative with probability one and nondegenerate, that is

$$X \geq 0 \quad \text{a.s.} \quad \text{and} \quad \mathrm{P}(X = 0) < 1, \tag{2.2}$$

where "a.s." stands for "almost surely". In what follows we assume that condition (2.2) holds. Other useful assumptions in renewal theory are the Kolmogorov condition

$$0 < \mu = \mathrm{E}X < \infty, \tag{2.3}$$

or the Marcinkiewicz-Zygmund condition

$$\mathrm{E}X^v < \infty. \tag{2.4}$$

A special case of the latter condition is $v = 2$, i.e.

$$\sigma^2 = \mathrm{var}[X] < \infty. \tag{2.5}$$

Below we list some almost sure properties of the renewal process as $t \to \infty$. The proofs can be found, e.g. in [14] or elsewhere.

(a) If (2.2) holds, then $N(t) < \infty$ almost surely for all $t \geq 0$;
(b) if (2.2) holds, then $N(t) \to \infty$ a.s.;
(c) if (2.3) holds, then

$$N(t) \sim \frac{t}{\mu} \qquad \text{a.s.} \tag{2.6}$$

(*the strong law of large numbers*);
(d) if (2.2), (2.3) and (2.4) ($\exists\ 1 \leq v < 2$) hold, then

$$N(t) - \frac{t}{\mu} = o\left(t^{1/v}\right) \qquad \text{a.s.} \tag{2.7}$$

(*Marcinkiewicz-Zygmund strong law of large numbers*);

(e) if (2.2), (2.3) and (2.5) hold, then

$$\liminf_{t\to\infty} \frac{N(t) - \frac{t}{\mu}}{\sqrt{2t \ln\ln t}} = -\frac{\sigma}{\mu^{3/2}} \qquad \text{a.s.,} \tag{2.8}$$

$$\limsup_{t\to\infty} \frac{N(t) - \frac{t}{\mu}}{\sqrt{2t \ln\ln t}} = \frac{\sigma}{\mu^{3/2}} \qquad \text{a.s.} \tag{2.9}$$

(*law of the iterated logarithm*).

3. Asymptotic behavior of renewal sequences for multidimensional time

One-dimensional case. Let X, $\{X_n, n \ge 1\}$ be independent identically distributed random variables, and put $S_n = X_1 + \ldots + X_n$, $n \ge 1$. The classical renewal theorem by Erdős, Feller and Pollard [8] says that, if X is nonnegative, integer-valued and aperiodic, then

$$u(k) \to \frac{1}{\mu} \qquad \text{as } k \to \infty$$

provided $0 < \mu \le \infty$, where $\mu = \mathrm{E}X$ and

$$u(k) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mathrm{P}(S_n = k). \tag{3.1}$$

The aperiodicity means that $\mathrm{E}e^{i\theta X} \ne 1$ for any $\theta \ne 0$. Later Chung and Pollard [7] relaxed the assumption of non-negativity of X.

Multidimensional case. Let $\mathbb{N}^r$ be the space of vectors $\overline{n} = (n_1, \ldots, n_r)$ with positive integer coordinates $n_1, \ldots, n_r$. The space $\mathbb{N}^r$ is equipped with the partial ordering "$\le$" acting coordinate wise, that is $\overline{m} \le \overline{n}$ if and only if $m_1 \le n_1, \ldots, m_r \le n_r$. Further let X, $\{X(\overline{n}),\ \overline{n} \in \mathbb{N}^n\}$ be a family of independent identically distributed random variables and let $\{S(\overline{n}),\ \overline{n} \in \mathbb{N}^r\}$ be the collection of their partial sums

$$S(\overline{n}) = \sum_{\overline{k} \le \overline{n}} X(\overline{k}).$$

In the case of $r = 2$, the function

$$u_r(t) = \sum_{\overline{n} \in \mathbb{N}^r} \mathrm{P}(S(\overline{n} = t)$$

has been considered by Ney and Wainger [25] for integer t and for integer valued random variables $X(\overline{n})$. They called u_r the *renewal sequence constructed by a random walk in multidimensional time.* The name for u_r is intuitively clear in view of the analogy with the corresponding definition in case of $r = 1$, where $u_1 \equiv u$. The sequence u_r, for $r = 1$, has a definite applied meaning, while that for $r > 1$ is defined in a purely analytic way. The problem for $r > 1$ mimics the one for $r = 1$, namely it is to investigate the asymptotic behavior of $u_r(t)$ as $t \to \infty$ (t integer). The difference between the cases $r = 1$ and $r > 1$ is obvious from the definition, since there is no "renewal" process behind u_r if $r > 1$. This is not the biggest difference, however, from an analytic point of view.

It was realized by Ney and Wainger in [25] that the asymptotic behavior of u_r, for $r > 1$, is different from what is seen in the case of $r = 1$. For example, u_r is no longer bounded. It is also mentioned in [25] that the classical method of a difference equation satisfied by $u(t)$ does not work for $r > 1$, since "... there does not appear to be a natural analog of this equation in dimension two, mainly because the lattice points of the plain are not linearly ordered under the natural order". The same, of course, is true for higher dimensions. Using Tauberian methods in [25], it was nevertheless possible to derive the asymptotics for both u_r and

$$U_r(t) = \sum_{k<t} u_r(k), \tag{3.2}$$

for the case of $r = 2$, and under some additional conditions (which, later on, turned out to be too restrictive). In case of $r = 1$, the function U_r is the mathematical expectation of the renewal process, while no stochastic interpretation is mentioned by [25] in the case of $r > 1$. We state one of the main results of [25] for the sake of completeness.

Theorem 3.1. *Let $r = 2$. If $\mathrm{E}|X|^4 < \infty$ and $\mathrm{E}X = \mu > 0$, then*

$$u_r(k) \sim \frac{\log k}{\mu}, \qquad k \to \infty. \tag{3.3}$$

The proof of (3.3) given in [25] is based on the following relationship:

$$u_2(k) = \sum_{n=1}^{\infty} d_2(n) \mathrm{P}(S_n = k), \tag{3.4}$$

where $d_2(n)$ is the divisor function, i.e the number of divisors of n.

Ney and Wainger [25] were perhaps the first to mention explicitly the relationship between the Dirichlet divisors problem in number theory and limit theorems for multiple sums in probability theory (see [17] for other examples of such relationships). One can even say more, namely that any improvement in the solution of the Dirichlet divisors problem will result in an improvement in the asymptotics of the renewal functions for multiple sums.

The technical tools used in [25] for the proof of (3.4) are a uniform local limit theorem (to obtain an approximation of $\mathrm{P}(S_n = k)$ in (3.4)) and the following estimate for the divisor problem:

$$\sum_{n\le x} d_2(n) = x\ln(x) + (2\gamma - 1)x + o\left(x^{1/3}\right),$$

where γ is Euler's constant.

The conditions of [25] are weakened by Maejima and Mori in [24]. Moreover, the case of general r is considered in [24]. It is also true that the conditions of [24] work effectively only for $r = 2, 3$. However, they mention that "... our results might be true for $r \ge 4$ if an order estimate in the divisor problem is improved for such r". Note also that the methods of proof in [24] are the same as in [25], except for making use of better (nonuniform) estimates of the rate of convergence in the local central limit theorem, which allow them to relax the conditions of [25]. Below is the main result of [24].

Theorem 3.2. *Let* $\mathrm{E}|X|^3 < \infty$ *and* $r = 2$ *or* $r = 3$. *Then*

$$u_r(k) \sim \frac{(\log k)^{r-1}}{(r-1)!\mu}, \qquad k \to \infty. \tag{3.5}$$

Further developments of the topic are due to Galambos and Kátai in [12]-[13]. By a new method of proof, Galambos and Kátai [12] establish the following extension of the result of [24].

Theorem 3.3. *Let X be integer valued, aperiodic and such* $\mathrm{E}|X|^3 < \infty$ *and* $\mathrm{E}X = \mu > 0$. *For an explicitly given polynomial $\mathcal{P}$ of order $r-1$*

$$u_r(k) = \mathcal{P}(\ln(k/\mu)) + R_k + O(1),$$

where $R_k \to 0$ for $r = 2$ and 3. Furthermore, for arbitrary r,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=N+1}^{2N} |R_k| < \infty.$$

It is mentioned in [12] that, under the assumptions of Theorem 3.3,

$$\sum_{k>\mu} \frac{1}{k} \left| \frac{u_r(k)}{\mathcal{P}(\ln(k/\mu))} - 1 \right| < \infty.$$

In what follows we consider renewal sequences u_r constructed from random walks with different distribution functions. To distinguish between these sequences we use a subscript denoting the underlying distribution function. That is, $u_{r,F}$ denotes the renewal sequence constructed from a random field of independent identically distributed random variables $\{X(\overline{n})\}$ having a distribution function F, and

$$u_{r,\Phi}(t) = \sum_{n=1}^{\infty} d_r(n) \varphi_{n\mu, n\sigma^2}(k), \qquad \varphi_{a,b^2}(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(x-a)^2}{2b^2}}.$$

The relation of $u_{r,F}$ and $u_{r,\Phi}$ is made more precise in [13]. As a matter of fact, the following result is proved in [13].

