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ON SOME PAIRS OF MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENTS

Bui Minh Phong (Budapest, Hungary)

Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. We proved that if multiplicative functions f, g and a positive integer k satisfy the condition

$$\sum_{n\le x} |g(n+k)-f(n)| = O(x),$$

then either

$$\sum_{n\le x} |f(n)| = O(x), \quad \sum_{n\le x} |g(n)| = O(x)$$

or there are functions $F,\ G\in\mathcal{M}$ and a complex constant s such that

$$f(n)=n^sF(n),\ g(n)=n^sG(n),\ 0\le \mathrm{Re}\ s<1$$

and $G(n+k)=F(n)$ are satisfied for all positive integers n.

1. Introduction

Let $\mathbb{N}$ and $\mathcal{P}$ denote the set of all positive integers and the set of all prime numbers, respectively. Let $\mathcal{M}$ ($\mathcal{M}^*$) be the set of complex-valued multiplicative (completely multiplicative) functions. (m,n) denotes the greatest common divisor of the integers m and n. Here $m \parallel n$ denotes that m is a unitary divisor of n, i.e. that $m|n$ and $(\frac{n}{m}, m) = 1$. We denote by $\mathcal{L}$ the subset of those functions $f \in \mathcal{M}$ for which the condition

$$\sum_{n \leq x} |f(n)| = O(x)$$

holds. It is obvious that for $f \in \mathcal{L}$ and $g \in \mathcal{L}$ the relation

$$\sum_{n \leq x} |g(n+k) - f(n)| = O(x) \tag{1}$$

holds for each $k \in \mathbb{N}$.

In [1], K-H. Indlekofer and I. Kátai proved that if $f \in \mathcal{M}^*$ and $g \in \mathcal{M}^*$ satisfy the condition (1), then either $f \in \mathcal{L}$, $g \in \mathcal{L}$ or there are a complex number s, functions F, $G \in \mathcal{M}^*$ such that

$$f(n) = n^s F(n), \; g(n) = n^s G(n) \quad (0 \leq \text{Re } s < 1)$$

and

$$G(n+k) = F(n)$$

hold for all $n \in \mathbb{N}$. The same result has been obtained in [2, 3] for the case when $f = g$ and $f \in \mathcal{M}$. For other generalization of this question we refer to the work [4] of K-H. Indlekofer and I. Kátai.

The main purpose of this note is to extend the above result of K-H. Indlekofer and I. Kátai. We shall characterize the functions $f \in \mathcal{M}$ and $g \in \mathcal{M}$ satisfying (1) with some fixed positive k. The general case concerning the characterization of those $f, g \in \mathcal{M}$ for which

$$\sum_{n \leq x} |g(An+B) - Ef(an+b)| = O(x),$$

where $a > 0, b, A > 0, B$ are fixed integers and E is a complex constant, seems to be a hard problem.

We shall prove the following

Theorem. *Assume that $f, g \in \mathcal{M}$ and $k \in \mathbb{N}$ satisfy the condition* (1). *Then either*

(a) $f \in \mathcal{L}, \quad g \in \mathcal{L}$

or

(b) *there are functions* $F, G \in \mathcal{M}$ *and a complex constant* s *such that*

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \ 0 \le \text{Re } s < 1$$

and

$$G(n+k) = F(n) \tag{2}$$

are satisfied for all $n \in \mathbb{N}$.

Remarks. (I) All solutions of (2) for $F, G \in \mathcal{M}$ have been determined in [5], [6], [7] and [8].

(II) We shall use the method that was used in [8] to reduce the problem to the case $f, g \in \mathcal{M}^*$ and apply the result of [1].

2. Auxiliary lemmas

In this section we assume that the conditions of the theorem are satisfied, i.e. the functions $f, g \in \mathcal{M}$ satisfy the condition (1) with some positive integer k.

We say that a function $f \in \mathcal{M}$ is of a finite support if

$$f(p^\alpha) = 0 \ \ (\alpha = 1, 2 \ldots)$$

holds for all but finitely many primes p.

Lemma 1. *If* f *or* g *is of a finite support, then* $f \in \mathcal{L}$, $g \in \mathcal{L}$.

