

**THE DISTRIBUTION OF  
AN ADDITIVE ARITHMETICAL FUNCTION  
ON THE SET OF SHIFTED INTEGERS  
HAVING  $k$  DISTINCT PRIME FACTORS**

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*To the memory of Professor Matukumalli Venkata Subbarao*

**Abstract.** It is proven that an additive arithmetical function has a limit law on the set of shifted integers having  $k$  prime factors if and only if the Erdős-Wintner condition holds.

## 1. Notations and results

We shall use the following notations:

$\mathcal{P}$  = set of prime numbers;  $p, q$  with or without suffixes always denote prime numbers,  $\omega(n)$  denote the number of distinct prime factors of  $n$ .  $P(n)$ ,  $p(n)$  denote the largest and the smallest prime divisor of  $n$  respectively.  $c$  denotes a constant, not certainly the same at different locations.

Let  $\mathcal{P}_k = \{n : \omega(n) = k\}$ ,  $\mathcal{P}_k(x) = \mathcal{P}_k \cap [1..x]$ , and  $\Pi_k(x) = \#\mathcal{P}_k(x)$ .  
 $\pi(x) = \#\{p \leq x \mid p \in \mathcal{P}\}$ ,

$$\pi(x, k, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} 1.$$

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Throughout this paper  $\pi_k$  denote integers having  $k$  distinct prime factors.

Let  $F_x$  be a sequence of distribution functions, and let  $F$  be a distribution function. We say that  $F_x$  converges weakly to  $F$ , in notation  $F_x \Rightarrow F$ , if  $\lim_{x \rightarrow \infty} F_x(z) = F(z)$  holds at all continuity points of  $F$ .

Let  $T$  be an assertion,  $A$  be a subset of  $\mathbb{N}$  and let

$$\nu_x(n \in A : T(n)) := \frac{1}{|A \cap [1, x]|} \#\{n \leq x, n \in A : T(n) \text{ holds true}\}.$$

Let  $f$  be a real additive function. We say that  $f$  possess a limit law on  $A$ , if for the sequence of distribution functions

$$F_x(z) = \nu_x(n \in A : f(n) \leq z),$$

there is a distribution function say  $F$ , such that  $F_x \Rightarrow F$ .

Erdős and Wintner [3] proved that an additive arithmetical function  $f$  possesses a limit law on  $\mathbb{N}$  if and only if the Erdős-Wintner condition holds, i.e. if and only if the three series

$$(1.1) \quad \sum_{|f(p)| > 1} \frac{1}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f^2(p)}{p}$$

converge.

Kátai [6] proved about 40 years ago that the convergence of the 3 series (1.1) implies the existence of the limit distribution of  $f$  on the set  $\mathcal{P} + 1$ .

The necessity of the convergence of (1.1) has been proved by Hildebrand [5] about 20 years ago.

The aim of this paper is to prove the following

**Theorem 1.** *Let  $f$  be a real additive function. Let  $2 \leq k \leq \epsilon(x) \cdot \sqrt{\log \log x}$ ,  $\epsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ). Let*

$$F_{k,x}(z) = \nu_x(n \in \mathcal{P}_k : f(n+1) \leq z).$$

*Assume that there is a sequence  $k = k_x$  and a distribution function  $F$  such that  $F_{k_x,x} \Rightarrow F$ . Then the 3 series given in (1.1) are convergent.*

*Conversely, assume that the series in (1.1) are convergent. Then with a distribution function  $G$*

$$\max_{2 \leq k \leq \epsilon(x) \sqrt{\log \log x}} |F_{k,x}(y) - G(y)| \rightarrow 0, \quad (x \rightarrow \infty)$$

if  $y$  is a continuity point of  $G$ . Consequently  $F = G$ .

The characteristic function of  $F$  is given by

$$\varphi(t) = \prod_p (1 + h(p)),$$

where

$$(1.2) \quad h(p) = -\frac{1}{p-1} + \sum_{m=1}^{\infty} \frac{e^{itf(p^m)}}{p^m}.$$

**Remark.** We know from the theorem of Paul Lévy that  $F$  is of pure type, it is continuous if and only if

$$\sum_{f(p) \neq 0} \frac{1}{p}$$

diverges.

One can conclude the sufficiency part of Theorem 1 from the next

**Theorem 2.** Let  $1 \geq \epsilon > 0$  and  $\varrho = \min\{\epsilon/4, 1/4\}$ . Let  $f$  be a real additive function and assume that

$$(1.3) \quad \sum_{|f(p)| > 1} \frac{1}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f^2(p)}{p}$$

are convergent.

Let

$$(1.4) \quad A(x) := \sum_{\substack{|f(p)| \leq 1 \\ p \leq x}} \frac{f(p)}{p},$$

$$(1.5) \quad a(m) := \sum_{\substack{|f(p)| \leq 1 \\ p|m}} \frac{f(p)}{p}.$$

Let  $K_D(x) = \{Dp + 1 \leq x \mid p \in \mathcal{P}\}$ ,

$$(1.6) \quad F_{D,x}(y) := \nu_x \left( n \in K_D(x), f(n) - \left( A \left( \left( \frac{x-1}{D} \right)^\varrho \right) - a(D) \right) \leq z \right),$$

with characteristic function  $\varphi_{D,x}(t)$ , and let

$$(1.7) \quad \varphi_D(t) := \prod_{\substack{|f(p)| > 1 \\ p|D}} (1 + h(p)) \prod_{\substack{|f(p)| \leq 1 \\ p|D}} (1 + h(p)) e^{-it \frac{f(p)}{p}}.$$

Then

$$(1.8) \quad \max_{1 \leq D \leq x^{1-\epsilon}} |\varphi_{D,x}(t) - \varphi_D(t)| \rightarrow 0 \quad (x \rightarrow \infty)$$

uniformly for all bounded values of  $t$ , i.e. if  $|t| < T$ ,  $T$  is an arbitrary constant.

## 2. Preliminaries

We need analogues of some-well known theorems related to prime numbers. In the proof of Theorem 1 it is allowed to drop not more than  $o(\Pi_k(x))$  elements. By this the limit distribution function does not change.

Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \quad p_1 < p_2 < \cdots < p_k$$

and

$$\pi_j = \pi_j(n) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} \quad (j = 1, \dots, k).$$

Let

$$\gamma_j(n) = \frac{\log p_{j+1}}{\log \pi_j} \quad (j = 1, \dots, k-1),$$

$$\Delta(n) = \min_{j=1,2,\dots,k-1} \gamma_j(n).$$

First we investigate some sets, which are unimportant in our case. We have the following

**Lemma 1.** *Let  $\epsilon(x) \rightarrow 0$  slowly. Let  $M = M_x$  be defined by  $\log \log M = \sqrt{\log \log(x)}$ . Let  $2 \leq k \leq \epsilon(x) \sqrt{\log \log x}$ . Then there exists a sequence  $A_x \rightarrow \infty$  ( $x \rightarrow \infty$ ) such that*

1.

$$\Delta(n) \geq A_x,$$

and

2.  $\alpha_1 = \alpha_2 = \dots = \alpha_k = 1$  and  $p_1 \geq M$  holds for all but  $\delta(x)\Pi_k(x)$  element of  $\mathcal{P}_k(x)$ , where  $\delta(x) \rightarrow 0$ . Let  $U_k(x)$  be the set of those elements of  $\mathcal{P}_k(x)$ , for which these conditions hold true.

**Proof.** The following sets have zero relative density in  $\mathcal{P}_k$ :

1.  $A_1 = \{n \in \mathcal{P}_k, n \leq x : \exists p^2 | n\}$ . We have

$$\begin{aligned} \#A_1 &\leq \sum_{\substack{p^\alpha \leq x^{1/2} \\ \alpha \geq 2}} \Pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha > x^{1/2} \\ \alpha \geq 2}} \frac{x}{p^\alpha} \ll \\ &\ll \Pi_k(x) \frac{k}{\log \log x} \sum_{\substack{p^\alpha \leq x^{1/2} \\ \alpha \geq 2}} \frac{1}{p^\alpha} + \mathcal{O}\left(x^{3/4}\right). \end{aligned}$$

Here we used

$$\frac{\Pi_{k-1}(x)}{\Pi_k(x)} \sim \frac{k}{\log \log x}$$

which is a direct consequence of

$$(2.1) \quad \Pi_k(x) = \frac{x}{\log x} \frac{\log \log^{k-1} x}{(k-1)!} \left(1 + \mathcal{O}\left(\frac{1}{\log \log x}\right)\right)$$

(see e.g in [7]).