Theorem 3.4. *Let X be integer valued, aperiodic, and such that* $\mathrm{E}X = = \mu > 0$. *If*

$$\int\limits_{|x|\geq z} x^2 \, dF(x) = O(z^{-a}), \qquad z \to \infty,$$

for a suitable $0 < a < 1$, then

$$u_{r,F}(t) = u_{r,\Phi}(t) + o(1).$$

Moreover

$$u_{r,\Phi}(t) > c_1 (\log k)^{r-1}$$

for a suitable $c_1 > 0$ which may depend on r.

Galambos, Indlekofer and Kátai [11] made a successful attempt to describe the asymptotics of $u_{r,F}$ via the asymptotics of $u_{r,\Phi}$, where Φ is the standard Gaussian distribution:

$$u_{r,F}(t) = u_{r,\Phi}(t) + o\left((\ln t)^{r-1}\right), \qquad t \to \infty \tag{3.6}$$

(t is integer).

Theorem 3.5. *Let X be integer valued, aperiodic, with finite positive mean μ and finite positive variance σ^2. Then (3.6) holds for any fixed $r \geq 2$. Consequently,*

$$\frac{u_{r,F}(t)}{u_{r,\Phi}(t)} \to 1, \qquad t \to \infty.$$

In particular, (3.5) holds for $r = 2$ *and* $r = 3$.

The idea of approximations in terms of the Gaussian distribution is common in probability theory. In the context of the renewal theorem in multidimensional time, it not only made it possible to relax conditions up to the existence of the second moment of $X(\overline{n})$ but also worked for all $r \geq 1$. It is also worthwhile mentioning that the asymptotics of $u_{r,\Phi}$ still depends on the asymptotics of the remainder term in the Dirichlet divisors problem.

4. Renewal functions and processes for multidimensional time

The results for renewal sequences u_r in multidimensional time can be completed with the corresponding counterparts for renewal functions and processes.

Renewal function in multidimensional time. In contrast to u_r, the asymptotic behavior of U_r (see definitions (3.1) and (3.2), respectively) can be evaluated for all $r \geq 1$. Moreover, the conditions for the asymptotics of U_r are weaker than those for u_r. For example, it is shown by Klesov in [19] that

$$\lim_{t\to\infty} \frac{U_r(t)}{(\ln t)^{r-1}} = \frac{1}{\mu(r-1)!} \tag{4.1}$$

provided the first moment exists and is positive. This result can also be derived from general studies on weighted renewal functions (see, e.g. [27]). The direct probabilistic methods developed in [19] allow one to prove the result for all $r \geq 1$ (not only for $r = 2$ or $r = 3$ as in Theorems 3.2 and 3.5 for u_r). A sharpening of (4.1) (also proved in [19]) reads as follows.

Theorem 4.1. *Let* $0 < \mu < \infty$ *and*

$$t(\log t)^{2(r-1)}\mathrm{P}(X \geq t) \to 0, \qquad t \to \infty. \tag{4.2}$$

Then there is a polynomial $\mathcal{P}$ *of degree* $r-1$ *such that*

$$\lim_{t\to\infty} \left[\frac{U_r(t)}{t} - \frac{1}{\mu}\mathcal{P}\left(\ln(t/\mu)\right)\right] = 0. \tag{4.3}$$

For $r = 1$, (4.3) simply coincides with (4.1) called the elementary renewal theorem in this case. For $r > 1$, (4.3) is much more informative than (4.1).

The first term in (4.3) is the same as in the case of $r = 1$, however the second one increases to infinity if $r > 1$, while it approaches the constant $1/\mu$ if $r = 1$.

The polynomial $\mathcal{P}$ is strongly related to the polynomial in the decomposition of the number of divisors in the Dirichlet problem. Namely, let

$$\mathcal{D}_r(x) = \sum_{k \le x} d_r(k).$$

As is well known (see, for example, [28], p. 263) there is a polynomial $\mathcal{Q}$ of degree $r - 1$ such that

$$\mathcal{D}_r(x) = x\mathcal{Q}(\ln(x)) + o(x^\rho), \qquad x \to \infty, \tag{4.4}$$

for some $0 < \rho < 1$. The leading coefficient of $\mathcal{Q}$ is known to be $\dfrac{1}{(r-1)!}$. The coefficients of $\mathcal{Q}$ can explicitly be evaluated from the Laurent series of the Riemann zeta function at $z = 1$ (see [23]). If $a_0, \dots, a_{r-1}$ are the coefficients of $\mathcal{Q}$ and $b_0, \dots, b_{r-1}$ are the coefficients of $\mathcal{P}$, then

$$b_m = \frac{(-1)^m}{m!} \sum_{0 \le i \le r-1} (-1)^i i! a_i, \qquad 0 \le m \le r - 1.$$

Since $b_{r-1} = a_{r-1}$, we obviously have $b_{r-1} = \dfrac{1}{(r-1)!}$ and thus (4.3) implies (4.1). Condition (4.2) is much weaker than the existence of the second moment. Moreover, the latter condition is even weaker that the main assumption (2.3) in the case of $r = 1$.

The approach of [19] differs from those in the preceding papers dealing with multidimensional time. It is based on a direct application of probabilistic methods that allow to reduce the problem for the renewal function to a study of the law of large numbers for the original sums $S(\overline{n})$.

A closely related problem is considered by Klesov and Steinebach in [22]. Namely, let D be a domain in $\mathbb{N}^r$ and

$$U_D(t) = \sum_{\overline{n} \in D} \mathrm{P}(S(\overline{n}) < t).$$

To relate the new notation with the old one we note that $U_r = U_{\mathbb{N}^r}$. The renewal function U_D is defined with respect to a random walk whose multidimensional time is restricted to the domain D. Put

$$A_D(t) = \operatorname{card}\{\overline{n} \in D : \ |\overline{n}| \le t\}. \tag{4.5}$$

Then under appropriate conditions on $X(\overline{n})$ and on the function A

$$\lim_{t\to\infty} \frac{U_D(t)}{A_D(t/\mu)} = 1. \tag{4.6}$$

The proof in [22] is completely "probabilistic" in the sense that (4.6) is derived from the law of large numbers for sums $S(\overline{n})$ (similarly to [19]). Conditions posed on A are discussed below.

Renewal process in multidimensional time. Up to now we were talking about the renewal function U_r and its density u_r. Nothing has been said about the renewal process itself. The reason is that relation (2.1) is meaningless for $r > 1$, since the set $\mathbb{N}^r$ is not linearly ordered, and thus the definition of the renewal process is not straightforward if $r > 1$. Another representation of the renewal process N via the sum of indicator functions (easily obtained for the case of $r = 1$) serves as the definition for an arbitrary $r \geq 1$:

$$N_r(t) = \operatorname{card}\{\overline{n} : \ S(\overline{n}) < t\} = \sum_{\overline{n}\in\mathbb{N}^r} \mathbb{I}S(\overline{n}) < t.$$

The asymptotics of N_r defined in this way is studied by Klesov and Steinebach in [21]. Again the proof in [21] is based on the idea of reducing the problem to known results in probability theory, namely to the strong law of large numbers for multiple sums. An expansion like (4.3) is also obtained in [21], however it holds for $r = 2$ and $r = 3$, while, for $r > 3$, it holds only under a (yet unproved) conjecture on the underlying rate of approximation in the Dirichlet divisors problem. However, a "rough" asymptotics of N_r is available for all r.

Theorem 4.2. *Let* $0 < \mu < \infty$ *and*

$$\mathrm{E}X\left(\ln^+ X\right)^{r-1} < \infty.$$

Then

$$\lim_{t\to\infty} \frac{N_r(t)}{t\left(\ln(t)\right)^{r-1}} = \frac{1}{\mu(r-1)!} \qquad a.s.$$

Just for the sake of completeness we state below a particular case of the result of [21], for $r = 2$, where the conjecture mentioned above is known to be true.