Proof. Assume that f is of a finite support, that is $f(p^\alpha) = 0$ ($\alpha = 1, 2, \ldots$) if $p \notin \mathcal{A} = \{p_1, \ldots, p_r\}$. Let $\Delta = p_1 \ldots p_r$. For an arbitrary positive integer n let $n = A_\Delta(n) E_\Delta(n)$, where $A_\Delta(n)$ is the product of those prime power divisors p^α of n for which $p \in \mathcal{A}$, and $E_\Delta(n)$ is coprime to Δ. Then by (1), we have

$$\sum_{m \le x, E_\Delta(m-k) > 1} |g(m)| = O(x). \tag{3}$$

If $g(n) = 0$ for all $n \ge 2$, then $f, g \in \mathcal{L}$. Assume that q^β is a prime power for which $g(q^\beta) \ne 0$. It is well-known that the greatest prime divisor of $q^\gamma - k$ tends to infinity as $\gamma \to \infty$, so

$$E_\Delta(q^\gamma - k) > 1 \text{ if } \gamma \ge \gamma_q(\mathcal{A}).$$

This together with (3) shows that for all large x

$$\sum_{q^\gamma \le x} |g(q^\gamma)| = O(x). \tag{4}$$

Assume that $g \notin \mathcal{L}$. Then it follows from (4) that there is an infinite sequence $m_1 < m_2 < \dots$ of positive integers coprime to q for which

$$\sum_{m_\nu \le x} |g(m_\nu)| \ne O(x)$$

and so, using the fact $g(q^\beta) \ne 0$, one can deduce that

$$\sum_{m_\nu \le x} |g(q^\beta m_\nu)| \ne O(x).$$

These together with (3) imply that

$$E_\Delta(m_\nu - k) = 1 \quad \text{and} \quad E_\Delta(q^\beta m_\nu - k) = 1$$

hold for every large ν. This contradicts Thue's theorem (see e.g. [9]), consequently $g \in \mathcal{L}$ and $f \in \mathcal{L}$.

The case when g is of a finite support can be treated similarly.

Lemma 2. *If there are positive integers Δ and D such that*

$$\sum_{n \le x, (n,\Delta)=1} |f(n)| = O(x) \quad \text{and} \quad \sum_{n \le x, (n,D)=1} |g(n)| = O(x), \tag{5}$$

then $f \in \mathcal{L}$ and $g \in \mathcal{L}$.

Proof. By using Lemma 1 we can assume that none of f and g is of finite support. If $\Delta = 1$ or $D = 1$, then the assertion is true. Let

$$\Delta = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r} \quad \text{and} \quad D = q_1^{\beta_1} \cdots q_s^{\beta_s},$$

where $r, s \in \mathbb{N}$ and $\alpha_i, \beta_j \in \mathbb{N}$, $\pi_i, q_j \in \mathcal{P}$ $(i = 1, \cdots, r;\ j = 1, \cdots, s)$.

We may assume that for each π_i there is at least one $l_i \in \mathbb{N}$ such that $f(\pi_i^{l_i}) \ne 0$. Since f is not of a finite support, there are positive integers $Q_1, \cdots, Q_s$ for which $(Q_i, Q_j) = 1$ $(1 \le i < j \le s)$ and $f(Q_i) \ne 0$, $(Q_i, \Delta) = 1$ $(i = 1, \cdots, s)$. For $u, v, j \in \mathbb{N}, u \ne v$ let

$$q_j^{\beta_{u,v,j}} \parallel Q_u - Q_v \quad \text{and} \quad T := \max_{u,v,j; u \ne v} \beta_{u,v,j}.$$

Then there is a $j_0 \in \{1, \cdots, s\}$ for which if $\pi_1^{t_1} \cdots \pi_r^{t_r} \le x$, then

$$q_j^{\gamma_j} \parallel \pi_1^{t_1} \cdots \pi_r^{t_r} Q_{j_0} + k$$

with

$$\gamma_j \le \begin{cases} T & \text{if } q_j \nmid \pi_1 \cdots \pi_r \quad \text{or} \quad q_j \nmid k, \\ \log k & \text{if } q_j | (\pi_1 \cdots \pi_r, k). \end{cases}$$

Thus, by (5) we infer that

$$\sum_{\pi_1^{t_1} \cdots \pi_r^{t_r} \le x} |g(\pi_1^{t_1} \cdots \pi_r^{t_r} Q_{j_0} + k)| = O(x),$$

consequently

$$\sum_{\pi_1^{t_1} \cdots \pi_r^{t_r} \le x} |f(\pi_1^{t_1} \cdots \pi_r^{t_r} Q_{j_0})| = O(x).$$

The last relation together with (5) shows that $f, g \in \mathcal{L}$. Lemma 2 is proved.

