LAUDATIO TO

Professor Imre Kátai

by A. Járai

Imre Kátai was born in Kiskunlacháza, May 13, 1938. He was three years old when he decided to live with his grandparents. His father died during World War II near the village of Osmolka (today belonging to Ukraine). After finishing primary school in Dunavarsány (1946-1952) he went to secondary school Fazekas Mihály Gimnázium, Budapest (1952-1956). His teacher Miklós Nagy recognised his extraordinary mathematical talent. He started his studies at József Attila University in Szeged. After one year he moved to Eötvös Loránd University and got his diploma as a mathematician and secondary school teacher in 1961.

He obtained his scientific degree C.Sc. (higher than Ph.D.) in 1966 and his D.Sc. degree in 1969. He became a corresponding member of Hungarian Academy of Sciences in 1979 and an ordinary member in 1985. His university carrier started at Eötvös Loránd University (ELTE) as a lecturer (1961-1963). During the years 1963-1965 he got a scholarship to work on his C.Sc. dissertation. Then he was assistant professor until 1966, associate professor until 1970, and the full professor. During 1970-1977 he served as the Dean of the Faculty of Sciences. From 1970 to 1983 he was the head of the Department of Numerical Analysis and between 1980 and 1993 the Director of Computer Center of ELTE. From 1993 to 2003 he was the Head of the Department of Computer Algebra and from 1995 to 1998 the Head of Department of Applied Mathematics and Computer Science at Janus Pannonius University, Pécs. Since 2005 he has been the Director of ELTE IK KKK. While serving as dean he introduced computer science as a new curriculum and was later the principal behind establishing the Faculty of Computer Science. He received the Gold Order of Labour (1978), the "Pro Universitate" Award (1978), Széchenyi Prize, the highest award possible to receive in Hungary for scientific work (1995), the Award of ELTE (2005), Commander's Cross of Civil Division of the Order of Merit of Hungarian Republic (2005) and the Award of ELTE IK (2007).

Professor Kátai started his scientific work as a student of P. Turán. He wrote wrote more than 320 scientific papers; the exact number cannot be determined because while you are counting them, he has already written a new one. Although most of the papers he wrote alone, he cooperated with 41 coauthors practically all over the world.

1. Comparative number theory: Ω-type theorems

Turán and his students have developed a method to investigate the change of signs of $\pi(x) - \mathrm{li}(x)$ and also of $\pi(x, q, l_1) - \pi(x, q, l_2)$, the difference of the number of primes in the different residue classes l_1 mod q and l_2 mod q. Turán called these kinds of investigation as comparative number theory. They are also called Ω-type theorems: we say that the function f is $\Omega(g)$, if

$$\limsup_{x\to\infty} |f(x)|/g(x) > 0,$$

if this is satisfied without taking the absolute value, we say f is $\Omega^+(g)$ and if it is satisfied for $-f$, then we say that f is $\Omega^-(g)$.

For several number theoretical functions it is relatively easy to prove $\Omega^{\pm}$-type theorems, but hard to prove effective results for change-of-sign intervals. Such theorems were proved by Turán, Knapowski and W. Stas with the help of the so-called "method of power sums". Kátai found a new and very powerful method based on a formula of Rodossky. An example of his results is the following theorem:

Suppose that the zeta function $\zeta(s)$ is different from zero if for $s = \sigma + i\tau$ we have $\sigma > 1/2$ and $|\tau| \leq B + 20$. Then there exist effectively computable constants c_1, c_2, k and δ, such that for each $T > c_1 B^{c_2}$ for the function $M(x) = \sum_{n\leq x} \mu(n)$ we have

$$\max_{x\in[T^k,T]} \frac{M(x)}{\sqrt{x}} > \delta, \qquad \min_{x\in[T^k,T]} \frac{M(x)}{\sqrt{x}} < -\delta.$$

Similar results were proved by Kátai, for example, for the functions $M_0(x)\sqrt{x}$, $m(x)/\sqrt{x}$, $S(x)\sqrt{x}$, $T(x)/\sqrt{x}$, $V(x)\sqrt{x}$ and $R(x)/\sqrt{x}$, where

$$M_0(x) = \sum_{n\leq x} \frac{\mu(n)}{n}, \qquad m(x) = \sum_{n=1}^{\infty} \mu(n)\exp(n/x),$$

$$S(x) = \sum_{n=1}^{\infty} \mu(n)\exp(-x^2/n^2), \qquad T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{((n-1)!\zeta(2n))},$$

and the functions V and R are the remainder terms in the approximation of the functions $\sum_{n\le x} \tau^2(n)$ and $\sum_{n\le x} r^2(n)$, respectively; here $\tau(n)$ is the number of divisors of n and $r(n)$ is the number of the positive integer solutions of the equation $n = u^2 + v^2$, see the papers [4], [9], [10], [11], [12], [13], [17], [26], [33], [34], [39], [40].

