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ON INTEGER-VALUED ARITHMETICAL FUNCTIONS SATISFYING CONGRUENCE PROPERTIES

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In memory of Professor M.V. Subbarao

Abstract. In this paper some results and problems concerning integer-valued arithmetical functions satisfying congruences are surveyed and the following theorem is proved: If $A > 0, B, a > 0, b, N > 0, C \neq 0$ are integers, $(aA, Ab + B) = (a, B) = 1$, f is an integer-valued completely multiplicative function, $f(B) \neq 0$ and

$$f\left[A(an+b)+B\right] \equiv f(B) \pmod{an+b}$$

for all $n > N, n \in \mathbb{N}$, then there are a non-negative integer α and a real-valued Dirichlet character $\chi \pmod{aA}$ such that $f(n) = \chi(n)n^{\alpha}$ holds for all $n \in \mathbb{N}$, $(n, aA) = 1$.

I. Notations

An arithmetical function $f(n) \not\equiv 0$ is said to be multiplicative if $(n, m) = 1$ implies

$$f(nm) = f(n)f(m)$$

and it is called completely multiplicative if this equation holds for all pairs of positive integers n and m. In the following we denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all integer-valued multiplicative and completely multiplicative functions, respectively.

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Let $\mathbb{N}$ and $\mathcal{P}$ denote the set of all positive integers and the set of all prime numbers, respectively. (m,n) denotes the greatest common divisor of the integers m and n. For every non-negative integer a let

$$\varphi_a(n) = n^a \quad (\text{for all} \quad n \in \mathbb{N})$$

and

$$\Phi := \{\varphi_0, \varphi_1, \varphi_2, \ldots\}, \quad \Phi^+ := \{\varphi_1, \varphi_2, \ldots\}.$$

II. The congruence $f(n+m) \equiv f(m) \pmod{n}$

The problem concerning the characterization of the functions n^α by a congruence property was studied by Subbarao in 1966:

Theorem 1. (Subbarao [24], 1966) *If $f \in \mathcal{M}$ satisfies*

$$f(n+m) \equiv f(m) \pmod{n} \quad \textit{for all} \quad n, m \in \mathbb{N},$$

then $f \in \Phi$.

Later, A. Iványi showed in 1972 the following

Theorem 2. (Iványi [6], 1972) *If $f \in \mathcal{M}^*$ and $M \in \mathbb{N}$ satisfy*

$$f(n+M) \equiv f(M) \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N},$$

then $f \in \Phi$.

We improved this result by proving the following

Theorem 3. (Phong and Fehér [21], 1990) *If $M \in \mathbb{N}$, $f \in \mathcal{M}$ satisfy $f(M) \neq 0$ and*

$$f(n+M) \equiv f(M) \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N},$$

then $f \in \Phi$.

Theorem 4. (Phong [17], 1993; Joó and Phong [10], 1992) *If integers $A > 0$, $B > 0$, $C \neq 0$, $N > 0$ with $(A,B) = 1$ and $f \in \mathcal{M}$ satisfy the relation*

$$f(An+B) \equiv C \pmod{n} \quad \textit{for all} \quad n \geq N,$$

then there are a positive integer α and a real-valued Dirichlet character $\chi \pmod{A}$ such that $f(n) = \chi(n) n^\alpha$ for all $n \in \mathbb{N}$, $(n, A) = 1$.

In 2004, I. Kátai proved the following

Theorem 5. (Kátai [11], 2004) *Let*

$$f_1, f_2 \dots, f_k \in \mathcal{M}^*$$

and

$$A_1, A_2, \dots, A_k \in \mathbb{Z}$$

such that

$$L(n) := A_1 f_1(n) + A_2 f_2(n) + \dots + A_k f_k(n) \not\equiv 0.$$

If

$$L(n+m) \equiv L(m) \pmod{n} \ \ \textit{for all} \ \ n, m \in \mathbb{N},$$

then

$$f_1, f_2, \dots, f_k \in \Phi.$$

Consequently

$$L(x) \in \mathbb{Z}[x] \ \ \textit{is a polynomial with integer coefficients.}$$