Lemma 3. *If there are positive integers Δ such that*

$$\sum_{n \le x, (n,\Delta)=1} |f(n)| = O(x), \tag{6}$$

then there is a positive integer D for which

$$\sum_{n \le x, (n,D)=1} |g(n)| = O(x). \tag{7}$$

Similarly, if (7) *holds for some positive D, then there is a positive integer Δ such that* (6) *also holds.*

Proof. We shall prove only the first assertion. We may assume that none of f and g is of a finite support.

Let $\Delta = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r}$ and $E_\Delta(n)$ as in the proof of Lemma 1. Since g is not of a finite support, there are $Q_1, \cdots, Q_t$ $(t > r)$ mutually coprime integers for which $g(Q_j) \ne 0$ $(j = 1, \cdots, t)$. Let

$$\pi_j^{\beta_{u,v,j}} \parallel Q_u - Q_v \quad \text{and} \quad T := \max_{u,v,j; u \ne v} \beta_{u,v,j}.$$

By (1) we have

$$\sum_{m\le x,(m,Q_1\cdots Q_t)=1} |g(Q_jm)-f(Q_jm-k)| = O(x).$$

We shall prove that

$$\sum_{m\le x,(m,Q_1\cdots Q_t)=1} |f(Q_jm-k)| = O(x), \tag{8}$$

which completes the proof of Lemma 3 with $D = Q_1\cdots Q_t$.

Since $t > r$, there is one Q_{j_0} such that

$$E_\Delta(Q_{j_0}m-k) \mid (\pi_1\cdots\pi_r)^T \tag{9}$$

holds for all $m\in\mathbb{N}$, $(m,Q_1\cdots Q_t)=1$. Consequently, (8) follows from (6) and (9).

The case when (7) holds for some positive integer D can be treated similarly. The proof of Lemma 3 is finished.

Lemma 4. *If*

$$\sum_{m\le x} |f(\delta m+1)| = O(x), \tag{10}$$

or

$$\sum_{m\le x} |g(dm+1)| = O(x) \tag{11}$$

are satisfied for some $\delta, d\in\mathbb{N}$, *then* $f\in\mathcal{L}, g\in\mathcal{L}$.

Proof. Assume that (10) holds for some $\delta\in\mathbb{N}$. We shall prove that there is a positive integer Δ such that

$$\sum_{n\le x,(n,\Delta)=1} |f(n)| = O(x)$$

and so the assertion of Lemma 4 follows from Lemma 3.

For every reduced residue class $l \pmod{\delta}$ let $E_1^{(l)},\cdots,E_{\varphi(\delta)-1}^{(l)}$ be coprime integers belonging to $l \pmod{\delta}$ and satisfying $f(E_j^{(l)})\neq 0$ ($j =$

$= 1, \cdots, \varphi(\delta) - 1$), if there are so many $E_j^{(l)}$. Then for each positive integer $t \equiv l \pmod{\delta}$, $(t, E_1^{(l)} \cdots E_{\varphi(\delta)-1}^{(l)}) = 1$, we have

$$tE_1^{(l)} \cdots E_{\varphi(\delta)-1}^{(l)} \equiv 1 \pmod{\delta}$$

and so by (10) we get

$$\sum_{t \le x, (t, E_1^{(l)} \cdots E_{\varphi(\delta)-1}^{(l)}) = 1} |f(t)| = O(x)$$

If for some l the maximal size h of the set $E_1^{(l)}, \cdots, E_h^{(l)}$ constructed above is less than $\varphi(\delta) - 1$, then

$$\sum_{t \le x, t \equiv l \pmod{\delta}, (t. E_1^{(l)} \cdots E_h^{(l)}) = 1} |f(t)| = O(x).$$

Hence the assertion of Lemma 4 follows.

The case (11) can be proved similarly as above.

Lemma 5. *If there is a positive integer n_0 or m_0 such that*

$$(n_0, k) = 1 \quad \textit{and} \quad f(n_0) = 0$$

or

$$(m_0, k) = 1 \quad \textit{and} \quad g(m_0) = 0,$$

then $f \in \mathcal{L}, g \in \mathcal{L}$.

Proof. We shall prove that for every positive integer N either $f(N) = = g(N+k) = 0$ or $f(N)g(N-k) \neq 0$.