In a joint paper Kátai and Corrádi [24] gave an $\Omega^{\pm}$-type estimate for the remainder term in the approximation of the number of points of $\mathbb{Z}^2$ in the circle around the origin having radius $\sqrt{x}$. Until today this is the best known result.

2. Distribution of additive number theoretical functions on the set $\mathbb{P}+1$

He started working in probabilistic number theory in the second half of sixties. The starting point was a well-known result of Erdős and Wintner.

Let $A \subset \mathbb{N}$ and let $A(x) = \{a \in A : a \le x\}$. We say that a function $f : A \to \mathbb{R}$ has a distribution on the set A, if

$$\lim_{x\to\infty} \frac{1}{\natural A(x)} \natural\{a \in A(x) : f(a) < y\} = F(y)$$

for almost all $y \in \mathbb{R}$ for a distribution function F.

The theorem of Erdős and Wintner says that an additive number theoretical function f has a limit distribution on $\mathbb{N}$ if and only if the series

$$\sum_{p\in\mathbb{P},|f(p)|\ge 1} \frac{1}{p}, \qquad \sum_{p\in\mathbb{P},|f(p)|<1} \frac{f(p)}{p}, \qquad \sum_{p\in\mathbb{P},|f(p)|<1} \frac{f^2(p)}{p}$$

are convergent. Kátai proved [51] in 1969 that the convergence of the three series is sufficient for the existence of the limiting distribution on the set $\mathbb{P}+1$. This must be considered in light of conjecture that the multiplicative properties of the "shifted primes" $\mathbb{P}+a$ are in essence the same as $\mathbb{N}$, the set of all naturals. His result was the first step to "prove" this conjecture.

The question of the necessity of the convergence of the three series remained open for several years. Kátai in [65] proved necessity under the condition of boundedness of f. P.D.T.A. Elliot proved it under non-negativity of f (*Acta Arithmetica*, **25** (1974), 259-264), and Hildebrand under supposing that the limit distribution exists for $\mathbb{P}+a$, $a \in \mathbb{N}$. Several other authors have results in this topic. In his paper [55], Kátai gives generalizations for joint distributions of additive and multiplicative functions.

3. Sets of uniqueness

Around 1967 Imre Kátai asked whether it is true that for a completely additive function $f(\mathbb{P}+1)=0$ implies $f(\mathbb{N})=0$. More generally, we can ask whether for a subset A of $\mathbb{N}$ it is true that $f(A)=0$ implies $f(\mathbb{N})=0$ for all completely additive functions. If yes, A is called a set of uniqueness for completely additive functions. Concerning the original function Kátai proved that there exist finitely many primes $q_1, q_2, \ldots, q_r$ such that $\{q_1, \ldots, q_r\} \cup (\mathbb{P}+1)$ is a set of uniqueness [49]. P.D.T.A. Elliot (A conjecture of Kátai, *Acta Arithm.,* **26** (1974), 11-20) answered the question of Kátai completely. Note that completely additive functions can be uniquely extended to a homomorphism of the multiplicative group $\mathbb{Q}^+$ of positive rationals by the definition $f(m/n) = f(m) - f(n)$ for $m, n \in \mathbb{N}$. If this homomorphism is continuous, then it can be further extended, by taking limit, to a homomorphism of the multiplicative group $\mathbb{R}^+$ of positive real numbers. D. Wolke (Bemerkung über Eindeutigkeitsmengen additiver Funktionen, *Elem. der Math.,* **33** (1978), 14-16) proved that A is a set of uniqueness for completely additive functions if and only if for all $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that n^k can be written as a finite product $\prod_i a_i^{\varepsilon_i}$, where $a_i \in A$ and $\varepsilon_i = \pm 1$.