2.  $A_2 = \{n \in \mathcal{P}_k, n \leq x : p(n) < M\}$ . We have

$$\begin{aligned} \#A_2 &\leq \\ &\leq \sum_{\substack{p^\alpha \leq x^{1/2} \\ p < M}} \Pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha > x^{1/2} \\ \alpha \geq 2}} \frac{x}{p^\alpha} \ll \Pi_k(x) \frac{k}{\log \log x} \sum_{p < M} \frac{1}{p} + \mathcal{O}\left(x^{3/4}\right). \end{aligned}$$

By means of these last two steps we can assume that  $p(n) > M$ , and  $n$  is squarefree.

Observe that, if  $n \in U_k(x)$  and  $pq|n, p < q$ , then  $p^{A_x} < q$ , which is an immediate consequence of the assumption  $\Delta(n) \geq A_x$ . The number of integers  $n \in \mathcal{P}_k(x)$  for which  $n \notin A_1$  or  $n \notin A_2$  or there are prime divisors  $p < q$ , such that  $q < p^{A_x}$  is not more than

$$\sum_{\substack{pq\pi_{k-2} \leq x \\ M < p < q < p^{A_x} \\ p < x^{1/2A_x}}} 1 + \sum_{\substack{pq\pi_{k-2} \leq x \\ M < p < q < p^{A_x} \\ p \geq x^{1/2A_x}}} 1.$$

The first sum is at most

$$\begin{aligned} \sum_{\substack{M < p < q < p^{A_x} \\ p < x^{1/2A_x}}} \Pi_{k-2} \left( \frac{x}{pq} \right) &\ll \frac{x}{\log x} \frac{(\log \log x)^{k-3}}{(k-3)!} \sum_{\substack{M < p < q < p^{A_x} \\ p < x^{1/2A_x}}} \frac{1}{pq} \ll \\ &\ll \Pi_k(x) \frac{k^2}{\log \log x} \log A_x, \end{aligned}$$

so if  $A_x$  tends to infinity suitably slowly, then this last expression is  $\Pi_k(x) o(1)$ .

For the second sum we use the sieve estimation for the number of integers not exceeding  $x$ , for which  $P(n) > z$ . The number of these integers is smaller than

$$\frac{x}{\log z} + c \frac{z^2}{\log^2 z}$$

if  $z > z_0$ . This estimation can be found for example in [4] (Theorem 3.6). Using this, we get that the second sum does not exceed

$$\begin{aligned} &\sum_{\substack{pq\pi_{k-2} \leq x \\ M < p < q < p^{A_x} \\ p > x^{1/2A_x}}} 1 \ll \\ &\ll \sum_{\pi_{k-2} < x^{1-\frac{1}{A_x}}} \sum_{\substack{n \leq \frac{x}{\pi_{k-2}} \\ P(n) > x^{1/2A_x}}} 1 \ll A_x \sum_{\pi_{k-2} < x^{1-\frac{1}{A_x}}} \frac{x}{\pi_{k-2} \log x} + A_x^2 \frac{x^{1/A_x}}{\log^2 x}, \end{aligned}$$

which is  $\Pi_k(x) o(1)$  with appropriate  $A_x$ . Here we used that

$$\sum_{\pi_{k-1} \leq x} \frac{1}{\pi_{k-1}} = \frac{\log \log^{k-1} x}{(k-1)!} (1 + o(1)).$$

The proof is ready.

Let us introduce the notations

$$B_k(x) := \#U_k(x)$$

and

$$B_k(x, d, l) = \sum_{\substack{n \in U_k(x) \\ n \equiv l \pmod{d}}} 1.$$

We shall prove some inequalities for the distribution of the elements of  $U_k(x)$  in arithmetic progressions.

We have

**Lemma 2.** *Let  $x \geq 2$ ,  $d < \log^A x$ , where  $A$  is a fixed positive number, and let  $(d, l) = 1$ . We have*

$$B_k(x, d, l) = \frac{B_k(x)}{\varphi(d)} \left(1 + \mathcal{O}\left(e^{-c\sqrt{\log x}}\right)\right)$$

uniformly for  $2 \leq k \leq \epsilon(x)\sqrt{\log \log x}$ , where  $\epsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ).

**Proof.** Let  $S_x$  be the set of those  $\pi_{k-1}$ , for which there exists at least one prime  $p > P(\pi_{k-1})$  such that  $\pi_{k-1}p \in U_k(x)$ . Let  $p^* = p_{\pi_{k-1}}$  be the smallest  $p$  with this property. Then  $\pi_{k-1}p \in U_k(x)$  for all  $p^* \leq p \leq \frac{x}{\pi_{k-1}}$ . Let  $\pi_{k-1}p \equiv l \pmod{d}$ . Then, using Lemma 1,  $\pi_{k-1} < x^\lambda$ , with an appropriate  $\lambda < 1/4$ ,  $P(\pi_{k-1}) < p$  and  $p(\pi_{k-1}) > \log^A x$ , when  $x$  is larger than  $x_A$  say. With this conditions we have  $(\pi_{k-1}, d) = 1$ . We get  $p \equiv l_{\pi_{k-1}} \pmod{d}$  with a unique  $l_{\pi_{k-1}} \pmod{d}$ .

We have that

$$(2.2) \quad \left| B_k(x, d, l) - \frac{B_k(x)}{\varphi(d)} \right| \ll \sum_{\pi_{k-1} \in S_x} \left| \left\{ \pi \left( \frac{x}{\pi_{k-1}}, d, l_{\pi_{k-1}} \right) - \pi(P(\pi_{k-1}), d, l_{\pi_{k-1}}) \right\} - \frac{1}{\varphi(d)} \left\{ \pi \left( \frac{x}{\pi_{k-1}} \right) - \pi(P(\pi_{k-1})) \right\} \right|.$$

The Theorem of Siegel-Walfisz for prime numbers is applicable for

$$\sum_{\pi_{k-1} \in S_x} \left| \pi \left( \frac{x}{\pi_{k-1}}, d, l_{\pi_{k-1}} \right) - \frac{1}{\varphi(d)} \pi \left( \frac{x}{\pi_{k-1}} \right) \right|.$$

We get that the right hand side of (2.2) does not exceed

$$\begin{aligned} & \frac{c}{\varphi(d)} \sum_{\pi_{k-1} \in S_x} \frac{x}{\pi_{k-1}} e^{-c\sqrt{\log x}} + c \sum_{\pi_{k-1} \in S_x} P(\pi_{k-1}) \ll \\ & \ll \frac{1}{\varphi(d)} x e^{-c\sqrt{\log x}} \frac{(\log \log x)^{k-1}}{(k-1)!} + \mathcal{O}\left(x^{1/2}\right), \end{aligned}$$

because  $\sum_{\pi_{k-1} < x} \frac{1}{\pi_{k-1}} \ll \frac{(\log \log x)^{k-1}}{(k-1)!}$ .

**Lemma 3.** *Let  $x \geq 2$ ,  $d < x^a$  with a fixed  $0 \leq a < 1$ . Then*

$$B_k(x, d, l) < c(a) \frac{B_k(x)}{\varphi(d)},$$

if  $(d, l) = 1$  and  $2 \leq k \leq \epsilon(x) \sqrt{\log \log x}$ , where  $\epsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ). Here  $c(a)$  depends only on  $a$ .