For the polynomial

$$P(x) = a_0 + a_1 x + \dots + a_k x^k \in \mathbb{Z}[x] \quad (a_k \neq 0)$$

and the function $f(n)$, let

$$P(E)f(n) := a_0 f(n) + a_1 f(n+1) + \dots + a_k f(n+k).$$

For any fixed subsets $A,\ B$ of $\mathbb{N}$ let $\mathcal{K}_P(A,B)$ denote the set of all $f \in \mathcal{M}$ for which

$$P(E)f(n+m) \equiv P(E)f(n) \pmod{m} \text{ for all } n \in A,\ m \in B.$$

It is obvious that $\Phi \subset \mathcal{K}_P(A,B)$ for all $P,\ A,\ B$. We are interested for a characterization of those triplets (P, A, B) for which

$$\mathcal{K}_P(A,B) = \Phi = \{\varphi_0, \varphi_1, \cdots, \varphi_a, \cdots\}.$$

We have proved the following results

Theorem 6. *The relation $\mathcal{K}_P(A,B) = \Phi$ holds in the following cases:*

(a) (B. M. Phong [13], 1991) $\quad P(x) = 1,\ A = \mathbb{N},\ B = \mathcal{P}$,

(b) (B. M. Phong [13], 1991) $P(x) = 1,\ A = \mathcal{P},\ B = \mathbb{N}$,

(c) (B. M. Phong [14],1990) $P(x) = (x-1)^k\ \ (k \in \mathbb{N}),\ \ A = \mathbb{N},\ \ B = \mathcal{P}$

and

(d) (B. M. Phong [15],1991) $P(x) = x^M - 1\ \ (M \in \mathbb{N}),\ \ A = \mathbb{N},\ \ B = \mathcal{P}$.

Theorem 7. (B. M. Phong [16], 2001) *Let $f \in \mathcal{M}^*$ with condition*

$$f(n) \neq 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Let $P(x)$ be a non-zero polynomial with rational coefficients for which there exists a suitable non-zero integer A_P such that

$$A_P P(E) f(n+m) \equiv A_P P(E) f(n) \pmod{m}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then $f \in \Phi$.

In the special case when $P(x) = (x-1)^k$, Theorem 7 is true under the assumption $f \in \mathcal{M}$.

Theorem 8. (B. M. Phong [16], 2001) *Let $f \in \mathcal{M}$ and let $A \neq 0,\ k \geq 0$ be integers. If $\Delta^k f(n)$ satisfies the relation*

$$A\Delta^k f(n+m) \equiv A\Delta^k f(n) \pmod{m}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then $f \in \Phi$.

III. The congruence $f(n+m) \equiv f(n) + f(m) \pmod{n}$

An another characterization of n^α by using congruence property was found by A. Iványi, namely he proved in 1972 the following

Theorem 9. (Iványi [6], 1972) *If $f \in \mathcal{M}$ satisfies*

$$f(n+m) \equiv f(n) + f(m) \pmod{n} \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $f \in \Phi^+$.

We improved this result by showing

Theorem 10. (Phong and Fehér [22], 1999) *Assume that $A > 0,\ B > 0,\ C,\ D \neq 0$ are fixed integers with $(A,\ B) = 1$ and a function $f \in \mathcal{M}^*$ satisfies the congruence*

$$f(An+B) \equiv Cf(n) + D \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Then the following assertions hold:

(I) *If* $f(p)=0$ *for some prime* p *with* $(p,\ A)=1$, *then*

$$p=2,\ C=-1,\ D=1,\ (2,AB)=1 \quad \text{and} \quad f(n)=\chi_2(n) \quad \text{for all} \quad n\in\mathbb{N}.$$