4. Sets of uniqueness modulo 1

A subset A of $\mathbb{N}$ is called a set of uniqueness modulo 1 for completely additive functions if $f(A) \equiv 0 \pmod 1$ implies $f(\mathbb{N} \equiv 0 \pmod 1)$ for all completely additive functions f. In [49] Kátai implicitly proved that there exist finitely many primes $q_1, q_2, \ldots, q_r$ such that $\{q_1, \ldots, q_r\} \cup (\mathbb{P}+1)$ is a set of uniqueness modulo 1. Sets of uniqueness modulo 1 were characterised by Dress and Volkmann and independently by Indlekofer, by Hoffmann and by Meyer: these are those sets for which any $n \in \mathbb{N}$ can be written as a finite product $\prod_i a_i^{\varepsilon_i}$, where $a_i \in A$ and $\varepsilon_i \in \mathbb{Z}$; in other words those sets which generate the multiplicative group $\mathbb{Q}^+$. It is not known whether $\mathbb{P}+1$ is a set of uniqueness modulo 1.

Clearly, more generally for any Abelian group G it is possible to ask whether a subset A of $\mathbb{N}$ is a set of uniqueness for the set of all completely additive functions $f : \mathbb{N} \to G$. In this setting modulo 1 uniqueness is the case $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. This type of problems has a large literature, several nice results were proved, among others by P.D.T.A. Elliot, Wirsing, Hildebrand; see the survey papers [253], 196-199 and [297], 123-124. For example, in [284] DeKoninck and Kátai (and independently Indlekofer and Timofeev) proved that the set $\{u^2 + v^2 + a : u, v \in \mathbb{N}\}$ is a set of uniqueness modulo 1 if $a \in \mathbb{N}$.

5. Characterization of regular additive and multiplicative functions

Imre Kátai proved in 1970 in a very elegant way [69] the following theorem: *If f is an additive function for which*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)| \to 0 \qquad \text{as} \quad x\to\infty,$$

then f is a constant times the logarithm function.

This theorem solved a problem of Erdős posed in 1946. Other statements characterising the logarithm as additive number theoretical function can be found in [70], [80], [102], [134], [142].

An analogous but harder-to-handle question is the characterization of multiplicative functions having regularity properties. Such theorems were proved by Kátai in the sequence of papers [136], [139], [140], [141], [146]. An example is the following:

Suppose that the complex valued multiplicative number theoretical function satisfies the condition

$$\sum_{n=1}^{\infty}\frac{1}{n}|f(n+1)-f(n)| < \infty.$$

Then either

$$\sum_{n=1}^{\infty}\frac{|f(n)|}{n} < \infty$$

or

$$f(n) = n^s \quad \text{for some } s \text{ with } \Re(s) < 1.$$

In one of his talks in 1978 Imre Kátai stated the conjecture that if for a complex valued multiplicative number theoretical function f the difference

$f(n+1) - f(n) \to 0$ as $n \to \infty$, then either $\lim_{n\to\infty} f(n) = 0$ or $f(n) = n^s$ for some s with $0 \le \Re(s) < 1$. This problem was solved in 1984 by Wirsing (see the paper Wirsing E., Thang Yuansheng and Shao Pintsung, On a conjecture of Kátai for additive functions, *J. Number Theory,* **56** (1996), 391-395).

Another conjecture posed by Kátai and Subbarao [297]: If for a completely multiplicative number theoretical function having values in $\mathbb{T}$ the set A of limit points of the sequence $f(n+1)\overline{f(n)}$ is finite, then A is the set of k'th roots for some $k \in \mathbb{N}$, the function f can be written in the form $f(n) = n^{i\tau}F(n)$ for some $\tau \in \mathbb{R}$, where $F(\mathbb{N}) = A$ and A is the set of limit points of the sequence $F(n+1)\overline{F(n)}$. This conjecture has been partially solved by Wirsing who proved that if A is finite, then there exist $\tau \in \mathbb{R}$ and $l \in \mathbb{N}$ such that f can be written in the form $f(n) = n^{i\tau}F(n)$, where $F^l(n) = 1$ (*Annales Sect. Comp.,* **24** (2004), 69-78).

Similar regularity results were proved by Daróczy and Kátai in a general setting for completely additive functions f mapping $\mathbb{N}$ to a metrisable Abelian group G: if $f(n+1) - f(n) \to 0$ as $n \to \infty$, then f is a restriction of a continuous homomorphism of the multiplicative topological group $\mathbb{R}^+$ into G (see [149], [160], [170], [172], [188], [192]).