**Proof.** With the notations of Lemma 2 we have

$$B_k(x, d, l) \leq c \sum_{\pi_{k-1} \in S_x} \pi \left( \frac{x}{\pi_{k-1}}, d, l_{\pi_{k-1}} \right).$$

Applying the Brun-Titchmarsh Theorem, the above sum does not exceed

$$c \frac{x}{\varphi(d) \log x} \sum_{\pi_{k-1} \in S_x} \frac{1}{\pi_{k-1}} \ll \frac{B_k(x)}{\varphi(d)}.$$

Let

$$B_k(x|d) = \sum_{\substack{n \in U_k(x) \\ (n, d) = 1}} 1.$$

**Lemma 4.** *Let  $x \geq 2$ ,  $A > 0$  an arbitrary number and  $\alpha < 1/2$ . For  $2 \leq k \leq \epsilon(x) \sqrt{\log \log x}$ , where  $\epsilon(x) \rightarrow 0$  ( $x \rightarrow \infty$ ), we have*

$$\sum_{d < x^\alpha} \max_{(l, d) = 1} \max_{z \leq x} \left| B_k(z, d, l) - \frac{B_k(z|d)}{\varphi(d)} \right| \ll B_k(x) \log^{-A} x.$$

The constant implied by  $\ll$  does not depend on  $k$ .

**Proof.** With the notations of Lemma 2 we have for  $x > x_0$  that

$$\begin{aligned} & \left| B_k(x, d, l) - \frac{B_k(x|d)}{\varphi(d)} \right| \leq \\ & \leq \sum_{\substack{\pi_{k-1} \in S_x \\ (\pi_{k-1}, d) = 1}} \left| \pi \left( \frac{x}{\pi_{k-1}}, d, l_{\pi_{k-1}} \right) - \frac{\pi \left( \frac{x}{\pi_{k-1}} | d \right)}{\varphi(d)} \right| + \mathcal{O}(x^{1/3}). \end{aligned}$$

Applying the Bombieri-Vinogradov theorem we get that the right hand side of this last inequality does not exceed

$$c \frac{x}{\log x} \log^{-A} x \sum_{\pi_{k-1} \in S} \frac{1}{\pi_{k-1}} + \mathcal{O}\left(x^{2/3}\right) \ll B_k(x) \log^{-A} x.$$

Here we used, that  $\pi(x) - \pi(x|d) \leq \omega(d) \leq \log x$ .

A more general version of this lemma was established by Wolke and Zhang [9].

**Lemma 5.** (Wolke-Zhang) *For any given  $A > 0$  and  $0 < \epsilon < 1/2$  there exist  $\eta > 0$  such that*

$$\sum_{d \leq x^{1/2-\epsilon}} \max_{(a,d)=1} \max_{y \leq x} \left| \Pi_k(y, d, a) - \frac{\Pi_k(y|d)}{\varphi(d)} \right| \ll \Pi_k(x) \log^{-A} x$$

holds uniformly for  $k \leq \eta \log x / \log \log^2 x$ , and the constant implied in  $\ll$  depends on  $A$  and  $\epsilon$  only.

**Corollary 1.** *Let  $0 \leq \eta < 1/4$ . We have*

$$\sum_{x^\eta < q < x^{2\eta}} B_k(x, q, -1) = \sum_{x^\eta < q < x^{2\eta}} \frac{B_k(x)}{q-1} + \mathcal{O}\left(\frac{B_k(x)}{\log^A x}\right).$$

**Proof.** From Lemma 4 we have

$$\sum_{x^\eta < q < x^{2\eta}} B_k(x, q, -1) = \sum_{x^\eta < q < x^{2\eta}} \frac{B_k(x|q)}{q-1} + \mathcal{O}\left(\frac{B_k(x)}{\log^A x}\right).$$

On the other hand

$$B_k(x) \geq B_k(x|q) \geq B_k(x) - \sum_{l=1}^{\infty} \sum_{\substack{aq^l < x \\ a \in P_{k-1}(x) \\ (a,q)=1}} 1 \geq B_k(x) + \mathcal{O}\left(\frac{x}{q}\right).$$

Putting it together

$$\begin{aligned} \sum_{x^\eta < q < x^{2\eta}} \frac{B_k(x|q)}{q-1} &= \sum_{x^\eta < q < x^{2\eta}} \frac{B_k(x)}{q-1} + \mathcal{O}\left(\sum_{x^\eta < q < x^{2\eta}} \frac{x}{q^2}\right) = \\ &= \sum_{x^\eta < q < x^{2\eta}} \frac{B_k(x)}{q-1} + \mathcal{O}(x^{1-\eta}). \end{aligned}$$

### 3. Proof of Theorem 2

We shall investigate the sequence of the characteristic functions. Let  $y_x$  be tending to infinity so slowly that  $y_x \leq \frac{1}{2} \log \log \log \log x$ , and  $x \geq c(\epsilon)$ , say. Let us define the additive function

$$f_0(p^\alpha) = \begin{cases} f(p^\alpha) & \text{if } p^\alpha \leq y_x, \\ 0 & \text{else,} \end{cases}$$

and consider the following distribution function:

$$G_{x,D}(z) := \nu_x(n \in K_D(x), f_0(n) - (A(y_x) - a(D)) \leq z).$$

Then the characteristic function of  $G_{x,D}$  is

$$(3.1) \quad \psi_{x,D}(t) = \frac{1}{\#K_D(x)} e^{-it(A(y_x) - a(D))} \sum_{n \in K_D(x)} e^{itf_0(n)}.$$

We define furthermore  $g_x$  by

$$(3.2) \quad g_x = \mu * e^{itf_0},$$

where  $\mu$  is the Mbius function. We have that  $g_x(p^\alpha) = e^{itf_0(p^\alpha)} - e^{itf_0(p^{\alpha-1})}$ , and by the inequality  $\pi(y_x) < 2 \frac{y_x}{\log y_x}$  we get that  $g_x(n) = 0$  for  $n > e^{2y_x}$ , i.e. if  $n > \log \log \log x$ . These together imply that

$$(3.3) \quad \sum_{n=1}^{\infty} \frac{g_x(n)}{\varphi(n)} = \\ = \prod_{p \leq y_x} \left( 1 + \sum_{\alpha \geq 1} \frac{e^{itf_0(p^\alpha)} - e^{itf_0(p^{\alpha-1})}}{p^{\alpha-1}(p-1)} \right) = \prod_{p \leq y_x} \left( 1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{e^{itf_0(p^\alpha)}}{p^\alpha} \right).$$

It is clear that

$$(3.4) \quad \sum_{n \in K_D(x)} e^{itf_0(n)} = \sum_{\substack{d \leq x \\ (D, d) = 1}} g_x(d) \pi \left( \frac{x-1}{D}, d, l_d \right),$$

where  $l_d$  is defined by  $Dl_d + 1 \equiv 0 \pmod{d}$ . Since  $g_x(d) = 0$  if  $d > \epsilon \log x$ , we can apply the Siegel-Walfisz Theorem, whence the right hand side of (3.4) is

$$\pi \left( \frac{x-1}{D} \right) \sum_{\substack{d=1 \\ (D,d)=1}}^{\infty} \frac{g_x(d)}{\varphi(d)} + \mathcal{O} \left( \pi \left( \frac{x-1}{D} \right) e^{-c\sqrt{\log x}} \sum_{\substack{d=1 \\ (D,d)=1}}^{\infty} \frac{|g_x(d)|}{\varphi(d)} \right).$$

In the error term

$$\begin{aligned} & \sum_{\substack{d=1 \\ (D,d)=1}}^{\infty} \frac{|g_x(d)|}{\varphi(d)} \leq \\ & \leq \prod_{p \leq y_x} \left( 1 + \sum_{\alpha=1}^{\infty} \frac{|g_x(p^\alpha)|}{\varphi(p^\alpha)} \right) \leq \prod_{p \leq y_x} \left( 1 + \frac{2p}{(p-1)^2} \right) \ll \log \log \log x, \end{aligned}$$

say. Furthermore

$$\sum_{\substack{d=1 \\ (d,D)=1}}^{\infty} \frac{g_x(d)}{\varphi(d)} = \prod_{\substack{p \leq y_x \\ p \nmid D}} \left( 1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{e^{itf_0(p^\alpha)}}{p^\alpha} \right) = (1+o_x(1)) \prod_{\substack{p \leq y_x \\ p \nmid D}} (1+h(p)),$$

where the implied constant in  $o_x(1)$  is absolute, it does not depend on  $t$ , and  $h(p)$  is defined by (1.2). Thus we obtain, that

$$\psi_{x,D}(t) = \prod_{\substack{p \leq y_x \\ |f(p)| > 1 \\ p \nmid D}} (1+h(p)) \prod_{\substack{p \leq y_x \\ |f(p)| \leq 1 \\ p \nmid D}} (1+h(p)) e^{-it \frac{f(p)}{p}} + o_x(1)$$

uniformly in  $t \in \mathbb{R}$ , and the constant implied by  $o_x(1)$  is absolute.