Theorem 4.3. *Let* $r = 2$. *Assume that* $0 < \mu < \infty$ *and*

$$\mathrm{E}X^v\left(\ln^+ X\right)^{r-1} < \infty \qquad \textit{for some} \ \ 1 \leq v < 2.$$

Then

$$\lim_{t\to\infty} \frac{1}{t^{1/v}} \left[\frac{N_r(t)}{\ln(t)} - \frac{t}{\mu} - \left(2\gamma - 1 + \frac{\ln(\mu)}{\mu}\right) \frac{t}{\ln(t)} \right] = 0 \qquad a.s.$$

The case of $r = 1$ in Theorem (4.3) is contained in (2.7). A similar result holds for all $r \geq 1$. The difference between $r \leq 3$ and $r > 3$ is that not all $1 \leq v < 2$ are allowed in the latter case. What can be said is that there exists a $v_0 \leq 2$ such that, if $1 \leq v < v_0$ and the assumptions of Theorem 4.3 hold, then

$$\lim_{t\to\infty} \frac{1}{t^{1/v}} \left[\frac{N_r(t)}{\ln(t)} - \frac{t}{\mu} \mathcal{P}(\ln(t/\mu)) \right] = 0 \qquad \text{a.s.}$$

for some polynomial $\mathcal{P}$ of degree $r - 1$.

Restricted domain. Let $D \subseteq \mathbb{N}^r$ and put

$$N_D(t) = \operatorname{card}\{\overline{n} \in D : \ S(\overline{n}) < t\} = \sum_{\overline{n} \in D} \mathbb{I}\{S(\overline{n}) < t\}. \tag{4.7}$$

Since there will be no confusion in notation we will omit the subscript D and write $N(t)$ rather than $N_D(t)$. The result below is due to Indlekofer and Klesov [18]. It holds for all $r \geq 1$ and all domains D for which the function A_D defined by (4.5) is *pseudo regularly varying* (see the defining property (4.9) below). The subscript D is also omitted for the function A.

Theorem 4.4. *Assume that X, $\{X(\overline{n}),\ \overline{n} \in \mathbb{N}^r\}$ are nonnegative independent identically distributed random variables such that condition (2.3) holds together with*

$$\mathrm{E}X \left(\ln^+ X\right)^{r-1} < \infty. \tag{4.8}$$

Let $D \subseteq \mathbb{N}^r$ be an infinite domain of $\mathbb{N}^r$ for which

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \frac{A(ct)}{A(t)} = 1, \tag{4.9}$$

where the function A is defined by (4.5) (we omit the subscript D). Then

$$\lim_{t\to\infty} \frac{N(t)}{A(t/\mu)} = 1 \qquad a.s.,$$

where the process N is defined by (4.7) (we omit the subscript D).

For $D = \mathbb{N}^r$, Theorem 4.4 reduces to Theorem 4.2, since

$$A(t) \sim \frac{1}{(r-1)!} t(\ln t)^{r-1}, \qquad t \to \infty,$$

(see (4.4)). The latter is a rough estimate in the Dirichlet divisors problem. Indeed, in this case, $A(t)$ represents the number of solutions (in $\mathbb{N}^r$) of the inequality $|\overline{n}| \leq t$. Several proofs of these results are known in the literature; an elementary proof is given in [22].

5. Equivalences in renewal theorems

Two branches of probability theory, namely random walks and renewal processes, were often developing independently of each other, although many results look very similar in both fields. In this section, we collect some of the equivalences between laws of large numbers and laws of the iterated logarithm for random walks and renewal processes obtained by Gut et al. in [15].

Renewal processes with linear drift. This case relies on condition (2.2). With $\mathrm{E}X = \mu$ we get $\mathrm{E}S_n = n\mu$. The latter property is one of the reasons for the notion "linear drift" in this case. Another explanation comes from the strong law of large numbers in (2.6).

It turns out that the sufficient moment conditions in (a)-(e) are, in fact, also necessary. This has been obtained in [15] by showing that these strong limit theorems hold simultaneously for both processes $\{S_n,\ n \geq 1\}$ and $\{N_t,\ t \geq 0\}$. So, necessary and sufficient moment conditions, which are well-known for $\{S_n, n \geq 1\}$, carry over to the renewal process $\{N_t, t \geq 0\}$, too. The results essentially follow from the fact that the two processes are inverses of each other. Hence it is natural to expect that strong (or weak) laws should hold simultaneously, and that even their normalizations should be inverses of each other. The latter idea has been extensively exploited in the monograph by [14]. An inspection of the proofs in [15] actually shows that the results, in fact, hold simultaneously for arbitrary positive summands and for (almost) arbitrary nonnegative summands. Earlier references developing this technique are e.g. [29], who proved invariance equivalences for the LIL for processes and their inverses, and [16] who derived functional central limit theorem equivalences for renewal processes.

Theorem 5.1. (SLLN) [15] *The following statements are equivalent:*

$$\lim_{t\to\infty}\frac{N(t)}{t}=a \qquad a.s. \qquad (\exists\, 0<a<\infty);$$
$$\lim_{n\to\infty}\frac{S(n)}{n}=\frac{1}{a} \qquad a.s. \qquad (\exists\, 0<a<\infty);$$
$$\mathrm{E}X=\frac{1}{a} \qquad (\exists\, 0<a<\infty).$$

In order to prove equivalence of the first two assertions of Theorem 5.1 the assumptions on the independence and identical distribution are not necessary. Moreover, the only assumption needed is that

$$\lim_{n\to\infty}\frac{R_n}{n}=0 \qquad \text{a.s.,} \tag{5.1}$$

together with $N(t)\to\infty$ a.s. $(t\to\infty)$ and $S_n\to+\infty$ a.s. $(n\to\infty)$, where

$$R_n=\begin{cases}\max\{k:\ X_{n+1}=\cdots=X_{n+k}\}, & \text{if } X_{n+1}=0,\\ 0, & \text{if } X_{n+1}>0.\end{cases}$$

In the case of independent identically distributed random variables condition (5.1) is easily checked.

Moreover, if, in particular, the X_n's are strictly positive, then we actually have the following, more general results.

Theorem 5.2. *Let $\{X_n\}$ be arbitrary strictly positive random variables. The following statements are equivalent:*

$$\lim_{t\to\infty}\frac{N(t)}{t}=a \qquad a.s. \qquad (\exists\, 0<a<\infty);$$
$$\lim_{n\to\infty}\frac{S(n)}{n}=\frac{1}{a} \qquad a.s. \qquad (\exists\, 0<a<\infty).$$

The assumption of positivity can be relaxed.

Theorem 5.3. *If $\{X_n\}$ are arbitrary nonnegative random variables such that (5.1) holds, then the following statements are equivalent:*

$$\lim_{t\to\infty}\frac{N(t)}{t}=a \qquad a.s. \qquad (\exists\, 0<a<\infty);$$
$$\lim_{n\to\infty}\frac{S(n)}{n}=\frac{1}{a} \qquad a.s. \qquad (\exists\, 0<a<\infty).$$

There is also equivalence of the rate of convergences.

Theorem 5.4. (Marcinkiewicz–Zygmund SLLN) *Let $1 \le v < 2$. The following statements are equivalent:*

$$\lim_{t\to\infty} \frac{N(t) - ta}{t^{1/v}} = 0 \qquad a.s. \qquad (\exists\ 0 < a < \infty);$$

$$\lim_{n\to\infty} \frac{S(n) - n\frac{1}{a}}{n^{1/v}} = 0 \qquad a.s. \qquad (\exists\ 0 < a < \infty);$$

$$\mathrm{E}X = \frac{1}{a}, \qquad \mathrm{E}X^r < \infty \qquad (\exists\ 0 < a < \infty).$$

Furthermore, laws of the iterated logarithm for random walks and renewal processes are also equivalent.

Theorem 5.5. (One-sided LIL) *The following statements are equivalent:*

$$\limsup_{t\to\infty} \frac{N(t) - ta}{\sqrt{2t \log\log t}} = b \qquad a.s. \qquad (\exists\ 0 < a, b < \infty);$$

$$\liminf_{n\to\infty} \frac{S_n - n\frac{1}{a}}{\sqrt{2n \log\log n}} = -\frac{b}{a^{3/2}} \qquad a.s. \qquad (\exists\ 0 < a, b < \infty);$$

$$\mathrm{E}X = \frac{1}{a}, \qquad \mathrm{var}[X] = \frac{b^2}{a^3} \qquad (\exists\ 0 < a, b < \infty);$$

$$\liminf_{t\to\infty} \frac{N(t) - ta}{\sqrt{2t \log\log t}} = -b \qquad a.s. \qquad (\exists\ 0 < a, b < \infty);$$

$$\limsup_{n\to\infty} \frac{S_n - n\frac{1}{a}}{\sqrt{2n \log\log n}} = \frac{b}{a^{3/2}} \qquad a.s. \qquad (\exists\ 0 < a, b < \infty).$$

Remark 5.1. By combining the statements of Theorem 5.5, it is obvious that the following "two-sided" versions are also equivalent:

$$\limsup_{t\to\infty} \frac{|N(t) - ta|}{\sqrt{2t \log\log t}} = b \qquad \text{a.s.} \qquad (\exists\ 0 < a, b < \infty);$$

$$\limsup_{n\to\infty} \frac{|S_n - n\frac{1}{a}|}{\sqrt{2n \log\log n}} = \frac{b}{a^{3/2}} \qquad \text{a.s.} \qquad (\exists\ 0 < a, b < \infty).$$

Renewal processes without linear drift. The equivalence statements concerning the SLLN, in fact, carry over to the case of infinite expectation

$EX = +\infty$. Marcinkiewicz-Zygmund and LIL type strong limit theorems, however, turn out to be entirely different here.