(II) *If* $f(n)\neq 0$ *for all* $n\in\mathbb{N}$, $(n,\ A)=1$, *then either*

$$C+D=1 \quad \text{and} \quad f(n)=1 \quad \text{for all} \quad n\in\mathbb{N}$$

or there are a positive integer α *and a real-valued Dirichlet character* χ $\pmod{A}$ *such that*

$$f(n)=\chi(n)n^{\alpha} \quad \text{for all} \quad n\in\mathbb{N},\ (n,A)=1.$$

In 2003, we improved this result as follows:

Theorem 11. (B. M. Phong [19], 2003) *Let* $A,\ B$ *be positive integers with the conditions*

$$(A,B)=1 \quad \text{and} \quad (A,2)=1.$$

Assume that a function $f\in\mathcal{M}$ *and an integer* $C\neq 0$ *satisfy the congruence*

$$f(An+B)\equiv f(An)+C \pmod{n} \quad \text{for all} \quad n\in\mathbb{N}.$$

We have:

(I) *If there is a prime power* $\pi^e>1$ *such that* $(\pi,A)=1$ *and* $f(\pi^e)=0$, *then*

(a) $\pi=2$ *and* $f(An)=-1$ *for all* $n\in\mathbb{N}$, $(n,2)=1$,

(b) $C=1$ *and* $f(2^{\gamma})=0$ *for all* $\gamma\in\mathbb{N}$ *in the case* $(B,2)=1$,

(c)

$$f(2^{\gamma})=\begin{cases}1 & \text{if } \gamma<\alpha,\\ 2-f(2^{\alpha}) & \text{if } \gamma>\alpha\end{cases} \quad \text{and} \quad f(2^{\alpha})=\begin{cases}2 & \text{if } e>\alpha,\\ 0 & \text{if } e=\alpha\end{cases}$$

in the case $2^{\alpha}\parallel B$ *with* $\alpha\in\mathbb{N}$, *furthermore* $e\geq\alpha$, $f(A)=-1$, $C=2$,

(II) *If* $f(n)f(Am)\neq 0$ *for all* $n,m\in\mathbb{N}$, $(n,A)=1$ *and*

$$|f(n)|=1 \quad \text{for all} \quad n\in\mathbb{N}, \quad n\equiv 1 \pmod{D}$$

holds for a some a fixed $D\in\mathbb{N}$, *then* *(i)* $f(A)+C=1$ *and* $f(An)=f(A)$ *for all* $n\in\mathbb{N}$ *in the case when* $f(A^m)\neq -1$ *for some* $m\in\mathbb{N}$,

(ii) $f(n)=1$ *for all* $n\in\mathbb{N}$, $(n,2A)=1$ *and*

$$f\left(2^{\alpha+\gamma}\right)=C-f\left(2^{\alpha}\right)\quad \textit{for all}\quad \gamma\in\mathbb{N},$$

where $2^\alpha \parallel B$, $\alpha\geq 0$. *Furthermore, if* $\alpha>0$, *then* $C=2$ *and* $f(2^\delta)=1$ *for* $\delta<\alpha$.

(III) If $f(n)\neq 0$ *for all* $n\in\mathbb{N}$, $(n,A)=1$ *and* $|f(N)|>1$ *for some* $N\in\mathbb{N},(N,A)=1$, *then there are a non-negative integer* α *and a real-valued Dirichlet character* $\chi \pmod A$ *such that*

$$f(n)=\chi(n)n^\alpha$$

holds for all $n\in\mathbb{N}$, $(n,A)=1$.

IV. Additive functions which satisfy some congruence conditions

We shall denote by $\mathcal{A}$ and $\mathcal{A}^*$ the set of all integer-valued additive and completely additive functions, respectively. A similar problem concerning the characterization of a zero-function as an integer-valued additive function satisfying a congruence condition have been studied by K. Kovács [12], P.V. Chung [2]-[4], I. Joó [9] and I. Joó-B.M. Phong [10]. It was proved by K. Kovács [12] that if $g\in\mathcal{A}^*$ satisfies the congruence