Now using the convergence of (1.3), one can easily see that

$$(3.5) \quad \max_{1 \leq D \leq x^{1-\epsilon}} |\psi_{x,D}(t) - \varphi_D(t)| \rightarrow 0 \quad (x \rightarrow \infty)$$

uniformly for all  $|t| < T$ , where  $\varphi_D(t)$  is given by (1.7).

Using Lemma 1.11 in [1] it follows immediately, that  $G_{x,D} \Rightarrow F_D$  for all fixed  $D$ .

Next we define another two additive functions as follows: let

$$f_1(p^\alpha) = \begin{cases} f(p) & \text{if } y_x < p \leq \left(\frac{x-1}{D}\right)^\theta, \alpha = 1 \text{ and } |f(p)| \leq 1, \\ 0 & \text{else,} \end{cases}$$

and let

$$f_2(p^\alpha) = \begin{cases} f(p) & \text{if } \left(\frac{x-1}{D}\right)^\varrho < p \leq \left(\frac{x-1}{D}\right)^{1-\vartheta_x}, \alpha = 1 \text{ and } |f(p)| \leq 1, \\ 0 & \text{else,} \end{cases}$$

where  $\varrho < \min(\epsilon/4, 1/4)$  and  $\vartheta_x$  is tending to zero slowly.

First we show that

$$(3.6) \quad \nu_x(n \in K_D(x), f(n) \neq f_0(n) + f_1(n) + f_2(n)) = o(1)\pi\left(\frac{x-1}{D}\right)$$

with an appropriate  $\vartheta_x$ .

To do this, let  $B = \{q \in \mathcal{P} \mid |f(q)| > 1\}$ , and  $B_y = \{q \in \mathcal{P} \mid q > y, q \in B\}$ . Then

$$(3.7) \quad \#\{n \in K_D(x) \mid \exists q|n, q \in B_{y_x}\} = \delta(y_x)\pi\left(\frac{x-1}{D}\right),$$

where  $\delta(y_x) \rightarrow 0$  ( $x \rightarrow \infty$ ). To see this we split the numbers in the set in (3.7) into two, not necessary distinct parts. In the first part we take numbers which have a prime divisor  $q$  with  $y_x < q \leq \left(\frac{x-1}{D}\right)^\varrho$ . The other part contains the numbers, which have a prime divisor  $q$  such that  $\left(\frac{x-1}{D}\right)^\varrho < q \leq \left(\frac{x-1}{D}\right)^{1-\vartheta_x}$ . Let us denote the number of integers in this two parts by  $\Sigma_1$  and  $\Sigma_2$ , respectively.

We have that

$$\Sigma_1 \leq \sum_{Dp+1 \leq x} \sum_{\substack{q|Dp+1 \\ y_x < q \leq \left(\frac{x-1}{D}\right)^\varrho \\ |f(q)| > 1}} 1 \leq \sum_{\substack{y_x < q \leq \left(\frac{x-1}{D}\right)^\varrho \\ |f(q)| > 1}} \pi\left(\frac{x-1}{D}, q, l_q\right),$$

and similarly

$$\Sigma_2 \leq \sum_{\substack{\left(\frac{x-1}{D}\right)^\varrho < q \leq \left(\frac{x-1}{D}\right)^{1-\vartheta_x} \\ |f(q)| > 1}} \pi\left(\frac{x-1}{D}, q, l_q\right).$$

Using the Brun-Titchmarsh theorem for  $\Sigma_1$  we get,

$$\Sigma_1 \leq c\pi\left(\frac{x-1}{D}\right) \sum_{\substack{y_x < q \\ |f(q)| > 1}} \frac{1}{q}.$$

Using sieve estimates for  $\Sigma_2$  we get,

$$\Sigma_2 \leq c \sum_{\substack{(\frac{x-1}{D})^e < q \leq (\frac{x-1}{D})^{1-\vartheta_x} \\ |f(q)| > 1}} \frac{x-1}{qD \log \frac{x-1}{qD}} \leq c \frac{1}{\vartheta_x} \frac{x-1}{D \log \left(\frac{x-1}{D}\right)} \sum_{\substack{(\frac{x-1}{D})^e < q \\ |f(q)| > 1}} \frac{1}{q}.$$

With the choice

$$(3.8) \quad \vartheta_x^2 = \max \left( \sum_{\substack{(\frac{x-1}{D})^e < q \\ |f(q)| > 1}} \frac{1}{q}, \sum_{\substack{(\frac{x-1}{D})^e < q \\ |f(q)| \leq 1}} \frac{f^2(q)}{q} \right),$$

we get  $\Sigma_1 + \Sigma_2 = o(1)\pi\left(\frac{x-1}{D}\right)$ .

The next assertion holds:

$$(3.9) \quad \nu_x \left( n \in K_D(x); \exists q|n, q > \left(\frac{x-1}{D}\right)^{1-\vartheta_x} \right) = o(1)\pi\left(\frac{x-1}{D}\right),$$

which can be obtained using sieve estimates. We have that the left hand side of (3.9) is at most

$$\sum_{Dp+1 \leq x} \sum_{\substack{Dp+1=aq \\ (\frac{x-1}{D})^{1-\vartheta_x} < q}} 1 \leq \sum_{a \leq (\frac{x-1}{D})^{\vartheta_x}} \sum_{Dp+1=aq} 1,$$

which does not exceed

$$c \sum_{a < (\frac{x-1}{D})^{\vartheta_x}} \frac{x}{aD \log^2 \frac{x}{aD}} \leq c \frac{x}{D \log^2 \frac{x}{D}} \sum_{a < (\frac{x-1}{D})^{\vartheta_x}} \frac{1}{a},$$

which is  $o(1)\pi\left(\frac{x-1}{D}\right)$ .

Similarly

$$(3.10) \quad \begin{aligned} & \#\{n \in K_D(x) \mid \exists q^2|n, q > y\} \leq \\ & \leq \sum_{y < q < (\frac{x-1}{D})^a} \pi\left(\frac{x-1}{D}, q^2, l_q\right) + \frac{x-1}{D} \sum_{q \geq (\frac{x-1}{D})^a} \frac{1}{q^2} = \delta(y)\pi\left(\frac{x-1}{D}\right), \end{aligned}$$

where  $\delta(y) \rightarrow 0$  ( $y \rightarrow \infty$ ).

(3.7) and (3.9) and (3.10) imply (3.6).

Let

$$A_{D,y_x} \left( \left( \frac{x-1}{D} \right)^e \right) = \sum_{\substack{y_x < p \leq \left( \frac{x-1}{D} \right)^e \\ |f_1(p)| \leq 1 \\ p|D}} \frac{f_1(p)}{p}.$$

Next we prove a Turn-Kubilius type inequality, namely that  
(3.11)

$$\sum_{Dp+1 \leq x} \left| f_1(Dp+1) - A_{D,y_x} \left( \left( \frac{x-1}{D} \right)^e \right) \right|^2 \leq c\pi \left( \frac{x-1}{D} \right) \sum_{\substack{y_x < p \\ |f(p)| \leq 1}} \frac{f^2(p)}{p},$$

and

$$(3.12) \quad \sum_{Dp+1 \leq x} f_2^2(Dp+1) \leq c\pi \left( \frac{x-1}{D} \right) \sum_{\substack{\left( \frac{x-1}{D} \right)^e < p \\ |f(p)| \leq 1}} \frac{f^2(p)}{p}.$$

**Proof of (3.11).** Let

$$B^2 \left( \left( \frac{x-1}{D} \right)^e \right) := \sum_{\substack{y_x < q \leq \left( \frac{x-1}{D} \right)^e \\ q|D}} \frac{f_1^2(q)}{q}.$$