Theorem 5.6. *The following statements are equivalent:*

$$\lim_{t\to\infty} \frac{N(t)}{t} = 0 \qquad a.s.,$$

$$\lim_{n\to\infty} \frac{S_n}{n} = +\infty \qquad a.s.,$$

$$EX = +\infty.$$

Theorem 5.7. *Let $v \geq 1$. The following statements are equivalent:*

$$\lim_{t\to\infty} \frac{N(t)}{t^{1/v}} = 0 \qquad a.s.,$$

$$\lim_{n\to\infty} \frac{S_n}{n^v} = +\infty \qquad a.s.,$$

$$\int_1^\infty \frac{1}{t} P(N(t) \geq ct^{1/v})\, dt < \infty \qquad (\forall\ 0 < c < \infty);$$

$$\sum_{n=1}^\infty \frac{1}{n} P(S_n \leq cn^v) < \infty \qquad (\forall\ 0 < c < \infty).$$

Remark 5.2. If one of the conditions of Theorem 5.7 is satisfied (and, hence, all of them), then $EX^{1/v} = +\infty$, but, in general, the converse does not hold for $v > 1$.

Theorem 5.8. *The following statements are equivalent:*

$$\limsup_{t\to\infty} \frac{N(t)}{\sqrt{t \log\log t}} = b_1 \qquad a.s. \quad (\exists\ 0 \leq b_1 < \infty); \tag{6a}$$

$$\liminf_{n\to\infty} \frac{S_n \log\log S_n}{n^2} = \frac{1}{b_1^2} \qquad a.s. \quad (\exists\ 0 \leq b_1 < \infty); \tag{6b}$$

$$\int_e^\infty \frac{1}{t} P(N(t) \geq c\sqrt{t \log\log t})\, dt < \infty \qquad (\exists\ 0 < c < \infty); \tag{6c}$$

(6*d*) $$\sum_{n=9}^{\infty} \frac{1}{n} \mathrm{P}\left(S_n \le c\frac{n^2}{\log\log n}\right) < \infty \qquad (\exists\ 0 < c < \infty).$$

Remark 5.3. a) If one of the conditions of Theorem 5.8 is satisfied (and, hence, all of them), then $\mathrm{E}\sqrt{X \log^+ \log^+ X} = +\infty$ but, in general, the converse does not hold.

b) The constant b_1 is determined as follows: $b_1 = c_1^{-1/2}$, where

$$c_1 = \sup\{0 < c < \infty : \text{(6d)holds}\}.$$

c) Note that, in case $b_1 = 0$, the equivalence of (6a) and (6b) means that

$$\lim_{t\to\infty} \frac{N(t)}{\sqrt{t\log\log t}} = 0 \quad \text{a.s.} \quad \text{iff} \quad \lim_{n\to\infty} \frac{S_n \log\log S_n}{n^2} = \infty \quad \text{a.s.}$$

Theorem 5.9. *The following statements are equivalent:*

(7*a*) $$\liminf_{t\to\infty} \frac{N(t)}{\sqrt{t\log\log t}} = b_0 \qquad a.s. \quad (\exists\ 0 < b_0 \le \infty);$$

(7*b*) $$\limsup_{n\to\infty} \frac{S_n \log\log S_n}{n^2} = \frac{1}{b_0^2} \qquad a.s. \quad (\exists\ 0 < b_0 \le \infty);$$

(7*c*) $$\mathrm{E}\sqrt{X \log^+ \log^+ X} < \infty.$$

Remark 5.4. In fact, if one of the conditions of Theorem 5.9 is satisfied (and, hence, all of them), then necessarily $b_0 = +\infty$ in the statements (7a) and (7b), where $1/\infty = 0$. This is an immediate consequence of Feller's (1946) strong law of large numbers. Moreover, liminf and limsup in (7a) and (7b), respectively, turn into lim's again.

Remark 5.5. The proofs of Theorems 5.1-5.9 make use of some classical results for partial sums. For some recent equivalence statements in the general case confer also [9].

6. Generalized renewal processes

A possible interpretation of the sequence $\{X_n, n \geq 1\}$ is that X_n represents the time between the $(n-1)$-th and n-th replacement (renewal) of (say) a machine part, so that $N(t)$ counts the number of replacements (renewals) up to time t.

In Section 5 we discussed limit theorems for the counting process $\{N(t),\ t \geq 0\}$. It turned out they are consequences of their corresponding counterparts for the underlying random walk via the following "duality":

$$(6.1)\quad \{N(t) = n\} \iff \{S_n < t,\ S_{n+1} \geq t\} \quad \text{for all } t \geq 0 \text{ and } n \in \mathbf{N}_0.$$

This method works perfectly under condition (2.2). Much less is known about renewal processes for which either identical distribution, or independence, or nonnegativity, or all of these assumptions are dropped. Note also that certain regularity assumptions are sometimes crucial for the applicability of a duality argument. There are situations in which a limit theorem for the renewal process is almost immediate from its partial sum counterpart. But there are other cases where the desired inversion requires more sophisticated techniques. Finally, there are also examples in which a duality argument does not work at all because either the partial sum sequence satisfies a certain limit theorem, but not so its corresponding renewal process, or vice versa.

Assuming that $S_{n+1} \sim S_n$, condition (6.1) can be rewritten in a nonrigorous way as $S(N(t)) \approx t$, where $S(x) = S_x$ for integer x. This observation suggests that S and N are perhaps "close" to be "inverses" to each other. Two questions arise:

(i) What is a rigorous explanation of the word "inverses" above?
(ii) What means "close"?

Some steps toward answering these questions are made by Klesov et al. in [20], where a general approach to deriving limit theorems for "renewal processes" from their corresponding counterparts for the underlying "partial sum sequence" is developed. In this section, we collect some results obtained in [20].

In order to avoid confusion with the independent identically distributed case, we change notation from now on and let $\{Z_n, n \geq 0\}$ be a *general* sequence of real-valued random variables. We then define a *general renewal process* $\{N(t), t \geq 0\}$ pointwise as

$$(6.2)\qquad N(t) = \sum_{n=1}^{\infty} I\{Z_n \leq t\}, \qquad t \geq 0.$$

In case of $Z_n \to \infty$ a.s., as $n \to \infty$, $N(t)$ is finite a.s. for every $t \ge 0$, since only a finite number of summands in (6.2) is nonzero. Along with $\{N(t), t \ge 0\}$ we introduce two other *general renewal processes,* that is

$$
\begin{aligned}
M(t) &= \sup\{n \ge 0 : \max(Z_0, Z_1, \dots, Z_n) \le t\} = \\
&= \sum_{n=1}^{\infty} I\{\max(Z_0, Z_1, \dots Z_n) \le t\}, \qquad t \ge 0,
\end{aligned} \tag{6.3}
$$

$\sup\emptyset = 0$, i.e. $M(t) + 1$ is the *first-passage time* of the sequence $\{Z_n, n \ge 0\}$ from the set $(-\infty, t]$, and

$$
\begin{aligned}
L(t) &= \sup\{n \ge 0 : Z_n \le t\} = \\
&= \sum_{n=1}^{\infty} I\{\inf(Z_n, Z_{n+1}, \dots) \le t\}, \qquad t \ge 0,
\end{aligned} \tag{6.4}
$$

i.e. $L(t) + 1$ is the *last-exit time* of $\{Z_n, n \ge 0\}$ from $(-\infty, t]$. Similar to $N(t)$, also $M(t)$ and $L(t)$ are both finite a.s. for every $t \ge 0$, provided $Z_n \to \infty$ a.s.