$$g(An+B)\equiv C \pmod n$$

for some integers $A>0,\ B>0, C$ and for all $n\in\mathbb{N}$, then $g(n)=0$ for all $n\in\mathbb{N}$ which are prime to A. This result was extended in [2], [9] and [10] for integer-valued additive function f. It follows from the results of [3] and [9] that for integers $A>0,\ B>0, C$ and functions $g_1\in\mathcal{A},\ \ g_2\in\mathcal{A}^*$ the congruence

$$g_1(An+B)\equiv g_2(n)+C \pmod n\quad \text{for all}\quad n\in\mathbb{N}$$

implies that $g_2(n)=0$ for all $n\in\mathbb{N}$ and $g_1(n)=0$ for all $n\in\mathbb{N}$ which are prime to A.

We improved this result by showing the following

Theorem 12. (B. M. Phong [20], 2003) *Assume that* $a \geq 1,\ b \geq 1,\ c \geq 1$ *and* d *are fixed integers and the functions* $f_1,\ f_2$ *are additive. Then the relation*

$$f_1(an+b) \equiv f_2(cn)+d \pmod{n}$$

holds for all $n \in \mathbb{N}$ *if and only if the equation*

$$f_1(an+b) = f_2(cn)+d$$

holds for all $n \in \mathbb{N}$.

Theorem 13. (B. M. Phong [20], 2003) *Assume that* $a \geq 1,\ b \geq 1,\ c \geq 1$ *and* d *are fixed integers. Let* $a_1 = \frac{a}{(a,\ b)},\ b_1 = \frac{b}{(a,\ b)}$ *and*

$$\mu := \begin{cases} 1 & \text{if } 2 \mid a_1 b_1, \\ 2 & \text{if } 2 \nmid a_1 b_1. \end{cases}$$

If the additive functions f_1 *and* f_2 *satisfy the equation* $f_1(an+b) = f_2(cn)+d$ *for all* $n \in \mathbb{N}$, *then*

$$f_1(n) = 0 \quad \text{for all} \quad n \in \mathbb{N},\ (n, \mu a b_1) = 1$$

and

$$f_2(n) = 0 \quad \text{for all} \quad n \in \mathbb{N},\ (n, \mu c b_1) = 1.$$

V. A problem of Fabrykowski and Subbarao

In 1985, Subbarao [25] introduced the concept of weakly multiplicative arithmetic function $f(n)$ (later renamed quasi multiplicative arithmetic functions) as one for which the property

$$f(np) = f(n)f(p)$$

holds for all primes p and positive integers n which are relatively prime to p. In the following let $\mathcal{QM}$ denote the set of all integer-valued quasi multiplicative functions. In [5] J. Fabrykowski and M. V. Subbarao proved that if $f \in \mathcal{QM}$ satisfies

$$f(n+p) \equiv f(n) \pmod{p}$$

for all $n \in \mathbb{N}$ and all $p \in \mathcal{P}$, then $f \in \Phi$. They also conjectured that this result continues to hold even if the above relation is valid for an infinity of primes instead of for all primes. This conjecture is still open.

Assume that for a set $\mathcal{B} \subset \mathcal{P}$ and for a function $f \in \mathcal{QM}$, the congruence relation

$$f(n+p) \equiv f(n) \pmod{p} \quad \text{for all} \quad n \in \mathbb{N}, \quad p \in \mathcal{B}$$

holds true. For a given subset $\mathcal{B}$ of $\mathcal{P}$ and for each positive integer n we define $\mathcal{B}(n)$ as follows:

$$\mathcal{B}(n) := \prod_{p \in \mathcal{B},\ p|n} p.$$

It is obvious from the definition that $\mathcal{B}(n)|\mathcal{B}(mn)$ holds for all positive integers n and m, furthermore one can deduce that if $f \in \mathcal{QM}$ satisfies the above congruence, then

$$f(n+m) \equiv f(m) \pmod{\mathcal{B}(n)} \quad \text{for all} \quad n,\ m \in \mathbb{N}.$$