Let

$$\begin{aligned} & \sum_{Dp+1 \leq x} \left| f_1(Dp+1) - A_{D,y_x} \left( \left( \frac{x-1}{D} \right)^e \right) \right|^2 = \\ & = S_1 - 2A_{D,y_x} \left( \left( \frac{x-1}{D} \right)^e \right) S_2 + A_{D,y_x}^2 \left( \left( \frac{x-1}{D} \right)^e \right) \pi \left( \frac{x-1}{D} \right), \end{aligned}$$

where

$$\begin{aligned} S_1 &= \sum_{Dp+1 \leq x} \left( \sum_{q|Dp+1} f_1(q) \right)^2 = \sum_{\substack{y_x < q \leq \left( \frac{x-1}{D} \right)^e \\ q|D}} f_1^2(q) \pi \left( \frac{x-1}{D}, q, l_q \right) + \\ &+ \sum_{\substack{y_x \leq q \leq \left( \frac{x-1}{D} \right)^e \\ q|D}} \sum_{\substack{y_x \leq q' \leq \left( \frac{x-1}{D} \right)^e \\ q'|D \\ q \neq q'}} f_1(q) f_1(q') \pi \left( \frac{x-1}{D}, qq', l_{qq'} \right) = \Sigma_{11} + \Sigma_{12}, \end{aligned}$$

and

$$S_2 = \sum_{\substack{y_x < q \leq \left(\frac{x-1}{D}\right)^e \\ q|D}} f_1(q) \pi\left(\frac{x-1}{D}, q, l_q\right).$$

Since  $\varrho < \epsilon/4$ , thus we can estimate  $\Sigma_{11}$  using the Brun-Titchmarsh theorem, and we get

$$\Sigma_{11} < cB^2 \left( \left( \frac{x-1}{D} \right)^e \right) \pi\left(\frac{x-1}{D}\right).$$

Moreover  $\Sigma_{12}$  equals

$$\begin{aligned} & \pi\left(\frac{x-1}{D}\right) \sum_{\substack{y_x \leq q \leq \left(\frac{x-1}{D}\right)^e \\ q|D}} \sum_{\substack{y_x \leq q' \leq \left(\frac{x-1}{D}\right)^e \\ q'|D \\ q \neq q'}} \frac{f_1(q) f_1(q')}{\varphi(qq')} + \\ & + \sum_{\substack{y_x \leq q \leq \left(\frac{x-1}{D}\right)^e \\ q|D}} \sum_{\substack{y_x \leq q' \leq \left(\frac{x-1}{D}\right)^e \\ q'|D \\ q \neq q'}} f_1(q) f_1(q') \left( \pi\left(\frac{x-1}{D}, qq', l_{qq'}\right) - \frac{\pi\left(\frac{x-1}{D}\right)}{\varphi(qq')} \right) = \\ & = \zeta + E \end{aligned}$$

such that

$$\Sigma_{12} \leq A_{D,y_x}^2 \left( \left( \frac{x-1}{D} \right)^e \right) \pi\left(\frac{x-1}{D}\right) + E.$$

An application of the Cauchy-Schwarz inequality shows that  $E^2$  is at most

$$\begin{aligned} & \sum_{\substack{y_x < q \leq \left(\frac{x-1}{D}\right)^e \\ q|D}} \sum_{\substack{y_x < q' \leq \left(\frac{x-1}{D}\right)^e \\ q'|D \\ q \neq q'}} \frac{f_1^2(q) f_1^2(q')}{\varphi(q) \varphi(q')} \times \\ & \times \sum_{\substack{y_x < q \leq \left(\frac{x-1}{D}\right)^e \\ q|D}} \sum_{\substack{y_x < q' \leq \left(\frac{x-1}{D}\right)^e \\ q'|D \\ q \neq q'}} \varphi(qq') \left| \frac{\pi\left(\frac{x-1}{D}\right)}{\varphi(qq')} - \pi\left(\frac{x-1}{D}, qq', l_{qq'}\right) \right|^2. \end{aligned}$$

Since  $qq' < \left(\frac{x-1}{D}\right)^{1/2}$ , the Brun-Titchmarsh theorem is applicable, and we get that  $E$  is at most

$$B^2 \left( \left( \frac{x-1}{D} \right)^e \right) \left( \pi\left(\frac{x-1}{D}\right) \right)^{1/2} \times$$

$$\times \left( \sum_{\substack{y_x < q \leq \left(\frac{x-1}{D}\right)^\varrho \\ q \nmid D}} \sum_{\substack{y_x < q' \leq \left(\frac{x-1}{D}\right)^\varrho \\ q' \nmid D \\ q' \neq q'}} \left| \frac{\pi\left(\frac{x-1}{D}\right)}{\varphi(qq')} - \pi\left(\frac{x-1}{D}, qq', l_{qq'}\right) \right| \right)^{1/2}.$$

The Bombieri-Vinogradov theorem is also applicable, and we get

$$\Sigma_{12} \leq A_{D,y_x}^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right) + \mathcal{O}(1)B^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right) \log x^{-A},$$

where  $A$  is an arbitrary big positive constant. We have

$$S_1 = A_{D,y_x}^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right) + \mathcal{O}(1)B^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right).$$

To estimate  $S_2$  we note, that using the Cauchy- Schwarz inequality, we have

$$\begin{aligned} A_{D,y_x} \left( \left(\frac{x-1}{D}\right)^\varrho \right) &= \\ &= \sum_{\substack{y_x < p \leq \left(\frac{x-1}{D}\right)^\varrho \\ p \nmid D}} \frac{f_1(p)}{p} \leq \left( \sum_{\substack{y_x < p \leq \left(\frac{x-1}{D}\right)^\varrho \\ p \nmid D}} \frac{f_1^2(p)}{p} \sum_{\substack{y_x < p \leq \left(\frac{x-1}{D}\right)^\varrho \\ p \nmid D}} \frac{1}{p} \right)^{1/2} \ll \\ &\ll B \left( \left(\frac{x-1}{D}\right)^\varrho \right) (\log \log x)^{1/2}, \end{aligned}$$

such that using the above method one can easily see, that

$$S_2 = 2A_{D,y_x}^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right) + o(1)B^2 \left( \left(\frac{x-1}{D}\right)^\varrho \right) \pi\left(\frac{x-1}{D}\right),$$

which implies (3.11).

**Proof of (3.12).** Since a positive integer  $n \leq x$  can have only a bounded number of distinct prime divisors  $q > \left(\frac{x-1}{D}\right)^\varrho$ , we have that (3.12) does not exceed

$$c \sum_{\left(\frac{x-1}{D}\right)^\varrho < q \leq \left(\frac{x-1}{D}\right)^{1-\vartheta_x}} f_2^2(q) \pi\left(\frac{x-1}{D}, q, l_q\right).$$

Using sieve estimates this is

$$c \frac{1}{\vartheta_x} \frac{x-1}{D \log \frac{x-1}{D}} \sum_{\left(\frac{x-1}{D}\right)^e < q \leq \left(\frac{x-1}{D}\right)^{1-\vartheta_x}} \frac{f_2^2(q)}{q},$$

which thanks to the choice (3.8) implies (3.12).

Using (3.11) and (3.12) we get that

$$\frac{1}{K_D(x)} \sum_{n \in K_D(x)} \left| e^{it(f_0(n)+f_1(n)+f_2(n)-(A\left(\left(\frac{x-1}{D}\right)^e\right)-a(D)))} - e^{it(f_0(n)-(A(y_x)-a(D)))} \right|^2 = o_x(1)$$

uniformly for all  $|t| < T$ , and  $1 \leq D \leq x^{1-\epsilon}$ .

Using this and (3.5) we have proved that

$$\sup_{D \leq x^{1-\epsilon}} \left| \frac{1}{\#K_D(x)} e^{-it(A\left(\left(\frac{x-1}{D}\right)^e\right)-a(D))} \sum_{n \in K_D(x)} e^{itf(n)} - \varphi_D(t) \right| \rightarrow 0 \quad (x \rightarrow \infty)$$

uniformly as  $|t| \leq T$ ,  $T$  is an arbitrary constant.

