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ON THE AVERAGE PRIME DIVISORS

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. Let $\kappa(n) = \sum_{p|n} p$, $\omega(n)$ = number of prime divisors of n, $\rho(n) = \dfrac{\kappa(n)}{\omega(n)}$. Refining the result of W.D. Banks and his coauthors we prove that

$$\#\{n \le x \mid \rho(n) = \text{integer}\} = (1 + o_x(1))c\frac{x}{\log\log x}$$

with some constant $c > 0$.

1. Introduction

Let $\mathcal{P}$ = set of primes. p with and without suffixes always denote primes, $\omega(n) := \sum_{p|n} 1$,

$$\kappa(n) := \sum_{p|n} p; \quad \rho(n) = \frac{\kappa(n)}{\omega(n)},$$

$$R(x) := \#\{n \le x \mid \rho(n) = \text{integer}\}.$$

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In [1] W.D. Banks and his coauthors proved that

$$c_1 < \frac{R(x)\log\log x}{x} < c_2 \tag{1.1}$$

holds for $x > x_0$, where x_0, c_1, c_2 are positive constants.

By using our method evaluated in [2], [3] we can prove the following

Theorem. *We have*

$$R(x) = (1 + o_x(1))c\frac{x}{\log\log x} \quad (x \to \infty), \tag{1.2}$$

where c is a suitable positive constant.

Let φ be the Euler's totient function, $\pi(x) =$ number of primes up to x,

$$\pi(x,k,l) = \sum_{\substack{p\le x\\ p\equiv l(\mathrm{mod}\ k)}} 1, \quad \pi_r(x) := \#\{n \le x \mid \omega(n) = r\}.$$

Let $z \ge 1$ and

$$A(n|z) := \prod_{\substack{p^\alpha \| n\\ p\le z}} p^\alpha, \quad B(n|z) = \frac{n}{A(n|z)}.$$

Let $P(n)$ be the largest prime factor of n. We shall write furthermore $x_1 = \log x,\ x_2 = \log x_1,\ x_3 = \log x_2, \ldots$. Let furthermore $e(\alpha) := e^{2\pi i\alpha}$.

2. Lemmata

Lemma 1. *Let*

$$c_k(n) := \sum_{\substack{h=1\\ (h,k)=1}}^{k} e\left(\frac{hn}{k}\right)$$

be the Ramanujan sum. Then

$$c_k(n) = \frac{\mu(t)\varphi(k)}{\varphi(t)}, \quad t = \frac{k}{(k,n)} \tag{2.1}$$

(see in Tenenbaum [4], p.35).

Lemma 2. *We have*

$$\#\left\{n \le x \;\middle|\; |\omega(n) - x_2| \ge \frac{1}{2}x_2\right\} \ll x/x_2^B, \tag{2.2}$$

where B is an arbitrary large constant.

This is an easy consequence of the Hardy–Ramanujan inequality, namely that

$$\pi_r(x) \le c_1 \frac{x}{x_2} \frac{(x_2 + c)^{r-1}}{(r-1)!}.$$

Lemma 3. *Let $U(x, w)$ be the number of those integers $n \le x$ for which there exists a square divisor d^2 such that $d > w$. Then*

$$U(x, w) \ll \frac{x}{w}. \tag{2.3}$$

The assertion is clear.

Lemma 4. *Let $G_L(x)$ be the number of integers $n \le x$ having two prime divisors p_1 and p_2 satisfying $L < p_1 < p_2 < 4p_1$. Then*

$$G_L(x) \ll \frac{x}{\log L}. \tag{2.4}$$

Proof. Since

$$G_L(x) \le \sum_{\substack{L < p_1 < p_2 \le 4p_1 \\ p_1 p_2 \le x}} \frac{x}{p_1 p_2} \le cx \sum_{L < p_1 \le \sqrt{x}} \frac{1}{p_1 \log p_1} \ll \frac{x}{\log L},$$

(2.4) holds.

Lemma 5. *Let $l_0 := \exp(x_2^A)$, where A is a large constant. Then*

$$\#\left\{n \le x \mid A(n|l_0) > \exp(x_2^{A+1})\right\} \ll \frac{x}{x_2^{2A}}. \tag{2.5}$$

This can be deduced easily from the wellknown estimate

$$\psi(x, y) := \#\{n \le x \mid P(n) \le y\} \ll x \exp\left(-\frac{x_1}{2 \log y}\right)$$

(see for instance Tenenbaum [4]).

Lemma 6. *Let A and l_0 be as in Lemma 5. Then there exists a constant $b > 0$ such that*

$$\#\{n \le x \mid \omega(A(n|l_0)) > bx_3\} \ll \frac{x}{x_2^{2A}}. \tag{2.6}$$

Proof. The left hand side is less than

$$\frac{1}{2^{bx_3}} \sum_{n \le x} \tau(A(n|l_0)) \le$$

$$\le \frac{x}{2^{bx_3}} \sum_{P(d) \le l_0} 1/d \ll \frac{x}{2^{bx_3}} \prod_{p<l_0} \frac{1}{1-1/p} \ll \frac{x}{2^{bx_3}} \exp(\log\log l_0),$$

whence the assertion follows.