Thus the conjecture of Fabrykowski and Subbarao is contained in the following

Conjecture. *Let A, B be fixed positive integers with the condition $(A,B)=1$ and $\mathcal{B}$ is an infinite subset of $\mathcal{P}$. If a function $f \in \mathcal{QM}$ and integer $C \neq 0$ satisfy the congruence*

$$f(An+B) \equiv C \pmod{\mathcal{B}(n)} \quad \textit{for all} \quad n \in \mathbb{N},$$

then there are a positive integer α and a real-valued Dirichlet character χ (mod A) *such that*

$$f(n) = \chi(n)n^{\alpha} \quad \textit{for all} \quad n \in \mathbb{N},\ (n,A)=1.$$

B.M. Phong and J. Fehér [23] proved this conjecture for the case, when $\mathcal{A} = \mathcal{P} \setminus \mathcal{B}$ is a finite set.

Theorem 14. (Phong and Fehér [23], 2000) *Let A, B be fixed positive integers with the condition $(A,B)=1$ and let $\mathcal{A} = \mathcal{P} \setminus \mathcal{B}$ be a finite set. Furthermore we assume that $(2,\ A)=1$ for the case $2 \notin \mathcal{B}$. If a function $f \in \mathcal{QM}$ and an integer $C \neq 0$ satisfy the congruence*

$$f(An+B) \equiv C \pmod{\mathcal{B}(n)} \quad \textit{for all} \quad n \in \mathbb{N},$$

then there are a non-negative integer α and a real-valued Dirichlet character χ (mod A) *such that* $f(n) = \chi(n)n^{\alpha}$ *holds for all* $n \in \mathbb{N}$, $(n,A)=1$.

This result was proved earlier in B.M. Phong [18] for the case when $A = B = 1$ and $|\mathcal{P} \setminus \mathcal{B}| = 1$.

Corollary 1. *Let A be a fixed positive integer and let $\mathcal{A} = \mathcal{P} \setminus \mathcal{B}$ be a finite set. If the functions $f_1,\ f_2 \in \mathcal{QM}$ satisfy the congruence*

$$f_1(An+m) \equiv f_2(m) \pmod{\mathcal{B}(n)} \quad \textit{for all} \quad n, m \in \mathbb{N},$$

then there are a non-negative integer α and a real-valued Dirichlet character $\chi \pmod{A}$ such that $f_1(n) = f_2(n) = \chi(n)n^{\alpha}$ holds for all $n \in \mathbb{N}$, $(n, A) = 1$.

Corollary 2. *Let A, B, D be fixed positive integers with the condition $(A,\ B) = 1$ and $(A,\ D,\ 2) = 1$. If a function $f \in \mathcal{QM}$ and an integer $C \neq 0$ satisfy the congruence*

$$f(An+B) \equiv C \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N}, \quad (n,\ D) = 1$$

then there are a non-negative integer α and a real-valued Dirichlet character $\chi \pmod{A}$ such that $f(n) = \chi(n)n^{\alpha}$ holds for all $n \in \mathbb{N}$, $(n, A) = 1$.

Corollary 3. *Let N be a fixed positive integer. If the function $f \in \mathcal{QM}$ satisfies the congruence*

$$f(n+p) \equiv f(n) \pmod{p} \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and for all primes} \quad p \geq N,$$

then there is a non-negative integer α such that $f(n) = n^{\alpha}$ holds for all $n \in \mathbb{N}$.

Remark. This corollary gives the answer to the conjecture of Fabrykowski and Subbarao for the case $\mathcal{B} = \{p \in \mathcal{P} \mid p \geq N\}$, where N is a given positive integer.

VI. New results

Now we shall prove the following

Theorem 15. *Let $A > 0$, B, $a > 0$, b, $N > 0$ and $C \neq 0$ be fixed integers. If a function $f \in \mathcal{M}^*$ satisfies the congruence*

$$f\left[A(an+b)+B\right] \equiv C \pmod{an+b} \quad \textit{for all} \quad n \in \mathbb{N},\ n > N,$$

then there are a non-negative integer e and a real-valued Dirichlet character $\chi \pmod{aA}$ such that $f(n) = \chi(n)n^{e}$ holds for all $n \in \mathbb{N}$, $(n, aA) = 1$.