Formally speaking, both $\{M(t)\}$ and $\{L(t)\}$ are particular cases of $\{N(t)\}$ with Z_n replaced by $\max(Z_0, Z_1, \dots, Z_n)$ and $\inf\{Z_n, Z_{n+1}, \dots,)$, respectively. Moreover,

$$M(t) \le N(t) \le L(t).$$

Note also the following inequalities being true for finite $M(t)$ and $L(t)$, respectively:

$$Z_{M(t)} \le t < Z_{M(t)+1}, \tag{6.5}$$

$$Z_{L(t)} \le t < Z_{L(t)+1}. \tag{6.6}$$

Remark 6.1. If $0 = Z_0 \le Z_1 \le Z_2 \le \cdots$, obviously

$$M(t) = N(t) = L(t).$$

But if (say) $Z_n > Z_{n+1}$ for some n, then for $Z_{n+1} \le t < Z_n$,

$$\mathbb{I}\{\max(Z_0, Z_1, \dots, Z_{n+1}) \le t\} = 0, \qquad \mathbb{I}\{Z_{n+1} \le t\} = 1,$$

where $\mathbb{I}\{A\}$ is the indicator of a random event A. Therefore $M(t) < N(t)$, and also

$$\mathbb{I}\{Z_n \le t\} = 0, \qquad \mathbb{I}\{\inf(Z_n, Z_{n+1}, \dots) \le t\} = 1,$$

so $N(t) < L(t)$.

Strong laws of large numbers. Assuming a strong law of large numbers for $\{Z_n, n \geq 0\}$, corresponding results for the general renewal processes are immediate from the inequalities (6.5) and (6.6) if the normalizing sequence satisfies certain regularity condition.

Theorem 6.1. *Assume*

$$\frac{Z_n}{a_n} \to 1 \qquad a.s., \tag{6.7}$$

where $\{a_n, n \geq 1\}$ is a nonrandom sequence such that $a_n \to \infty$ as $n \to \infty$ and

$$\frac{a_{n+1}}{a_n} \to 1. \tag{6.8}$$

Then, as $t \to \infty$,

$$\frac{a_{M(t)}}{t} \to 1 \qquad a.s.,$$
$$\frac{a_{L(t)}}{t} \to 1 \qquad a.s.$$

Moreover, if $\{a_n, n \geq 1\}$ is nondecreasing, also

$$\frac{a_{N(t)}}{t} \to 1 \qquad a.s.$$

Introducing some more notation we obtain the strong laws of large numbers for the renewal processes. If $\{a_n, n \geq 0\}$ is strictly increasing, let $\{a(t) : t \geq 0\}$ be an extension of it, i.e. $a(n) = a_n$ for all $n = 0, 1, 2, \ldots$, such that

$$\begin{gathered} a(\cdot) \text{is continuous and strictly increasing with} \\ a(t) \to \infty, \quad t \to \infty. \end{gathered} \tag{6.9}$$

Define the generalized inverse function

$$a^{-1}(u) = \inf\{t : \ a(t) = u\}, \qquad u > u_0 = a_0.$$

Obviously $a^{-1}(\cdot)$ is also continuous and strictly increasing, with $a^{-1}u \to \infty$ as $u \to \infty$.

Assume

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \left| \frac{a^{-1}((1 \pm \varepsilon)t)}{a^{-1}(t)} - 1 \right| = 0. \tag{6.10}$$

Then, the following strong laws hold true:

Corollary 6.1. *Assume (6.7) and (6.8), (6.9), (6.10). Then, as $t \to \infty$,*

$$\frac{M(t)}{a^{-1}(t)} \to 1 \quad a.s., \tag{6.11}$$

$$\frac{L(t)}{a^{-1}(t)} \to 1 \quad a.s., \tag{6.12}$$

$$\frac{N(t)}{a^{-1}(t)} \to 1 \quad a.s. \tag{6.13}$$

Remark 6.2. Condition (6.10) can be rewritten in several equivalent forms. Denote a^{-1} by f. In terms of f, (6.10) is rewritten as follows

$$\lim_{c\to 1} \limsup_{t\to\infty} \left| \frac{f(ct)}{f(t)} - 1 \right| = 0. \tag{6.14}$$

In turn, (6.14) is equivalent to

$$\lim_{c\downarrow 1} f^*(c) = 1, \tag{6.15}$$

$$\lim_{c\uparrow 1} f_*(c) = 1, \tag{6.16}$$

where

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}.$$

Other sets of conditions, equivalent to (6.14), are

$$\lim_{c\downarrow 1} f^*(c) = \lim_{c\downarrow 1} f_*(c) = 1,$$

and

$$\lim_{c\uparrow 1} f^*(c) = \lim_{c\uparrow 1} f_*(c) = 1$$

(see [1]). If one assumes that f is nondecreasing, then conditions (6.15) and (6.16) become equivalent and each of them is equivalent to (6.14).

Example 6.1. Any regularly varying function a of positive index satisfies condition (6.10). Any quickly growing function, like $a(t) = e^t$ also satisfies (6.10). A less trivial example of a function f satisfying (6.14) is given by

$$f(t) = \begin{cases} 0, & t = 0, \\ t^\alpha e^{\sin(\ln(t))}, & t > 0. \end{cases}$$

Note that f is not regularly varying for $\alpha > 0$, however f is increasing, unbounded, and continuous if $\alpha \geq 1$. On the other hand, f is not monotone if $0 < \alpha < 1$, although it is still continuous and unbounded in this case.

So, the growth condition (6.10) on the normalizing sequence $\{a_n,\ n \geq 1\}$ is crucial for deriving the strong laws of Corollary 6.1 from their counterparts in Theorem 6.1. Yet, this condition can be avoided, and thus the regularity assumptions can be weakened, by applying a totally different technique of proof. Such a method was introduced in [21] for the case of renewal processes constructed from random walks with multidimensional time.

Theorem 6.2. *Assume (6.7), (6.9) and (6.10). Then, as $t \to \infty$, assertions (6.11)-(6.13) retain.*

Remark 6.3. Unfortunately, there are also situations in which the inversion techniques applied in Theorems 6.1-6.2 and Corollary 6.1 cannot work at all. Consider, for instance, a max-scheme of independent identically distributed random variables X, $\{X_n,\ n \geq 1\}$ with distribution function $F(t) = \mathrm{P}(X \leq t)$, $t \in \mathbf{R}$. For $Z_n = \max(X_1, \ldots, X_n)$, $n \geq 1$, $Z_0 = 0$, the corresponding renewal processes $\{M_t,\ t \geq 0\}$, $\{N_t,\ t \geq 0\}$ and $\{L_t,\ t \geq 0\}$ coincide. Moreover, for any $t \geq 0$, $N(t)$ has a geometric distribution, i.e.

$$\begin{aligned} \mathrm{P}(N(t) = n) =& \mathrm{P}(\max(X_1, \ldots, X_n) \leq t,\ X_{n+1} > t) = F^n(t)(1 - F(t)), \\ &\mathrm{P}(N(t) \geq n) = F^n(t), \qquad n = 0, 1, \ldots . \end{aligned}$$

So, if $F(t) < 1$ for all $t \geq 0$, then for all fixed $x \geq 0$

$$\mathrm{P}\left(N(t) > x/(1 - F(t))\right) = \mathrm{P}\left(N(t) \geq [x/(1 - F(t))] + 1\right) =$$

$$= \exp\left\{x(\log F(t))/(1 - F(t)) + O(1) \log F(t)\right\}.$$

Since $\log(1 - x)/x \to -1$, as $x \to 0$, the right-hand side tends to $\exp\{-x\}$ as $t \to \infty$. Hence

$$N(t)(1 - F(t)) \overset{w}{\leftarrow} E, \qquad t \to \infty,$$

where E has an exponential Exp(1)-distribution. In view of this fact it is impossible that

$$\frac{N(t)}{b(t)} \to 1 \qquad \text{a.s.,} \quad t \to \infty,$$

for any (nonrandom) normalizing family $\{b(t), t > 0\}$. Because otherwise, for each $\varepsilon > 0$,

$$\mathrm{P}(N(t) > (1 + \varepsilon)b(t)) \to 0, \qquad t \to \infty,$$

which requires $b(t)(1 - F(t)) \to \infty$, $t \to \infty$, by the consideration above. This, however, in turn implies

$$\mathrm{P}(N(t) \le (1 - \varepsilon)b(t)) \to 1, \qquad t \to \infty,$$

so that not even a *weak* law of large numbers applies to $\{N_t,\ t \ge 0\}$.

Nevertheless, the underlying "renewal sequence" $\{Z_n,\ n \ge 1\}$ may satisfy a *strong* law of large numbers. For example, in case of an Exp(1)-distribution, i.e. $F(t) = 1 - e^{-t}$ for $t \ge 0$, and $F(t) = 0$ otherwise, it is well-known (cf. [10]) that

$$\frac{Z_n}{\log n} = \frac{\max(X_1, \dots, X_n)}{\log n} \to 1 \qquad \text{a.s.}, \quad n \to \infty.$$

Note that, in the latter case, all assumptions of Theorems 6.1-6.2 and Corollary 6.1 are fulfilled with $a_n = \log n$, $a(t) = \log t$, $a^{-1}(t) = e^t$, with the exception of (6.10). So, the latter condition cannot be dropped in general.