Assume first that there is $N \in \mathbb{N}$ such that $f(N) = 0$ and $g(N+k) \neq 0$. Applying (1) with $n = N\left[N(N+k)^2 m + 1\right]$, we have

$$\sum_{n \le x} |g(N^2(N+k)m + 1)| = O(x).$$

This relation with Lemma 4 shows that $f \in \mathcal{L}$ and $g \in \mathcal{L}$.

Assume now that there is a positive integer N such that $f(N) \neq 0$ and $g(N+k) = 0$. Since

$$N^2(N+k)^2 m + N + k = (N+k)\left[N^2(N+k)m + 1\right]$$

and

$$(N+k, N^2(N+k)m+1) = 1,$$

it follows from (1) and our assumptions that

$$\sum_{m \le x} |f(N(N+k)m+1)| = O(x).$$

Hence, we infer from Lemma 4 that $f \in \mathcal{L}$ and $g \in \mathcal{L}$.

Thus, we have proved that for every $N \in \mathbb{N}$ either

$$f(N) = g(N+k) = 0 \quad \text{or} \quad f(N)g(N+k) \neq 0.$$

Let

$$F(n) = \begin{cases} 1 & \text{if } f(n) \neq 0, \\ 0 & \text{if } f(n) = 0 \end{cases} \quad \text{and} \quad G(n) = \begin{cases} 1 & \text{if } g(n) \neq 0, \\ 0 & \text{if } g(n) = 0. \end{cases}$$

Then

$$F \in \mathcal{M}, G \in \mathcal{M} \quad \text{and} \quad G(n+k) = F(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

If there is $n_0 \in \mathbb{N}$ for which $(n_0, k) = 1$ and $f(n_0) = 0$, then Theorem 2 of [5] shows that

$$\mathcal{S}_F := \{n \in \mathbb{N} \mid F(n) \neq 0\} \quad \text{and} \quad \mathcal{S}_G := \{n \in \mathbb{N} \mid G(n) \neq 0\}$$

are finite sets. Hence $f \in \mathcal{L}$ and $g \in \mathcal{L}$.

In the case when there is $m_0 \in \mathbb{N}$ such that $(m_0, k) = 1$ and $g(m_0) = 0$, we also have $f \in \mathcal{L}$ and $g \in \mathcal{L}$. The proof of Lemma 5 is complete.

3. Proof of the theorem

In this section we assume that $f \in \mathcal{M}$ and $g \in \mathcal{M}$ satisfy the condition (1) and $f \notin \mathcal{L}$, $g \notin \mathcal{L}$. Then, it follows from Lemma 5 that $f(n)g(n) \neq 0$ for all $n \in \mathbb{N}$, $(n,k) = 1$. Let

$$H(n) := \frac{f(n)}{g(n)} \quad \text{on the set} \quad n \in \mathbb{N},\ (n,k) = 1.$$

By using the method of [7]-[8], one can deduce from (1) and our assumption that the functions f, g and H are completely multiplicative functions on the set $(n,k)=1$, furthermore

$$H(n) = \chi_k(n),$$

where χ_k denotes a character (mod k). Let

$$f^*(n) := \chi_k(n)f(n) \quad \text{and} \quad g^*(n) := \chi_k(n)g(n).$$

Then $f^* \in \mathcal{M}^*$, $g^* \in \mathcal{M}^*$ and

$$\sum_{n\le x} |g^*(n+k) - f^*(n)| = \sum_{n\le x,(n,k)=1} |g(n+k) - f(n)| = O(x).$$

By the theorem of [1], the last relation implies that there are a complex number s and functions $U, V \in \mathcal{M}^*$ such that

$$f^*(n) = n^s U(n) \quad \text{and} \quad g^*(n) = n^s V(n) \quad (\text{Re } s < 1),$$

where

$$V(n+k) = U(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

Finally, let

$$f(n) = n^s F(n) \quad \text{and} \quad g(n) = n^s G(n). \tag{12}$$

Then $F \in \mathcal{M}$, $G \in \mathcal{M}$ and

$$G(n+k) = F(n) = \chi_k(n) \quad \text{for all} \quad n \in \mathbb{N}, (n,k) = 1. \tag{13}$$

One can deduce from (1) and (12) that

$$\sum_{n\le x} |G(n+k) - F(n)| = O(x). \tag{14}$$

By using the method of [1], we get from (13) and (14) that

$$G(n+k) = F(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

The theorem is proved.

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Bui Minh Phong
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
bui@compalg.inf.elte.hu