Directly from Theorem 15 follows

Corollary. *Let $A > 0$, B, $N > 0$ and D be fixed integers. If a function $f \in \mathcal{M}^*$ and an integer $C \neq 0$ satisfy the congruence*

$$f(An+B) \equiv C \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N},\ n > N,\ (n,\ D) = 1$$

then there are a non-negative integer e and a real-valued Dirichlet character χ $\pmod{AD}$ such that $f(n) = \chi(n)n^e$ holds for all $n \in \mathbb{N}$, $(n, AD) = 1$.

In order to prove the theorem, we show first the following key lemma.

Lemma. *Let $U \geq 1$, V, $u \geq 1$, v, $\alpha > 1$, $\beta > 1$, $k \geq 1$, l and $F \neq 0$ be fixed integers. If*

$$U\alpha^{kn+l} + V \;\Big|\; F \cdot \left(u\beta^{kn+l} + v\right) \tag{1}$$

for all $n \in \mathbb{N}$, then there is a positive integer e such that

$$\beta = \alpha^e \quad \textit{and} \quad u(-V)^e + vU^e = 0.$$

Remark 1. This result was proved by M. Cavachi in [1] for the case when $U = -V = u = -v = k = F = 1$ and $l = 0$. Our proof is used in the method of [1].

Remark 2. This lemma continues to hold even if the above relation (1) is valid for all large $n \in \mathbb{N}$ instead of for all $n \in \mathbb{N}$.

Proof. Assume that the integers $U \geq 1$, V, $u \geq 1$, v, $\alpha > 1$, $\beta > 1$, $k \geq \geq 1$, l and $F \neq 0$ satisfy (1). For every non-negative integer m we define a sequence of polynomials $Q_m(x) \in \mathbb{Z}[x]$ as follows:

$$Q_0(x) := v \tag{2}$$

and

$$Q_{m+1}(x) = \alpha^{k(m+1)}(Ux+V)Q_m(\alpha^k x) - \beta^k(U\alpha^{k(m+1)}x + V)Q_m(x). \tag{3}$$

It is clear from (2) and (3) that $\deg(Q_m(x)) \leq m$ for all integers $m \geq 0$.

Let

$$R_0(n) := F\frac{u\beta^{kn+l} + v}{U\alpha^{kn+l} + V} \quad \text{for all} \quad n \in \mathbb{N} \tag{4}$$

and for each integer $m \geq 0$ we define $R_{m+1}(n)$ by the relation

$$R_{m+1}(n) := \alpha^{k(m+1)}R_m(n+1) - \beta^k R_m(n). \tag{5}$$

One can check from (1) and (4)-(5) that

$$R_m(n) \in \mathbb{Z} \ \text{ for all integers } \quad m \geq 0, n \geq 1. \tag{6}$$

Finally, we define a sequence $\{P_0, P_1, \ldots, P_m, \cdots\}$ of integers as follows:

$$P_0 := u \ \text{ and } \ P_{m+1} := \beta^k V \left(\alpha^{k(m+1)} - 1\right) P_m.$$

We shall prove that

$$R_m(n) = F \cdot \frac{P_m \beta^{kn+l} + Q_m(\alpha^{kn+l})}{\left(U\alpha^{k(n+m)+l} + V\right) \cdots (U\alpha^{kn+l} + V)}. \tag{7}$$