Another example would be $F(t) = \Phi(t)$, $t \in \mathbf{R}$, a standard normal distribution function, in which case

$$\frac{Z_n}{\sqrt{2 \log n}} = \frac{\max(X_1, \dots, X_n)}{\sqrt{2 \log n}} \to 1 \qquad \text{a.s.}, \quad n \to \infty,$$

(cf. [10]). Here $a^{-1}(t) = \exp\{t^2/2\}$ also violates assumption (6.10).

So, there are (renewal) sequences $\{Z_n,\ n \ge 0\}$ satisfying a SLLN for which their corresponding renewal processes $\{N_t,\ t \ge 0\}$, $\{M_t,\ t \ge 0\}$ and $\{L_t,\ t \ge 0\}$ do not possess any (nondegenerate) strong limiting behavior.

Remark 6.4. Just for the sake of completeness we should like to mention that there are also cases in which the renewal process satisfies a SLLN, but not so its sequence of renewal times. Consider, for example, a nonhomogeneous Poisson process $\{N_t,\ t \ge 0\}$ with cumulative intensity function $\{\lambda_t,\ t \ge 0\}$, i.e. $\lambda(t) = \mathrm{E}N(t)$, $t \ge 0$. If e.g. $\lambda(t)$ is continuous and strictly increasing to infinity, it is well-known that

$$\{N_t,\ t \ge 0\} \overset{\mathcal{D}}{=} \{\widetilde{N}(\lambda(t)),\ t \ge 0\},$$

where $\{\widetilde{N}_t,\ t \ge 0\}$ is a homogeneous Poisson process with renewal times $\widetilde{S}_0 = 0$, $\widetilde{S}_n = X_1 + \ldots + X_n$, $n \ge 1$, based on a sequence $\{X_n,\ n \ge 1\}$ of independent identically distributed Exp(1)-random variables.

Choose

$$\lambda(t) = \begin{cases} \log t, & t \ge e, \\ \\ t/e, & 0 \le t \le e. \end{cases}$$

Then, from the SLLN for $\{\widetilde{N}(t),\ t \geq 0\}$, as $t \to \infty$,

$$\frac{N(t)}{\log t} = \frac{\widetilde{N}(\log t)}{\log t} \to 1 \qquad \text{a.s.}$$

But, since $Z_n = \exp(\widetilde{S}_n)$ are the renewal times of $\{N(t),\ t \geq 0\}$, it holds, in view of the LIL for the partial sums $\{\widetilde{S}_n,\ n \geq 1\}$, that

$$\frac{Z_n}{e^n} = \exp\left\{\widetilde{S}_n - n\right\}$$

oscillates between 0 and $+\infty$ a.s., as $n \to \infty$.

Convergence rate results. It may also be interesting to collect general conditions under which convergence rate statements hold for the laws of large numbers in Theorems 6.1-6.2 and Corollary 6.1.

Theorem 6.3. *Assume that, as $n \to \infty$,*

$$\frac{Z_n - a_n}{b_n} \to 0 \qquad a.s., \tag{6.17}$$

where

$$a_n \to \infty, \quad \textit{but} \quad a_n - a_{n-1} = o(b_n), \tag{6.18}$$

$$0 < b_n \to \infty, \quad \textit{but} \quad b_n = o(a_n), \tag{6.19}$$

$$\frac{b_{n+1}}{b_n} = O(1). \tag{6.20}$$

Then, as $t \to \infty$,

$$\frac{a_{M(t)} - t}{b_{M(t)}} \to 0 \qquad a.s., \tag{6.21}$$

$$\frac{a_{L(t)} - t}{b_{L(t)}} \to 0 \qquad a.s. \tag{6.22}$$

Moreover, if $\{a_n\}$ is nondecreasing, then

$$\frac{a_{N(t)} - t}{\max(b_{M(t)}, b_{L(t)})} \to 0 \qquad a.s. \tag{6.23}$$

Now assume that $a(\cdot)$ has a continuous derivative $a'(\cdot)$ on (t_0, ∞) satisfying

$$a'(t) \asymp a'(s) \quad \text{if} \quad t \asymp s, \tag{6.24}$$

i.e. $|a'(t)/a'(s)|$ is bounded away from 0 and ∞, if $|t/s|$ is bounded away from 0 and ∞, as $t, s \to \infty$. Moreover, let $\{b_t,\ t \geq 0\}$ be an extension of $\{b_n\}$ such that

$$b(t) \asymp b(s) \quad \text{if} \quad t \asymp s. \tag{6.25}$$

Corollary 6.2. *Assume (6.17) together with (6.9), (6.10), (6.18), (6.19), (6.24) and (6.25). Then, as $t \to \infty$,*

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}\big(M(t) - a^{-1}(t)\big) \to 0 \qquad a.s., \tag{6.26}$$

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}\big(N(t) - a^{-1}(t)\big) \to 0 \qquad a.s., \tag{6.27}$$

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}\big(L(t) - a^{-1}(t)\big) \to 0 \qquad a.s. \tag{6.28}$$

Corollary 6.3. *Assume that, for some $a > 0$ and $r > 1$, as $n \to \infty$,*

$$\frac{Z_n - na}{n^{1/r}} \to 0 \qquad a.s. \tag{6.29}$$

Then, as $t \to \infty$,

$$\frac{M(t) - t/a}{t^{1/r}} \to 0 \qquad a.s., \tag{6.30}$$

$$\frac{N(t) - t/a}{t^{1/r}} \to 0 \qquad a.s., \tag{6.31}$$

$$\frac{L(t) - t/a}{t^{1/r}} \to 0 \qquad a.s. \tag{6.32}$$

Similar to Theorem 6.2 the regularity assumptions of Corollary 6.2 can be considerably weakened if a different technique of proof is applied.

Theorem 6.4. *Assume (6.17) together with*

$$b(t) \uparrow \infty \quad as \quad t \to \infty, \tag{6.33}$$

$$\frac{a(t)}{b(t)} \uparrow \quad as \quad t \to \infty, \tag{6.34}$$

$$a'\left(a^{-1}(t)\right) \asymp a'\left(a^{-1}(s)\right) \quad if \quad t \asymp s, \tag{6.35}$$

$$b\left(a^{-1}(t)\right) \asymp b\left(a^{-1}(s)\right) \quad if \quad t \asymp s, \tag{6.36}$$

where $a(t)$ be continuously differentiable on (t_0, ∞) with

$$a'(t) = o(b(t)) \quad as \quad t \to \infty. \tag{6.37}$$

Then, as $t \to \infty$, assertions (6.26)-(6.28) retain.

Example 6.2. There are still situations in which the assumptions of Theorems 6.2 and 6.4 are not fulfilled, but yet a strong law of large numbers may be available. Consider, for instance, a sequence $\{Z_n,\ n \geq 1\}$ satisfying

$$\frac{Z_n}{\log n} \to 1 \qquad a.s.$$

as $n \to \infty$, and assume a rate of convergence therein, e.g.

$$\limsup_{n \to \infty} \left| \frac{Z_n - \log n}{b(n)} \right| \leq B,$$

with some nonrandom constant $B > 0$. Let the function $\{b(t),\ t > 0\}$ be such that, for any $A > B$,

$$a_{\pm}(t) = \log t \pm Ab(t)$$

have inverse functions (say) $a_{\pm}^{-1}(t)$ satisfying

$$a_{\pm}^{-1}(t) = e^t \mp o\left(e^t\right)$$

as $t \to \infty$. Then the SLLN for $\{N_t,\ t \geq 0\}$ retains, i.e.

$$\lim_{t\to\infty} \frac{N(t)}{e^t} = 1 \qquad \text{a.s.}$$

The proof is similar to that of Theorem 6.4 by dividing $N(t) = \sum_{n=1}^{\infty} I\{Z_n \leq t\}$ into four subseries according to the conditions $n \leq a_+^{-1}(t)$, $a_+^{-1}(t) < n \leq e^t$, $e^t < n \leq a_-^{-1}(t)$, $a_+^{-1}(t) < n$. Details are omitted.

Note that the function $a(t) = \log t$ with $a^{-1}(t) = e^t$ violates conditions (6.10) and (6.35), so that neither Theorem 6.2 nor Theorem 6.4 is applicable in this situation. Nevertheless, a SLLN for the renewal process holds true.

Below we demonstrate possible applications of the above results. Some further examples can be found in [20].