Lemma 7. *Let c, B be arbitrary constants,*

$$\frac{x}{x_1^c} \le y \le x, \ \ k \le x_1^B, \ \ (l,k) = 1.$$

Then

$$\pi(x+y,k,l) - \pi(x,k,l) = \frac{\mathrm{li}\,(x+y) - \mathrm{li}\,x}{\varphi(k)} \left(1 + O(\exp(-c_2\sqrt{x_1}))\right) \tag{2.7}$$

uniformly in k, l.

Lemma 8. *Let $\mathbb{Z}_q^*$ be the set of reduced residues classes* mod q, *$\lambda_{q,h}(s)$ be the number of solutions of $l_1 + l_2 + \ldots + l_h \equiv s$ (mod q), where l_ν run over $\mathbb{Z}_q^*$ independently. Then*

$$\lambda_{q,h}(s) = \frac{1}{q} \sum_{a=0}^{q-1} e\left(-\frac{sa}{q}\right) c_q(a)^h. \tag{2.8}$$

Consequently, if $q =$ odd, p^* is its smallest prime divisor, then

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \right| \le \frac{(q-1)}{q} \cdot \frac{1}{\varphi(p^*)^h}, \tag{2.9}$$

while if $q =$ even, then

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \left\{1 + (-1)^{h+s}\right\} \right| \le \frac{q-2}{q} \cdot \frac{1}{\varphi(p^*)^h}, \tag{2.10}$$

where p^* is the smallest odd prime factor of q.

Proof. Clear.

3. Proof of the Theorem

3.1. Let $\underline{v}$ be a fixed large number, and assume that $1 \le h \le Bx_2$, $\frac{x_2}{3} \le q \le 2x_2$. Let us choose a large constant A, and a positive constant c_0. We shall consider the set

$$\mathcal{L} = \{l_j : j = 0, 1, 2, \ldots\}, \tag{3.1}$$

where

$$l_0 = \exp(x_2^A),\ l_{j+1} = l_j + \frac{l_j}{(\log l_j)^{c_0}} \quad (j = 0, 1, 2, \ldots). \tag{3.2}$$

Let $I(l_j) = [l_j, l_{j+1}),\ \beta(l_j) = \operatorname{li} l_{j+1} - \operatorname{li} l_j$, where $\operatorname{li} y = \int\limits_2^y \frac{du}{\log u}$. If $u \in \mathcal{L},\ u = l_\nu$, then $\Delta u := l_{\nu+1} - l_\nu$, and so $I(u) = [u, u + \Delta u]$.

3.2. Let $Y \in [x^{1/2}, x]$. We shall consider such h-tuples $(u_1, \ldots, u_h)$ for which

$$l_0 \le u_1 < \ldots < u_h, \quad u_\nu \in \mathcal{L} \quad (\nu = 1, 2, \ldots, h). \tag{3.3}$$

We say that it is feasible if $u_1, \ldots, u_h \le Y$, it is well spaced if $u_{j+1} \ge u_j$ $(j = 1, \ldots, h-1)$, and that it is completely suitable if

$$(u_1 + \Delta u_1) \ldots (u_h + \Delta u_h) \le Y.$$

3.3. Let

$$E_h(u_1, \ldots, u_h) := \#\{p_1 \ldots p_h \mid p_\nu \in I(u_\nu),\ \nu = 1, \ldots, h\}. \tag{3.4}$$

In [3] we proved

Lemma 9. *If* $(u_1,\dots,u_h)$ *is a well-spaced feasible* h*-tuple, then*

$$E_h(u_1,\dots,u_h)=\prod_{\nu=1}^{h}\beta(u_\nu)\cdot\left(1+O\left(e^{-c_3x_2^{A/2}}\right)\right). \tag{3.5}$$

3.4. For a given q and prime $p>l_0$ let $H(p)=H_q(p)\in[1,q-1]$ such that $H(p)\equiv p \pmod q$, and if $(l_0<)p_1<p_2<\dots<p_h$ are primes then let $H(p_1\dots p_h)=H(p_1)\dots H(p_h)$ be the word standing from the letters $H(p_1),\dots,H(p_h)$ concatenating them.

Let $\alpha=l_1l_2\dots l_h$, $l_\nu\in\mathbb{Z}_q^*=\{u\in[1,q-1],\ (u,q)=1\}$. Furthermore let

$$E_h^{(q)}(u_1,\dots,u_h\mid\alpha)=\#\{p_1\dots p_h\mid p_j\in I(u_j),\ H_q(p_j)=l_j,\ j=1,\dots,h\}. \tag{3.6}$$

In [3] we proved also

Lemma 10. *We have*

$$E_h^{(q)}(u_1,\dots,u_h\mid\alpha)=\frac{1}{\varphi(q)^h}\prod_{\nu=1}^{h}\beta(u_\nu)\cdot(1+O(\exp(-c_2\sqrt{x_1}))), \tag{3.7}$$

if $u_1,\dots,u_h$ *is a well-spaced feasible* h*-tuple.*

3.5. Let $\kappa_q(n)\equiv\sum_{p|n}p \pmod q$, especially $\kappa_q(p_1\dots p_h)\equiv l_1+l_2+\dots+$ $+l_h \pmod q$, if $H_q(p_j)=l_j$ $(j=1,\dots,h)$. Thus the value of $\kappa_q(p_1\dots p_h)$ does depend only on the value of $\alpha(=H_q(p_1\dots p_h))$. We shall write $\kappa_q(\alpha):=$ $:=l_1+\dots+l_h \pmod q$.