By using the facts $P_0 = u$ and $Q_0(x) = v$, it follows from (2) that (7) is true for $m = 0$ and for all $n \in \mathbb{N}$. Assume that (7) holds for m and for all $n \in \mathbb{N}$. Now, by using (5) and the assumption of induction, we have

$$\begin{aligned} R_{m+1}(n) &:= \alpha^{k(m+1)} R_m(n+1) - \beta^k R_m(n) = \\ &= F\alpha^{k(m+1)} \frac{P_m \beta^{k(n+1)+l} + Q_m(\alpha^{k(n+1)+l})}{\left(U\alpha^{k(n+1+m)+l} + V\right) \cdots \left(U\alpha^{k(n+1)+l} + V\right)} - \\ &\quad - F\beta^k \frac{P_m \beta^{kn+l} + Q_m(\alpha^{kn+l})}{\left(U\alpha^{k(n+m)+l} + V\right) \cdots (U\alpha^{kn+l} + V)}. \end{aligned} \tag{8}$$

Since

$$\alpha^{k(m+1)} P_m \beta^{k(n+1)+l} \left(U\alpha^{kn+l} + V\right) - \beta^k P_m \beta^{kn+l} \left(U\alpha^{k(n+1+m)+l} + V\right) =$$
$$= V\alpha^{k(m+1)} P_m \beta^{k(n+1)+l} - V\beta^k P_m \beta^{kn+l} = \beta^{kn+l} P_{m+1}$$

and

$$\alpha^{k(m+1)} \left(U\alpha^{kn+l} + V\right) Q_m(\alpha^{k(n+1)+l}) - \beta^k \left(U\alpha^{k(n+1+m)+l} + V\right) Q_m(\alpha^{kn+l})$$
$$= Q_{m+1}(\alpha^{kn+l}),$$

we infer from (8) that

$$R_{m+1}(n) = F \frac{P_{m+1} \beta^{kn+l} + Q_{m+1}(\alpha^{kn+l})}{\left(U\alpha^{k(n+1+m)+l} + V\right) \left(U\alpha^{k(n+m)+l} + V\right) \cdots (U\alpha^{kn+l} + V)}.$$

Hence (7) is proved.

Now we prove that there is a positive integer e such that $\beta = \alpha^e$. Assume in contrary that $\beta \neq \alpha^i$ for all $i \in \mathbb{N}$. Then let $m \in \mathbb{N}$ such that $\alpha^m < \beta < \alpha^{m+1}$. Since

$$\begin{aligned}&\Big(U\alpha^{k(n+m)+l} + V\Big)\cdots\big(U\alpha^{kn+l} + V\big) =\\ &= (U\alpha^{km} + V\alpha^{-(kn+l)})\cdots(U\alpha^{k} + V\alpha^{-(kn+l)})(U + V\alpha^{-(kn+l)})\alpha^{(m+1)(kn+l)}\\ &\qquad \geq \frac{1}{2}\alpha^{(m+1)(kn+l)}\end{aligned}$$

holds for all larger $n \in \mathbb{N}$, we infer from (7) that

$$\begin{aligned}|R_m(n)| &\leq |F|\frac{|P_m\beta^{kn+l} + Q_m(\alpha^{kn+l})|}{\frac{1}{2}\alpha^{(m+1)(kn+l)}} \leq\\ &\leq 2|F|\Big[|P_m|\Big(\frac{\beta}{\alpha^{m+1}}\Big)^{kn+l} + \frac{|Q_m(\alpha^{kn+l})|}{\alpha^{(m+1)(kn+l)}}\Big]\end{aligned}$$

also holds for all larger $n \in \mathbb{N}$. Since $\frac{\beta}{\alpha^{m+1}} < 1$ and $\deg(Q_m(x)) \leq m$, the right side tends to zero as $n \to \infty$, therefore the last relation with (6) implies that $R_m(n) = 0$ for all larger $n \in \mathbb{N}$. Hence, we get from (7) that $P_m\beta^{kn+l} + Q_m(\alpha^{kn+l}) = 0$, consequently

$$P_m\left(\frac{\beta}{\alpha^m}\right)^{kn+l} + \frac{Q_m(\alpha^{kn+l})}{\alpha^{m(kn+l)}} = 0$$

for all large $n \in \mathbb{N}$. Thus $P_m = 0$, since otherwise the left side is unbounded as $n \to \infty$. This is impossible, because from the definition of the sequence $\{P_m\}_{m=0}^{\infty}$ we have $P_m \neq 0$ for all $m \geq 0$.