Example 6.3. (Nonlinear renewal process: cf. [14], pp. 133-138) Consider, as before, a sequence X, $\{X_n,\ n \geq 1\}$ of independent identically distributed random variables with $\mathrm{E}X = a > 0$, but set now $Z_n = S_n/\alpha(n)$, where $\{\alpha(t),\ t > 0\}$ is a positive, continuous function such that

$$\frac{t}{\alpha(t)} \uparrow \infty, \qquad t \uparrow \infty.$$

For example, the first-passage time

$$M(t) + 1 = \inf\{n : Z_n > t\} = \inf\{n : S_n > t\alpha(n)\},$$

$\inf \emptyset = +\infty$, is of some statistical importance in sequential analysis and plays a key role in what is called nonlinear renewal theory (cf., e.g. [30], [26]). Now, by Theorem 6.2, if $a(t) = ta/\alpha(t)$ with inverse function $a^{-1}(t)$ satisfying (6.10), then, as $t \to \infty$,

$$\lim_{t\to\infty} \frac{M(t)}{a^{-1}(t)} = 1, \qquad \lim_{t\to\infty} \frac{N(t)}{a^{-1}(t)} = 1, \qquad \lim_{t\to\infty} \frac{L(t)}{a^{-1}(t)} = 1 \quad a.s.,$$

where $L(t)$, $M(t)$, $N(t)$ are as in (6.2)-(6.4).

If, additionally, $\mathrm{E}|X|^r < \infty$ for some $1 < r < 2$, then, with $b(n) = n^{1/r}/\alpha(n)$, as $n \to \infty$,

$$\frac{Z_n - a(n)}{b(n)} = \frac{S_n - na}{n^{1/r}} \to 0 \qquad a.s.,$$

so that, by Theorem 6.4,

$$\lim_{t\to\infty} \frac{M(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = \lim_{t\to\infty} \frac{N(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = \lim_{t\to\infty} \frac{L(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = 0 \quad a.s.,$$

as $t \to \infty$ (cf. [14], Theorem 5.5 in Chapter IV).

7. Renewal theory and regular variation

It is a rare case in probability theory that an almost sure limiting behavior can effectively be studied "pointwise", that is, for individual ω's from a random set Ω_1 with $\mathrm{P}(\Omega_1) = 1$. The strong law of large numbers for renewal processes is such a case, since, as we shall see below, if ω is such that the strong law of large numbers holds for the underlying partial sums, then ω is also such that the strong law of large numbers holds for the corresponding renewal process. Before we give some exact results we collect some definitions introduced in the paper [1] by Buldygin et al.

Definition 7.1. Let f be a function such that $\limsup_{t\to\infty} f(t) = \infty$. A function $f^{(-1)}$ is called a quasi-inverse function for f, if

(f_1) $f^{(-1)}$ is nondecreasing,

(f_2) $f^{(-1)}(s) \to \infty$ as $s \to \infty$,

(f_3) there exists s_0 such that $f\left(f^{(-1)}(s)\right) = s$ for $s \geq s_0$.

For any continuous function f with $\lim_{t\to\infty} f(t) = \infty$, a quasi-inverse exists. One version of a quasi-inverse in this case is given by $f_1^{(-1)}(s) = \min\{t : f(t) = s\}$. Another version is $f_2^{(-1)}(s) = \max\{t : f(t) = s\}$. If f is not monotone, then $f_1^{(-1)}$ and $f_2^{(-1)}$ do not coincide. This shows that a quasi-inverse is not unique.

Definition 7.2. A function g is said to be of *positive order of variation* (POV) if

$$\liminf_{t\to\infty} \frac{g(ct)}{g(t)} > 1 \qquad \text{for all } c > 1.$$

Any regularly varying function g of positive order is POV. Any quickly growing function like $f(t) = e^t$ is POV. On the other hand, any slowly varying function is not POV. In what follows such functions g appear as normalizations in limit theorems. We require that they are POV, therefore the case of slowly varying normalizations is not covered by this approach.

Definition 7.3. Let $\{x_n,\ n \geq 1\}$ be a sequence of real numbers. Put $x_0 = 0$. The function

$$\widehat{x}(t) = ([t] + 1 - t)\, x_{[t]} + (t - [t])\, x_{[t]+1}, \qquad t \geq 0,$$

is called the (piecewise) linear interpolation of the sequence $\{x_n,\ n \geq 1\}$.

Theorem 7.1. *Let g be a function and let $\{x_n,\ n \geq 1\}$ be a sequence. Assume that the function g is continuous, increasing, unbounded and POV. Assume that the sequence $\{x_n,\ n \geq 1\}$ is such that*

$$\limsup_{n\to\infty} x_n = \infty.$$

By $\widehat{x}$ we denote the linear interpolation of the sequence $\{x_n,\ n \geq 1\}$. By $\widehat{x}^{(-1)}$ we denote a quasi-inverse function for $\widehat{x}$. Then

$$\lim_{n\to\infty} \frac{x_n}{g(n)} = a \in (0, \infty) \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\widehat{x}^{(-1)}(s)}{g^{(-1)}(s/a)} = 1.$$

Below we describe a typical application of Theorem 7.1 to renewal processes.

Example 7.1. Let $\{S_n,\ n \geq 1\}$ be partial sums of nonnegative random variables $\{X_n,\ n \geq 1\}$ such that

$$\lim_{n\to\infty} S_n = infty \qquad \text{a.s.} \tag{7.1}$$

Let a function g be given as in Theorem 7.1. Assume that the strong law of large numbers holds, that is

$$\lim_{n\to\infty} \frac{S_n}{g(n)} = a \in (0, \infty) \qquad \text{a.s.} \tag{7.2}$$

Denote by Ω_1 the random event, where both (7.1) and (7.2) hold. Fix $\omega \in \Omega_1$. Starting from the sequence $S_n(\omega)$, $n \geq 1$, construct its linear interpolation $\widehat{S}$ and then a quasi-inverse function $\widehat{S}^{(-1)}$ for $\widehat{S}$. Now, apply Theorem 7.1 with $x_n = S_n(\omega)$ and obtain

$$\lim_{n\to\infty} \frac{\widehat{S}^{(-1)}(s)}{g^{(-1)}(s/a)} = 1$$

at this point ω. Finally, if $\{N(t)\}$ is the renewal process constructed from the partial sums $\{S_n,\ n \geq 1\}$, then

$$\widehat{S}^{(-1)}(t) \leq N(t) \leq \widehat{S}^{(-1)}(t) + 1$$

and therefore, at the point ω,

$$\lim_{t\to\infty} \frac{N(t)}{g^{-1}(t/a)} = 1. \tag{7.3}$$

Since $\omega \in \Omega_1$ is arbitrary and $\mathrm{P}(\Omega_1) = 1$, the relation (7.3) holds almost surely.

The simplest case is given by a power function $g(t) = t^r$, $r > 0$. An application of the above reasoning to this case shows that, if

$$\lim_{n\to\infty} \frac{S_n}{n^r} = a \in (0, \infty) \qquad \text{a.s.},$$

then

$$\lim_{t\to\infty} \frac{N(t)}{t^{1/r}} = \frac{1}{a^{1/r}} \qquad \text{a.s.}$$

In particular, if

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu \in (0, \infty) \qquad \text{a.s.},$$

then

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \qquad \text{a.s.}$$

In the case of independent identically distributed random variables $\{X_n,\ n \geq \geq 1\}$, the latter result coincides with (2.6).

Other renewal processes. The approach described above works well for other functionals of partial sums. The renewal process defined in (2.1) is only one of them.

Another one is

$$F(t) = \min\{n : S_n \geq t\}.$$

For every $t \geq 0$, $N(t)$ describes the *last* time, when the random walk belongs to the set $A = (0, t)$, while $F(t)$ is the *first* moment when the walk exits from the set $(0, t)$.

These definitions can be extended to the case of an arbitrary sequence or function. Let f be a continuous function. The functionals we define below depend on the underlying function f, but we omit f in the notation, since it will be clear to which function the functionals correspond to. Define

$$F(t) = \inf\{s : f(s) \geq t\} = \inf\{s : f(s) = t\},$$
$$L(t) = \sup\{s : f(s) \leq t\} = \sup\{s : f(s) = t\}.$$

Another functional of interest is the time spent in the set $(0, t]$:

$$T(t) = \text{meas}\{s : f(s) \leq t\} = \int_0^\infty \mathbb{I}(f(s) \leq t)\, ds,$$

where "meas" is the Lebesgue measure and $\mathbb{I}(A)$ is the indicator of a set A. The fourth functional of interest is M constructed from the monotonization $\tilde{f}$ of f, i.e. $\tilde{f}(t) = \max_{0 \leq u \leq t} f(u)$. Then

$$M(t) = \sup\{s : \tilde{f}(s) \leq t\} = \sup\{s : \tilde{f}(s) = t\}.$$

It is clear that

$$F(t) \leq M(t) \leq T(t) \leq L(t).$$

The functional M is an obvious analog of the renewal process N. The reasoning of Example 7.1 of applies to both F and L, and therefore for M and T, too.