From Lemma 8, 9, 10 we obtain

Lemma 11. *Let* $u_1,\dots,u_h$ *be a well-spaced feasible* h*-tuple. Then, for every* s *(mod* q*)*

a) in the case $q=$ *odd*

$$\sum_{\kappa_q(\alpha)\equiv s \pmod q}\alpha\ E_h^{(q)}(u_1,\dots,u_h\mid\alpha)=\frac{1}{q}E_h(u_1,\dots,u_h)(1+O(-c_2(\sqrt{x_1})))=$$
$$=O\left(\frac{1}{\varphi(p^*)^h}E_h(u_1,\dots,u_h)\right),$$

where p^* *is the smallest prime divisor of* q *(then* $p^*\ge 3!$*);*

b) in the case $q =$ even,

$$\sum_{\substack{\alpha \\ \kappa_q(\alpha)\equiv s \pmod q}} E_h^{(q)}(u_1, \ldots, u_h \mid \alpha) =$$

$$= \frac{1}{q}\{1 + (-1)^{h+s}\}E_h(u_1, \ldots, u_h)(1 + O(e^{-c_2\sqrt{x_1}}))+$$

$$+O\left(E_h(u_1, \ldots, u_h) \cdot \frac{1}{\varphi(p^*)^h}\right),$$

where p^* is the smallest odd prime factor of q.

Lemma 12. *Let us summarize $E_h(u_1, \ldots, u_h)$ over those feasible well spaced h-tuples for which $\prod_{\nu=1}^{h} u_\nu < Y < \prod_{\nu=1}^{h}(u_\nu + \Delta u_\nu)$, and then over all $h \le Bx_2$. The amount is less than $\ll Y/x_2 Ac$.*

This assertion is proved in [3].

4. Completion of the proof

Let us consider the integers $n \le x$. Let l_0 be as in (3.2). Drop those integers for which (1) $\left|\omega(n) - x_2\right| \ge \frac{x_2}{x}$, or (2) $B(n|l_0)$ is not square free, or (3) $B(n|l_0)$ has two prime divisors p_1, p_2 such that $p_1 < p_2 < 4p_1$, or (4) $A(n|l_0) > \exp(x_2^{A+1})$, or (5) $\omega(A(n|l_0)) > bx_3$, or (6) if $n = A(n|l_0)B(n|l_0)$, $B(n|l_0) = p_1 \ldots p_h$, and by $p_\nu \in I(n_\nu)$, $\nu = 1, \ldots, h$, then by $Y = \frac{x}{A(n|l_0)}$ $\prod u_\nu < Y < \prod(u_\nu + \Delta u_\nu)$.

By using our lemmas we obtain that the size of the dropped integers is $O\left(\frac{x}{x_2^2}\right)$. Now we classify the others.

Let D be a fixed integer, for which $P(D) \le l_0$, $D \le \exp(x_2^{A+1})$, $\omega(D) \le bx_3$. Let $Y = \frac{x}{D}$. Let $\mathcal{E}_D$ be the set of those integers $n \le x$, for which $A(n|l_0) = D$, and which are not dropped. Let $\mathcal{E}_D^{(h)}$ be that subset for which $B(n|l_0)$ contains exactly h prime divisors, $B(n|l_0) = p_1 \ldots p_h$. If n is not dropped, then $p_\nu \in I(u_\nu)$, $\nu = 1, \ldots, h$ such that $(u_1, \ldots, u_h)$ is a feasible, well spaced, completely suitable h tuple.

Let now $u_1, \ldots, u_h$ be fixed, $\mathcal{E}_D^{(h)}(u_1, \ldots, u_h)$ be the numbers $p_1 \ldots p_h$ such that $p_\nu \in I(u_\nu)$, $\nu = 1, \ldots, h$.

Let $t = \omega(D)$, $q = t + h$. It is clear that $n = Dp_1 \ldots p_h$ satisfies $\kappa(n) \equiv \equiv O(\text{mod } \omega(n))$, if $\kappa(D) + p_1 + \ldots + p_h \equiv O(\text{mod } q)$. Let $\kappa(D) \equiv -s \pmod q$. Then $\kappa(n) \equiv O(\text{mod } q)$ holds if and only if $p_j \equiv l_j \pmod q$ $(j = 1, \ldots, h)$ and $l_1 + \ldots + l_h \equiv s \pmod q$.

Our assertion easily follows from Lemma 11.

References

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