Thus we have prove that $\beta = \alpha^e$ for a suitable positive integer e. Hence

$$R_0(n) = F\frac{u\beta^{kn+l} + v}{U\alpha^{kn+l} + V} = F\frac{u\alpha^{e(kn+l)} + v}{U\alpha^{kn+l} + V},$$

and

$$F\frac{u(-V)^e + vU^e}{U\alpha^{kn+l} + V} = U^eR_0(n) - F\frac{u\Big[\big(U\alpha^{kn+l}\big)^e - (-V)^e\Big]}{U\alpha^{kn+l} + V}$$

are integers for all $n \in \mathbb{N}$. This shows that

$$u(-V)^e + vU^e = 0.$$

The proof of the lemma is complete.

The proof of Theorem 15. Let $A > 0,\ B,\ a > 0,\ b,\ N > 0$ and $C \neq 0$ be integers. Let $f \in \mathcal{M}^*$ be integer-valued, for which the congruence

$$f\,[A(an+b)+B] \equiv C \pmod{an+b}$$

holds for every $n \in \mathbb{N},\ n > N$.

Assume first that $f(p) = 1$ for all primes $p,\ (p, aA) = 1$. In this case we get from the fact $f \in \mathcal{M}^*$ that $f(n) = 1$ for all $n \in \mathbb{N},\ (n, aA) = 1$. Thus Theorem 15 is true with $e = 0$.

In the following, let p be a suitable prime number such that $(p, aA) = 1$, $|f(p)| > 1$ and let $M \in \mathbb{N},\ M > N$. Then

$$p^{\varphi(aA)m}\Big(A(aM+b)+B\Big) = A\left(p^{\varphi(aA)m}(aM+b) + B\frac{p^{\varphi(aA)m}-1}{A}\right) + B,$$

$$p^{\varphi(aA)m}(aM+b) + B\frac{p^{\varphi(aA)m}-1}{A} \equiv b \pmod{a}$$

and

$$p^{\varphi(aA)m}(aM+b) + B\frac{p^{\varphi(aA)m}-1}{A} \geq aN+b$$

are true for all $m \in \mathbb{N}$.

From the assumption of Theorem 15, we have

$$\begin{aligned} f(p)^{\varphi(aA)m} f\Big(A(aM+b)+B\Big) &= f\left(p^{\varphi(aA)m}\Big(A(aM+b)+B\Big)\right) \equiv \\ &\equiv C \left(\bmod\ \frac{p^{\varphi(aA)m}\Big(A(aM+b)+B\Big)-B}{A}\right) \end{aligned} \tag{9}$$

holds for all $m \in \mathbb{N}$. Thus, by setting $m = 2n$ into (9), we have

$$\Big(A(aM+b)+B\Big)p^{2\varphi(aA)n} - B \ \Big|\ A\left(f\Big(A(aM+b)+B\Big)f(p)^{2\varphi(aA)n} - C\right)$$

holds for all $n \in \mathbb{N}$. Since $f(p)^2 > 1$, we infer from our lemma that there is a positive integer $e = e_p$ such that

$$f(p)^2 = p^{2e}$$

and

$$f\Big(A(aM+b)+B\Big)B^e - C\Big(A(aM+b)+B\Big)^e = 0. \tag{10}$$

Thus, we have proved that (10) holds for each fixed $M \in \mathbb{N},\ M > N$, and so (10) holds for all $M \in \mathbb{N},\ M > N$.

Now consider the function

$$g(n) = \frac{f(n)}{n^e} \quad (n = 1, 2, \ldots).$$

It is obvious that $g \in \mathcal{M}^*$ and it follows from (10) that

$$g\left(A(aM + b) + B\right) = \frac{C}{B^e}$$

holds for all $M \in \mathbb{N},\ M > N$, which shows that $g(n) = \chi(n)$, where χ is a real-valued Dirichlet character $\pmod{aA}$. Theorem 15 is proved.

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