6. q-additive and q-multiplicative functions

The notion of q-additive functions was introduced by A.O. Gelfond: let for a fixed $q \in \mathbb{N}, q > 1$ be

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \varepsilon_2(n)q^2 + \dots$$

the (unique) q-ary expansion of $n \in \mathbb{N}$ with digits $\varepsilon_i \in \{0, 1, \dots, q-1\}$. A function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is called q-additive if $f(0) = 0$ and for each $n \in \mathbb{N}$ we have

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j).$$

An important example for q-additive function is the sum-of-the-digits function $f(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)$. Replacing sum by product (and supposing $f(0) = 1$) we obtain the notion of the q-multiplicative function.

In 1972 Delange proved that a q-additive function has a limit distribution if and only if the series

$$\sum_{j=0}^{\infty}\sum_{\alpha=1}^{q-1} f(\alpha q^j) \quad \text{and} \quad \sum_{j=0}^{\infty}\sum_{\alpha=1}^{q-1} f^2(\alpha q^j)$$

are convergent. The same year Kátai proved that the same statement remains true if we consider the limit distribution only on the prime numbers. His main tool was the following lemma:

Let $h < c_1 \log(N)$, $1 \le l_1 < l_2 < \ldots < l_h < N$, *and*

$$b_1, b_2, \ldots, b_h \in \{0, 1, \ldots, q-1\}.$$

Then the number of all primes $p < q^N$ *for which* $\varepsilon_{l_j}(p) = b_j$ *for* $j = 1, 2, \ldots, h$ *is*

$$(1 + o(1))\frac{\pi(q^N)}{q^h} \qquad (N \to \infty).$$

The original proof is not satisfactory; a complete proof of this statement will be published in a joint paper by G. Harman and I. Kátai. An unproved conjecture of Kátai says that this is uniformly true for all $h < N/3$. The lemma can be proved for fixed h and for positive values of polynomials $P \in \mathbb{Z}[X]$. This is done in a joint work of Bassily and Kátai, and helped characterising q-additive functions for which the values $|P(n)|$ have a limit distribution (see [231], [233]).

Very recently, substantially advancing the method of Kátai and Bassily, M. Drmota, C. Maudit and J. Rivat found the asymptotic of the distribution of the sum-of-digits function (Primes with an avarage sum of digits, in print by *Compositio Math.*), also solving an old problem of Gelfond.

We know much less about the asymptotic distribution of values of q-multiplicative functions (see the survey papers [232] and [264] for results and open problems). Papers of Imre Kátai concerning this topic are [30], [36], [88], [101], [102], [105], [129], [135], [143], [152], [162], [163], [216], [231], [233].

7. Number systems

Imre Kátai wrote his first paper about number systems jointly with Szabó [84]. They characterised in the ring of gaussian integers all those bases α for which each element β of the ring can be written uniquely as

$$\beta = \varepsilon_0 + \varepsilon_1\alpha + \ldots + \varepsilon_k\alpha^k,$$

where the digits ε_j belong to $\{0, 1, \ldots, |\alpha|^2 - 1\}$. They proved that the necessary and sufficient condition for α is that $\Re(\alpha) < 0$ and $\Im(\alpha) = \pm 1$. The analogous question for quadratic number fields was solved by Kátai and Kovács in [120] and [128]. In [226] Kátai investigated the question whether for an algebraic integer α in the ring of algebraic integers of the number field $\mathbb{Q}(\alpha)$ there exists a complete residue system modulo α such that using these as digits and α as base, there is a unique representation for all algebraic integers $\beta \in \mathbb{Q}(\alpha)$. The conditions that all the conjugates of α have to be greater than 1 in absolute value and different from $1 - \varepsilon$, where ε is a unit, are necessary. They were proved also sufficient by Kátai [226] in imaginary quadratic number fields. The question for general number fields seems to be very hard. A result in this direction is Theorem 4 in Kátai [248]. The notion of number system has been generalised to higher dimensions in several ways. For example, in $\mathbb{Z}^n$ the base can be an integer matrix and the digits are elements of $\mathbb{Z}^n$.

The "unit ball" in these exotic number systems is the set of sums

$$\varepsilon_0 + \varepsilon_1 \alpha^{-1} + \varepsilon_2 \alpha^{-2} + \ldots$$

It has interesting topological and measure theoretical properties. For example, its boundary has fractal properties (see the results in [217], [218], [222], [228], [229]). Some of the results were conjectured by the help of computer experiments and some of the results were proved using computer programmes.