An application to stationary sequences. The results above apply to random limits as well. Let X_n, $-\infty < n < \infty$, be a strictly stationary sequence of random variables such that $\mathrm{E}|X_0| < \infty$. Let $\mathcal{F}$ be the σ-algebra of shift invariant events and let $\mathrm{E}\,[X_0|\mathcal{F}] \stackrel{\text{def}}{=} a > 0$ with probability one. Put $S_n = X_1 + \ldots + X_n$. Then

$$\frac{S_n}{n} \longrightarrow a \qquad \text{a.s.}$$

Applying an appropriate extension of Theorem 7.1 we derive that

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} \frac{N(s)}{s} = \frac{1}{a} \qquad \text{a.s.},$$

where the functionals F and N are constructed from the sequence $\{S_n\}$ as explained above and a may be random. The same result holds for the time spent by the sequence $\{S_n\}$ below a certain level.

References

[1] **Buldygin V.V., Klesov O.I. and Steinebach J.G.,** Properties of a subclass of Avakumović functions and their generalized inverses, *Ukrainian Math. J.,* **54** (2) (2002), 179-206.

[2] **Buldygin V.V., Klesov O.I. and Steinebach J.G.,** On factorization representations for Avakumovič-Karamata functions with nondegenerate groups of regular points, *Anal. Math.,* **30** (2004), 161-192.

[3] **Булдигін В.В., Клесов О.І . і Штейнебах Й.Г.,** Про деякі властивості асимтотично квазі-інверсних функцій та їх застосування І., *Теорія ймовір. мат. стат.,* **70** (2005), 11-28. (Ukrainian) (Buldygin V.V., Klesov O.I. and Steinebach J.G., On some properties of asymptotically quasi-inverse functions and their applications I., *Theory Probab. Math. Stat.)*

[4] **Булдигін В.В., Клесов О.І . і Штейнебах Й.Г.,** Про деякі властивості асимтотично квазі-інверсних функцій та їх застосування II., *Теорія ймовір. мат. стат.,* **71** (2005), 37-52. (Ukrainian) (Buldygin V.V., Klesov O.I. and Steinebach J.G., On some properties of asymptotically quasi-inverse functions and their applications II., *Theory Probab. Math. Stat.)*

[5] **Булдигін В.В., Клесов О.І . і Штейнебах Й.Г.,** Властивість PRV для функцій та асимптотична поведінка розв'язків стохастичних диференційних рівнянь, *Теорія ймовір. мат. стат.,* **72** (2006), 11-25. (Ukrainian) (Buldygin V.V., Klesov O.I. and Steinebach J.G., The PRV property of functions and the asymptotic behavior of solutions of stochastic differential equations, *Theory Probab. Math. Stat.)*

[6] **Buldygin V.V., Klesov O.I. and Steinebach J.G.,** On some extensions of Karamata's theory and their applications, *Publ. Inst. Math. Nouv. Sér.,* **80 (94)** (2006), 59-96.

[7] **Chung K.L. and Pollard H.,** An extension of renewal theory, *Proc. Amer. Math. Soc.,* **3** (1952), 303-309.

[8] **Erdős P., Feller W. and Pollard H.,** A property of power series with positive coefficients, *Bull. Amer. Math. Soc.,* **55** (1949), 201-204.

[9] **Frolov A., Martikainen A. and Steinebach J.,** Limit theorems for maxima of sums and renewal processes, *Zap. Nauch. Sem. LOMI,* **278** (2001), 261-274.

[10] **Galambos J.,** *Asymptotic theory of extreme order statistics,* Wiley, New York, 1978.

[11] **Galambos J., Indlekofer K.-H. and Kátai I.,** A renewal theorem for random walks in multidimensional time, *Trans. Amer. Math. Soc.,* **300** (1987), 759-769.

[12] **Galambos J. and Kátai I.,** A note on a random walk in multidimensional time, *Math. Proc. Camb. Phil. Soc.,* **99** (1986), 163-170.

[13] **Galambos J. and Kátai I.,** Some remarks on random walks in multidimensional time, *Proc. 5th Pannonian Sympos. Math. Statist. Visegrád, Hungary, 1985,* Reidel, Dordrecht, 1986, 65-74.

[14] **Gut A.,** *Stopped random walk,* Springer Verlag, Berlin, 1978.

[15] **Gut A., Klesov O.I. and Steinebach J.G.,** Equivalences in strong limit theorems for renewal counting processes, *Stat. Probab. Letters,* **35** (1997), 381-394.

[16] **Iglehart D.I. and Whitt W.,** The equivalence of functional central limit theorems for counting processes and associated partial sums, *Ann. Math. Statist.,* **42** (1971), 1372-1378.

[17] **Indlekofer K.-H. and Klesov O.I.,** Dirichlet's divisors in probability theory, *Theory Stoch. Processes,* **3 (19)** (1997), 208-215.

[18] **Indlekofer K.-H. and Klesov O.I.,** The asymptotic behavior of the renewal process constructed from a random walk with a restricted multidimensional time domain, *Annales Univ. Sci. Budapest. Sect. Comp.,* **24** (2004), 209-221.

[19] **Клесов О.И.,** Теорема восстановления для случайного блуждания с многомерным временем, *Укр. мат. журн.,* **43** (9) (1991), 1161-1167. (Klesov O.I., Renewal theorem for random walks with multidimensional time, *Ukrainian Math. J.,* **43** (9) (1991), 1089-1094.

[20] **Klesov O.I., Rychlik Z. and Steinebach J.G.,** Strong limit theorems for general renewal processes, *Probab. Math. Statist.,* **21** (2) (2001), 329-349.

[21] **Клесов О.И. і Штейнебах Й.Г.,** Посилені теореми відновлення для випадкових блукань з багатовимірним часом, *Теорія ймовір. мат. стат.,* **56** (1997), 105-111. (Ukrainian) English transl., Asymptotic behavior of renewal processes defined by random walks with multidimensional time, *Theory Probab. Math. Statist.,* **56** (1998), 107-113.

[22] **Klesov O.I. and Steinebach J.G.,** The asymptotic behavior of the renewal function constructed from a random walk in multidimensional time with restricted time domain, *Annales Univ. Sci. Budapest. Sect. Comp.,* **22** (2003), 181-192.

[23] **Лаврик А.Ф.,** О главном члене в проблеме делителей степенного ряда дзета функции Римана в окрестности ее полюсов, *Труды Матем. инст. АН СССР,* **142** (1976), 165-173. (Lavrik A.F., On the

main terms of the divisor problem in the power series of the Riemann zeta function around its pole, *Trudy Mat. Inst. Akad. Nauk SSSR* (in Russian))

[24] **Maejima M. and Mori T.**, Some renewal theorems for random walks in multidimensional time, *Math. Proc. Camb. Phil. Soc.,* **95** (1984), 149-154.

[25] **Ney P. and Wainger S.**, The renewal theorem for a random walk in two-dimensional time, *Studia Math.,* **44** (1972), 71-85.

[26] **Siegmund D.**, *Sequential analysis. Tests and confidence intervals,* Springer Verlag, New York, 1985.

[27] **Smith W.L.**, On the elementary renewal theorem for non-identically distributed variables, *Pacific J. Math.,* **14** (1964), 673-699.

[28] **Titchmarsh E.C.**, *The theory of the Riemann zeta function,* Oxford University Press, Oxford, 1951.

[29] **Vervaat W.**, Functional central limit theorems for processes with positive drift and their inverses, *Z. Wahrsch. verw. Gebiete,* **23** (1972), 245-253.

[30] **Woodroofe M.**, *Nonlinear renewal theory in sequential analysis,* CBS-NSF Regional Conf. Ser. Appl. Math. **39**, SIAM, Philadelphia, 1982.

V.V. Buldygin
Department of Math. Analysis
and Probability Theory
National Technical University
of Ukraine (KPI)
pr. Peremogy 37
03056 Kiev, Ukraine
valbuld@comsys.ntu-kpi.kiev.ua

K.-H. Indlekofer
Faculty of Computer Science,
Electrical Engineering and
Mathematics
University of Paderborn
Warburger Str. 100
D-33098 Paderborn, Germany
k-heinz@uni-paderborn.de

O.I. Klesov
Department of Math. Analysis
and Probability Theory
National Technical University
of Ukraine (KPI)
pr. Peremogy 37
03056 Kiev, Ukraine
tbimc@ln.ua

J.G. Steinebach
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
D-50931 Köln, Germany
jost@uni-koeln.de