## 4. Proof of Theorem 1

### 4.1. Concluding the sufficiency part of Theorem 1 from Theorem 2

Consider first the following sequence of distribution functions:

$$F_{k,x}(z) = \nu_x(n \in U_k(x) : f(n+1) - A(x) \leq z),$$

where

$$A(x) = \sum_{\substack{p \leq x \\ |f(p)| \leq 1}} \frac{f(p)}{p}.$$

With the notations of Lemma 2, an application of Lemma 1 shows that the characteristic function of  $F_x$  is

$$\frac{1}{B_k(x)} e^{-itA(x)} \sum_{\pi_{k-1} \in S_x} \sum_{P(\pi_{k-1}) < p \leq \frac{x}{\pi_{k-1}}} e^{itf(\pi_{k-1}p+1)}.$$

Since  $A(x)$  is convergent, using Theorem 2 and Lemma 1, we can express it in the following form

$$(4.1) \quad \begin{aligned} & \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) (\varphi_{\pi_{k-1}}(t) + o_x(1)) - \\ & - \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi(P(\pi_{k-1})) (\varphi_{\pi_{k-1}}(t) + o_x(1)), \end{aligned}$$

for all  $|t| < T$ . We use the estimation  $\pi(P(\pi_{k-1})) \leq x^{1/A_x}$  in the second term, and we get that this is

$$\mathcal{O}(x^{2/A_x}).$$

Using the identity

$$\varphi(t) = \varphi_D(t) K_D(t),$$

where

$$K_D(t) = \prod_{\substack{p|D \\ |f(p)|>1}} (1+h(p)) \prod_{\substack{p|D \\ |f(p)|\leq 1}} (1+h(p)) e^{-it\frac{f(p)}{p}}$$

we get that the main term in (4.1) is

$$(4.2) \quad \begin{aligned} \Sigma_1 = & \varphi(t) \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + \\ & + \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \varphi_{\pi_{k-1}}(t) (1 - K_{\pi_{k-1}}(t)) + \\ & + o(1) \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right). \end{aligned}$$

Since

$$(4.3) \quad \frac{1}{B_k(x)} \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \rightarrow 1 \quad (x \rightarrow \infty),$$

we get

$$\Sigma_1 = \varphi(t)(1 + o(1)).$$

To see this last identity we calculate the second term in (4.2). If  $p|\pi_{k-1}$ , then  $p > M_x$ , so for big enough  $x$

$$\begin{aligned} \log K_{\pi_{k-1}}(t) &= \sum_{\substack{p|\pi_{k-1} \\ |f(p)| > 1}} \log(1 + h(p)) + \sum_{\substack{p|\pi_{k-1} \\ |f(p)| \leq 1}} \log(1 + h(p)) - \frac{itf(p)}{p} = \\ &= \sum_{\substack{p|\pi_{k-1} \\ |f(p)| > 1}} \frac{e^{itf(p)} - 1}{p} + \mathcal{O}\left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| > 1}} \frac{1}{p^2}\right) + \\ &\quad + \sum_{\substack{p|\pi_{k-1} \\ |f(p)| \leq 1}} \frac{e^{itf(p)} - 1 - itf(p)}{p} + \mathcal{O}\left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| \leq 1}} \frac{1}{p^2}\right) = \\ &= \mathcal{O}\left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| > 1}} \frac{1}{p}\right) + \mathcal{O}\left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| > 1}} \frac{1}{p^2}\right) + \\ &\quad + \mathcal{O}(1)|t|^2 \left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| \leq 1}} \frac{f^2(p)}{p}\right) + \mathcal{O}\left(\sum_{\substack{p|\pi_{k-1} \\ |f(p)| \leq 1}} \frac{1}{p^2}\right), \end{aligned}$$

so

$$K_{\pi_{k-1}}(t) - 1 = o_x(1)$$

for all  $|t| < T$ , which together with the convergence of  $A(x)$  implies our assertion.

### 4.2. Proof of the necessity part of Theorem 1

In the proof we shall use some ideas of Hildebrand [5] combining these with ours.

**Corollary 2.** *Let  $f$  be an additive function. Assume that  $f(n) = c \log n + g(n)$  and the series*

$$\sum_{|g(p)| > 1} \frac{1}{p-1}, \quad \sum_{|g(p)| \leq 1} \frac{g^2(p)}{p-1}$$

converge. Define

$$U(x) = c \log x + \sum_{\substack{p \leq x \\ |g(p)| \leq 1}} \frac{g(p)}{p-1}.$$

Then  $\nu_x(n \in \mathcal{P}_k, n \leq x : f(n+1) - U(x) \leq z)$  converge weakly to a limit distribution as  $x \rightarrow \infty$ . The characteristic function of the limit law is

$$\begin{aligned} \chi(t) &:= \frac{1}{1+itc} \prod_{|g(p)| > 1} \left( 1 - \frac{1}{p-1} + \sum_{m \geq 1} \frac{e^{itg(p^m)}}{p^m} \right) \times \\ &\times \prod_{|g(p)| \leq 1} \left( 1 - \frac{1}{p-1} + \sum_{m \geq 1} \frac{e^{itg(p^m)}}{p^m} \right) e^{-it \frac{g(p)}{p}}, \end{aligned}$$

and the limit distribution is continuous if and only if

$$\sum_{f(p) \neq 0} \frac{1}{p}$$

diverges.

**Proof.** Let

$$A(x) = \sum_{\substack{p \leq x \\ |g(p)| \leq 1}} \frac{g(p)}{p-1}.$$

Then this last lemma shows, that the distributions

$$\nu_x(n \in \mathcal{P}_k, n \leq x : g(n+1) - A(x) \leq z)$$

possess a limit law with characteristic function  $\xi(t)$ , say. Let

$$\varphi_x(t) = \frac{1}{\Pi_k(x)} \sum_{\pi_k \leq x} e^{itg(\pi_k+1)}.$$

We have

$$\varphi_x(t) e^{-itA(x)} \rightarrow \xi(t) \quad (x \rightarrow \infty).$$

Consider next the following sum

$$(4.4) \quad \frac{1}{\Pi_k(x)} \sum_{\pi_k \leq x} e^{itf(\pi_k+1)} = \frac{1}{\Pi_k(x)} \sum_{\pi_k \leq x} (\pi_k + 1)^{itc} e^{itg(\pi_k+1)}.$$

Remember that  $k$  may depend on  $x$ . Let us introduce the following notation

$$\Psi_{k,x}(y, t) := \sum_{\pi_{k_x} \leq y} e^{itg(\pi_k+1)}.$$

Applying the Abel summation formula for the right hand side for (4.4) we get

$$(4.5) \quad \varphi_x(t) x^{itc} - itc \frac{1}{\Pi_k(x)} \int_1^x y^{itc-1} \Psi_{k,x}(y, t) dy.$$

Since for  $x^\lambda < y < x$  using (2.1), we have

$$\lim_{x \rightarrow \infty} \frac{1}{\Pi_{k_x}(y)} \Psi_{k,x}(y, t) e^{-itA(y)} = \xi(t),$$

thus for this  $y$

$$\Psi_{k,x}(y, t) = \xi(t) e^{itA(x)} + \xi(t) e^{itA(y)} \left\{ 1 - e^{it(A(x)-A(y))} \right\} + o(1).$$

Since

$$A(x) - A(y) = o(1)$$

if  $x^\lambda < y < x$ , we have

$$(4.6) \quad \frac{1}{\Pi_{k_x}(y)} \Psi_{k,x}(y, t) = \xi(t) e^{itA(x)} + o(1).$$

Using this we have that the integral in (4.5) equals

$$(4.7) \quad \int_1^{x^\lambda} y^{itc-1} \Psi_{k,x}(y, t) dy + \xi(t) e^{itA(x)} \int_{x^\lambda}^x y^{itc-1} \Pi_k(y) dy + o(x).$$