8. Rényi-Parry type expansions

A. Rényi in 1957 and W. Parry in 1960 considered the expansions of numbers $0 \le x \le 1$ in the form

$$x = \varepsilon_0 + \varepsilon_1/q + \varepsilon_2/q^2 + \ldots,$$

where $q > 1$ is a real number which is not integer and the digits ε_j are from $\{0, 1, \ldots, \lfloor q \rfloor\}$. There are many more x's whose expansion is not unique, as in the case $q \in \mathbb{N}$. The expansion depends on the "strategy" we apply to choose the next digit when there are more than one possibilities. Fixing one or more strategies, we may generalise the notion of q-additive functions for this situation and several questions may be asked. There is a central role of the expansion of 1 and the so-called "univoque numbers". We may also consider a "base depending on n", i.e. we may substitute q^{-n} with λ_n, where λ_n is a strictly monotonic sequence of positive real numbers for which $\lambda_n \to 0$. Further generalisations, for example, to complex numbers are also possible. Large part of the important results have been published in joint papers of Daróczy and Kátai (see survey papers [272] and [319]).

9. Random walks in multidimensional time

Let $\mathbb{N}^r$ be the "r-dimensional time" with coordinate-wise partial ordering and $X : n \mapsto X(n)$, $n \in \mathbb{N}^r$ a family of independent identically distributed integer valued random variables, a "random walk". Using the partial sums

$$S(n) = \sum_{k \leq n, k \in \mathbb{N}^r} X(k) \quad \text{for} \quad n \in \mathbb{N}^r$$

we define the renewal sequence $u_r(k)$ of X as the sum for $n \in \mathbb{N}^r$ of the probabilities that $S(n)$ takes the value $k \in \mathbb{N}$. The following theorem, generalising earlier results, was proved by Galambos and Kátai [156]:

Suppose that X has finite expected value $\mathbb{E}(X) = \mu > 0$ and $\mathbb{E}(|X|^3)$ is also finite, moreover X is aperiodic (i.e. $\mathbb{E}(e^{itX})$ not equal to 1 for any $t \in \mathbb{R}$). Then there exists an explicitly given polynomial p having degree $r-1$ such that

$$u_r(k) = P(\ln(k/\mu)) + R_k + O(1),$$

where $R_k \to 0$ for $r = 2$ and $r = 3$, and for arbitrary r,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{k=N+1}^{2N} |R_k| < \infty.$$

Further results have been obtained in the joint paper by Galambos, Indlekofer and Kátai [168].

10. Theorem of Daboussi

The surprising result that for any irrational $\alpha \in \mathbb{R}$ and for any complex valued multiplicative arithmetical function bounded by 1

$$\lim_{y \to \infty} \sup_{x} \frac{1}{x} \sum_{n \leq x} f(n) e^{2\pi i n \alpha} = 0$$

was proved in the paper of H. Daboussi and H. Delange (Quelques propriétés des fonctions multiplicatives de module au plus égal 1, *C.R. Acad. Sci. Paris Ser. A*, **278** (1974), 657-660).

Kátai generalized this in the following form:

Let $t : \mathbb{N} \to \mathbb{R}$ *and* $p_1 < p_2 < \dots$ *be a sequence of prime numbers such that* $\sum 1/p_j$ *is divergent. Suppose that for the sequences*

$$\eta_{j,k}(m) = t(p_j m) - t(p_k m)$$

we have

$$\frac{1}{x} \sum_{m \le x} e^{2\pi i \eta_{j,k}(m)} \to 0 \quad as \quad x \to \infty$$

whenever $j \neq k$. *Then*

$$\lim_{x \to \infty} \sup_{f \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{n < x} f(n) e^{2\pi i t(n)} \right| = 0,$$

where $\mathcal{M}_1$ *denotes the set of all complex valued multiplicative arithmetical functions bounded by 1.*

He also published further papers in this topic, in part with coauthors Indlekofer, DeKoninck and Bassily.

I could continue the list of his important, interesting and inspiring results and works. Since I have known him I have always been fascinated and astonished by his proving power and his fantastic ability to find new and interesting problems - and I am sure I am not the only one. I wish you, Imre, a happy birthday.