The first term is

$$\mathcal{O}(x^\lambda) = \Pi_k(x) o(1).$$

Using the estimation (2.1) we get that the second term in (4.7) is

$$\xi(t) e^{itA(x)} \int_{x^\lambda}^x y^{itc-1} \left( \frac{y}{\log y} \frac{\log \log^{k-1} y}{(k-1)!} (1 + o(1)) \right) dy,$$

which is

$$\frac{1}{(itc+1)} \xi(t) e^{itA(x)} \Pi_k(x) + \Pi_k(x) o(1).$$

Here we used, that  $k$  depends only upon the upper bound of the integration, and that

$$\begin{aligned} & \left( \frac{x^{itc+1} \log \log^{k-1} x}{\log x (k-1)!} \right)' = \\ & = (itc+1) \frac{x^{itc} \log \log^{k-1} x}{\log x (k-1)!} + \mathcal{O} \left( \frac{\log \log^{k-2} x}{\log^2 x} \frac{1}{(k-2)!} \right). \end{aligned}$$

We had shown, that (4.5) equals

$$\xi(t) \frac{e^{itA(x)} x^{itc}}{1+itc} + o(1).$$

We get

$$\frac{1}{\Pi_k(x)} \sum_{\pi_k \leq x} e^{it(f(\pi_k+1))} = \frac{x^{ict} e^{itA(x)}}{1+ict} \xi(t) + o(1),$$

and so

$$\frac{1}{\Pi_k(x)} \sum_{\pi_k \leq x} e^{it(f(\pi_k+1)-U(x))} = \frac{\xi(t)}{1+ict} + o(1),$$

and our lemma immediately follows.

**Lemma 6 (Sieve estimate).** *Let  $q, q'$  be prime numbers, and  $x > x_0$ . We have*

$$\begin{aligned} (4.8) \quad A &= \#\{n \leq x : qn+1, q'n+1 \in U_k(x)\} \ll \\ &\ll \frac{x}{\log^2 x} \left( \frac{(\log \log x)^{k-1}}{(k-1)!} \right)^2 \Psi(|q-q'|), \end{aligned}$$

where

$$\Psi(n) = \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}.$$

**Proof.** Using Lemma 1 we need only to count the elements of the set

$$A = \#\{n \leq x : qn+1 = \pi_{k-1}p, q'n+1 = \pi'_{k-1}p' \text{ and } \pi_{k-1} \leq x^\beta, \pi'_{k-1} \leq x^\beta\}.$$

Recall that  $P(qn + 1) = p$ ,  $P(q'n + 1) = p'$  and  $p(\pi_{k-1}) \geq M_x$ ,  $p(\pi'_{k-1}) \geq M_x$ . We have

$$\begin{aligned}qn + 1 &= \pi_{k-1}p, \\q'n + 1 &= \pi'_{k-1}p',\end{aligned}$$

so by the Chinese remainder theorem  $n \equiv l_{\pi_{k-1}\pi'_{k-1}} \pmod{\pi_{k-1}\pi'_{k-1}}$  with a unique  $l_{\pi_{k-1}\pi'_{k-1}}$ . So  $n = t\pi_{k-1}\pi'_{k-1} + l_{\pi_{k-1}\pi'_{k-1}}$ , where  $t \leq \frac{x}{\pi_{k-1}\pi'_{k-1}}$ . For  $p, p'$  we have

$$\begin{aligned}p &= qt\pi'_{k-1} + \frac{ql + 1}{\pi_{k-1}}, \\p' &= q't\pi_{k-1} + \frac{q'l + 1}{\pi'_{k-1}},\end{aligned} \quad t \leq \frac{x}{\pi_{k-1}\pi'_{k-1}}.$$

Using sieve estimates we have, that the number of such  $p, p'$  is not more than

$$\begin{aligned}A &\ll \\ \sum_{\substack{\pi_{k-1} \leq x^\beta \\ \pi'_{k-1} \leq x^\beta}} \Psi \left( q\pi'_{k-1}q'\pi_{k-1} \left| q\pi'_{k-1} \frac{q'l + 1}{\pi'_{k-1}} - q'\pi_{k-1} \frac{ql + 1}{\pi_{k-1}} \right| \right) \frac{x}{\log^2 x} \frac{1}{\pi_{k-1}\pi'_{k-1}} &\ll \\ (4.9) \quad &\ll \Psi(|q - q'|) \frac{x}{\log^2 x} \sum_{\substack{\pi_{k-1} \leq x^\beta \\ \pi'_{k-1} \leq x^\beta}} \Psi(\pi'_{k-1}\pi_{k-1}) \frac{1}{\pi_{k-1}\pi'_{k-1}}.\end{aligned}$$

Here we observed, that

$$q\pi'_{k-1} \frac{q'l + 1}{\pi'_{k-1}} - q'\pi_{k-1} \frac{ql + 1}{\pi_{k-1}} = q - q'.$$

The sum in the last expression of (4.9) is

$$\begin{aligned}&\sum_{\pi_{k-1} \leq x^\beta} \prod_{p_i | \pi_{k-1}} \frac{p_i - 1}{(p_i - 2)p_i} \sum_{\pi'_{k-1} \leq x^\beta} \prod_{p'_i | \pi'_{k-1}} \frac{p'_i - 1}{(p'_i - 2)p'_i} \ll \\ &\ll \sum_{\pi_{k-1} \leq x^\beta} \prod_{p_i | \pi_{k-1}} \frac{1}{(p_i - 2)} \sum_{\pi'_{k-1} \leq x^\beta} \prod_{p'_i | \pi'_{k-1}} \frac{1}{(p'_i - 2)} \ll \\ &\ll \left( \frac{1}{(k-1)!} \left( \sum_{p \leq x^\beta} \frac{1}{p} + \sum_{p > M_x} \frac{1}{p^2} \right)^{k-1} \right)^2,\end{aligned}$$

which implies our assertion.

The next theorem can be found in [5] for shifted primes.

**Lemma 7.** *With suitable constant  $\delta_1$  and  $c_1$ , and multiplicative  $g : \mathbb{N} \rightarrow \mathbb{C}$  such that  $|g| = 1$ , and*

$$\max_{1 \leq l \leq c_1} \left| \frac{1}{x} \sum_{n \leq x} g(n)^l \right| \leq \delta_1$$

we have

$$(4.10) \quad \left| \frac{1}{B_k(x)} \sum_{n \in U_k(x)} g(n+1) \right| \leq 1 - \delta_1.$$

**Proof.** We use the ideas of Hildebrand without any important changes. Therefore we shall give an outline of the proof, only.

It is enough to prove that if the conditions hold, then

$$1 - \left| \frac{1}{B_k(x)} \sum_{\pi_k \in U_k(x)} g(\pi_k + 1) \right| \gg 1.$$

Some computation shows that

$$(4.11) \quad 1 - \left| \frac{1}{B_k(x)} \sum_{\pi_k \in U_k(x)} g(\pi_k + 1) \right| \geq \frac{1}{2B_k(x)} \sum_{\pi_k \in U_k(x)} |1 - wg(\pi_k + 1)|^2$$

with an appropriate complex  $w$ , with absolute value 1. Setting

$$R(Q) = \frac{1}{B_k(x)} \sum_{\pi_k \in U_k(x)} |1 - wg(\pi_k + 1)|^2 \sum_{\substack{q|\pi_k+1 \\ Q < q < 2Q}} 1,$$

we get with  $0 < \eta < 1/4$  and  $x \geq 2^{1/\eta}$  after some computation that the right hand side of (4.11) is at least

$$\frac{\eta^2}{2 \log 2} \log x \min_{x^\eta \leq Q \leq x^{2\eta}} R(Q),$$

so it is enough to prove that  $R(Q) \gg \frac{1}{\log x}$  uniformly with  $x > x_0$ ,  $x^\eta \leq Q \leq x^{2\eta}$ . Let  $\delta < 1/4$ . There is an  $\omega$  complex number with absolute value 1 such that

$$(4.12) \quad \sum_{\substack{Q < q < 2Q \\ |g(q) - \omega| \leq \delta}} 1 \geq \frac{\delta}{10} \frac{Q}{\log Q}.$$

Set

$$S = \sum_{\pi_k \in U_k(x)} \sum'_{\substack{q | \pi_k + 1 \\ Q < q < 2Q}} 1.$$

The ' in the inner sum means the restriction to prime numbers  $q$  for which  $|g(q) - \omega| \leq \delta$  with this appropriate  $\omega$ . We get a lower bound for this sum applying Lemma 4 and Corollary 1.

$$S = \sum'_{Q < q < 2Q} B_k(x, q, -1) \geq \frac{B_k(x)}{4Q} \sum'_{Q < q < 2Q} 1 + \mathcal{O}\left(\frac{B_k(x)}{\log^A x}\right).$$

If  $\pi_k + 1 = nq$  and  $q^2 \nmid \pi_k + 1$ , then after some computation we have

$$|1 - wg(\pi_k + 1)| \geq |g(n) - \overline{w\omega}| - |g(q) - \omega|.$$

Thus  $|g(q) - \omega| \leq \delta$  implies  $|g(n) - \overline{w\omega}| \leq 2\delta$  or  $|1 - wg(\pi_k + 1)| > \delta$ .

Let

$$S_1 = \sum_{\pi_k \in U_k(x)} \sum'_{Q < q < 2Q} \sum''_{\substack{n \\ qn = \pi_k + 1}} 1,$$

where '' in the inner sum means the restriction to integers for which  $|g(n) - \overline{w\omega}| \leq 2\delta$ . Let

$$S_2 = \sum^*_{\pi_k \in U_k(x)} \sum'_{\substack{q | \pi_k + 1 \\ Q < q < 2Q}} 1,$$

where \* in the outer sum means the restriction to the prime numbers for which  $|1 - wg(\pi_k + 1)| > \delta$ . Let

$$S_3 = \sum_{\pi_k \in U_k(x)} \sum'_{\substack{q^2 | \pi_k + 1 \\ Q < q < 2Q}} 1.$$

We have  $S \leq S_1 + S_2 + S_3$ . It is easy to see that  $S_3 \leq x^{1-\eta}$ . We use  $R(Q)$  to estimate  $S_2$ .

$$S_2 \leq \sum_{\pi_k \in U_k(x)} \frac{|1 - wg(\pi_k + 1)|^2}{\delta^2} \sum_{\substack{q|\pi_k+1 \\ Q < q < 2Q}} 1 = \frac{B_k(x)}{\delta^2} R(Q).$$

Putting it all together it is enough to prove that  $S_1 \leq \frac{B_k(x)}{8Q} \sum'_{Q < q < 2Q} 1$ .

We have

$$S_1 \leq \sum''_{n \leq \frac{x+1}{Q}} \sum'_{\substack{Q < q < 2Q \\ qn-1=\pi_k}} 1.$$

Applying the Cauchy-Schwarz inequality we get

$$(S_1)^2 \leq \underbrace{\left( \sum''_{n \leq \frac{x+1}{Q}} 1 \right)}_{S_{11}} \underbrace{\left\{ \sum_{n \leq \frac{x+1}{Q}} \left( \sum'_{\substack{Q < q < 2Q \\ qn-1=\pi_k}} 1 \right)^2 \right\}}_{S_{12}}.$$

We get

$$S_{12} = \sum''_{Q < q, q' < 2Q} \sum'_{\substack{n \leq \frac{x+1}{Q} \\ qn-1=\pi_k \\ q'n-1=\pi'_k}} 1.$$

If  $q = q'$  then this sum is  $\mathcal{O}(x)$ . For the other case we can use Lemma 6, and we get

$$S_{12} \ll \frac{x}{Q \log^2 x} \left( \frac{(\log \log x)^{k-1}}{(k-1)!} \right)^2 \sum'_{Q < q < 2Q} \sum'_{Q < q' < q} \Psi(q - q') + x.$$

After some computation we get that the inner sum does not exceed

$$c \left( \frac{\delta}{10} \right)^{-1/2} \sum'_{Q < q < 2Q} 1$$

(see Hildebrand).

We have

$$S_{12} \ll \delta^{-1/2} \frac{x}{Q \log^2 x} \left( \frac{(\log \log x)^{k-1}}{(k-1)!} \right)^2 \left( \sum'_{Q < q < 2Q} 1 \right)^2 + x.$$

It is not so hard to prove that if the conditions of the lemma hold, then  $S_{11} \leq \frac{x}{Q} \delta \log \frac{1}{\delta}$ .

Putting it together we get

$$S_1 \ll \left( \delta^{1/5} \frac{\Pi_k(x)}{Q} + \frac{x \log Q}{\delta Q^{3/2}} \right) \sum'_{Q < q < 2Q} 1.$$

Choosing small  $\delta$ , we finished the outline of the proof.

The following two lemmas can be found in [5], thus we give only remarks accordingly to our case.

**Lemma 8.** *Assume that*

$$\min_{\substack{1 \leq l \leq c_1 \\ \tau \leq c_2}} \operatorname{Re} \sum_{p \leq x} \frac{1 - g(p)^l p^{-i\tau}}{p} > c_2$$

with a suitable  $c_2 > 0$ . Then (4.10) holds.

**Remark.** One can prove that if this condition holds, then the condition of Lemma 7 holds, too, with a suitable large  $c_2$ .

**Lemma 9.** *There are constants  $\delta_3, c_3$  such that for fixed  $x \geq 2, h > 0$  and*

$$\max_{a \in \mathbb{R}} \{n \in U_k(x) : f(n+1) \in [a, a+h]\} \geq (1 - \delta_3) B_k(x)$$

we have

$$\min_{|\lambda| \leq c_3 h^2} \sum_{p \leq x} \frac{(\min(h, f(p) - \lambda \log p))^2}{p} \leq c_3 h^2.$$

**Proof.** We must show that the condition of the previous lemma is violated if  $h = 1$ . We have

$$\left| \frac{1}{B_k(x)} \sum_{\pi_k \in U_k(x)} e^{itf(\pi_k+1)} \right| = \left| \frac{1}{B_k(x)} \sum_{\pi_k \in U_k(x)} e^{it(f(\pi_k+1)-a)} \right| \geq 1 - |t| - 2\delta_3.$$

Set  $\delta_3 \leq \frac{\delta_1}{4}$  and  $|t| \leq \frac{\delta_1}{2}$ . The result of Lemma 7 is violated. We have from Lemma 8 that for  $k = k(t) \leq c_1$  and  $\tau = \tau(t), |\tau| \leq c_2$

$$\sum_{p \leq x} \frac{\operatorname{Re} \left( 1 - e^{itf(p^k)} p^{-i\tau} \right)}{p} \leq c_2.$$

The following lemma can be found in [1] Lemma 1.9.

**Lemma 10.** *Let  $F_x$  be a sequence of distribution functions, and let  $\alpha, \beta$  be real functions. If  $F_x(z + \alpha(x))$  and  $F_x(z + \beta(x))$  converge weakly to a limit distribution with  $x \rightarrow \infty$ , then  $\lim_{x \rightarrow \infty} (\alpha(x) - \beta(x))$  exists, and is finite.*

**Proof of necessity part of Theorem 1.** Assume  $f$  has a limit distribution. Then

$$\#\{n \leq x, n \in \mathcal{P}_k : f(n+1) \in [-z, z]\} \geq (1 - \delta_3) \Pi_k(x)$$

holds for some  $z$  real number. We get using Lemma 9 with  $h = 2z$

$$\sum_{p \leq x} \frac{(\min(h, f(p)) - \lambda_x \log p)^2}{p} \leq c_3 h^2$$

with  $|\lambda_x| \leq c_3 h^2$ . It follows that

$$\sum_p \frac{(\min(h, f(p)) - \lambda \log p)^2}{p}$$

converge for some  $\lambda$ . We get immediately that for  $g(n) = f(n) - \lambda \log n$  the series

$$\sum_{|g(p)| > 1} \frac{1}{p-1}, \quad \sum_{|g(p)| \leq 1} \frac{g(p)^2}{p-1}$$

converge. From Corollary 2 and Lemma 10 we get that

$$\lambda \log x + \sum_{\substack{p \leq x \\ |g(p)| \leq 1}} \frac{g(p)}{p-1}$$

converge. This implies  $\lambda = 0$  and the convergence of

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p-1},$$

and the proof of Theorem 1 is completed.

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