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ON REGULAR ψ–CONVOLUTIONS II.

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. We prove the equivalence of regular ψ-convolutions (satisfying the condition $\psi(x, y) \geq \max\{x, y\}$ for all $(x, y) \in T$) and Lehmer-Narkiewicz convolutions. We prove that several classical results in regular Narkiewicz convolutions can be extended to Lehmer-Narkiewicz convolutions.

1. Introduction

An arithmetic function is a complex-valued function whose domain is the set of positive integers $\mathbb{Z}^+$. Let F denote the set of arithmetic functions. A binary operation B in F is called a regular convolution if the following conditions hold:

(i) The triple $(F, +, B)$ is a commutative ring with unity (here "+" denotes the usual pointwise addition). (1.1)

(ii) B is multiplicativity preserving, that is fBg is multiplicative whenever $f, g \in F$ are multiplicative (as usual $f \not\equiv 0$, $f \in F$ is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever m and n are relatively primes). (1.2)

(iii) The function $1 \in F$ defined by $1(n) = 1$ for all $n \in \mathbb{Z}^+$ has an inverse μ_B with respect to B and μ_B is 0 or -1 at prime powers. (1.3)

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For each positive integer n, if $A(n)$ is a non-empty subset of positive divisors of n, Narkiewicz [10] defines the binary operation A in F (called the A-convolution) by

$$(fAg)(n) = \sum_{d \in A(n)} f(d)g(n/d) \tag{1.4}$$

for each $n \in \mathbb{Z}^+$. He then calls the A-convolution as regular if A satisfies (1.1)-(1.3). If $A(n)$ is the set of all divisors of n, then A reduces to the familiar Dirichlet convolution D. As usual, a divisor d of n is called a unitary divisor if d and n/d are relatively prime; that is $\gcd(d, n/d) = 1$. If $A(n)$ denotes the set of all unitary divisors of n, the binary operation A in (1.4) reduces to the well-known unitary convolution U studied extensively by E. Cohen ([2], [3]). The unitary convolution was originally introduced by R. Vaidyanathaswamy (cf. [25]) under the name of compounding operation. To make a distinction between a general regular convolution as defined in (1.1)-(1.3) and that of the special operation A defined in (1.4) which is regular, we call the latter as *regular-Narkiewicz convolution.*

It is well-known that the Dirichlet convolution is regular. Also, from the results established by Cohen [2], it follows that unitary convolution is also a regular convolution.

Narkiewicz obtained necessary and sufficient conditions for the convolution A in (1.4) to be a regular convolution in terms of the sets $A(n)$ (cf. [10]). A more useful characterization of regular-Narkiewicz convolutions obtained by Narkiewicz is given below:

Theorem 1.1. (cf. [10], Theorem II) *Let K be the class of all decompositions of the set of non-negative integers into arithmetic progressions (finite or not) containig zero and such that no two arithmetic progressions belonging to the same decomposition have a positive integer in common. Let us associate with every prime number p an element π_p of K. Let the sets $A(n)$ be defined by $\prod_i p_i^{\alpha_i} \in A(n)$ where $n = \prod_i p_i^{\beta_i}$ if and only if for every $i : \alpha_i \leq \beta_i$ and α_i, β_i belong in the decomposition π_{p_i} to the same progression. Then these sets $A(n)$ define a regular convolution and conversely every regular A-convolution can be obtained in this way.*

From the above theorem it is clear that every regular A-convolution is uniquely determined by a sequence $\{\pi_p\}$ of elements of K. This is described by writing $A \sim \{\pi_p\}$. If $A \sim \{\pi_p\}$, where $\pi_p : \{0, 1, 2, 3, \ldots\}$ for each prime p, then A is the Dirichlet convolution and if $\pi_p : \{0, 1\}, \{0, 2\}; \{0, 3\}; \ldots$, then these decompositions determine the unitary convolution. Apart from these two Theorem 1.1 shows that there are infinitely many examples of regular A-convolutions.

In 1930 R. Vaidyanathaswamy (see [24]; also [25], [26]) established the following remarkable identity valid for any multiplicative function and known as the identical equation for multiplicative functions: if f is any multiplicative function, then for any positive integers m and n, we have

$$f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)f(n/b)f^{-1}(ab)G(a,b), \tag{1.5}$$

where f^{-1} is the inverse of f with respect to the familiar Dirichlet convolution, that is

$$\sum_{d|m} f(d)f^{-1}(m/d)(m) = e(m)$$

for all positive integers m, where

$$e(m) = \begin{cases} 1, & \text{if } m = 1, \\ 0, & \text{if } m > 1, \end{cases} \tag{1.6}$$

and

$$G(a,b) = \begin{cases} (-1)^{\omega(a)} & \text{if } \gamma(a) = \gamma(b), \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

$\omega(a)$ being the number of distinct prime factors of a and $\gamma(a)$ the product of distinct prime factors of a with $\omega(1) = 0$ and $\gamma(1) = 1$.

It has been observed by M.V. Subbarao and A.A. Gioia [23] that the unitary analogue of (1.5) is true, that is whenever m and n are relatively prime and f is multiplicative we have

$$f(mn) = \sum_{\substack{a\|m \\ b\|n}} f(m/a)f(n/b)f_U^{-1}(ab)G(a,b), \tag{1.8}$$

where f_U^{-1} denotes the inverse of f with respect to the unitary convolution, that is

$$\sum_{d\|m} f(d)f_U^{-1}(m/d) = e(m)$$

for all positive integers m.

In fact, M.V. Subbarao and A.A. Gioia (cf. [23], p.70) noted that the identity in (1.8) reduces to a triviality in the sense that the right hand side of (1.8) can be evaluated without much difficulty since $f_U^{-1}(m) = (-1)^{\omega(m)} f(m)$. They also established a non-trivial identity (cf. [23], Theorem 2) in the case of unitary products.

It is interesting to note that the A-analogue of (1.5) is also true which has in fact been established by P. Haukkanen (cf. [5], Theorem 1.4.8, $G = \mathbb{Z}^+$) in a slightly more general setting. However, we mention here only the A-analogue of (1.5): if f is a multiplicative function, then we have for $m \in A(mn)$

$$f(mn) = \sum_{\substack{a\in A(m)\\ b\in A(n)}} f(m/a)f(n/b)f_A^{-1}(ab)G(a,b), \tag{1.9}$$

where f_A^{-1} is the inverse of f with respect to the regular Narkiewicz A-convolution, so that

$$\sum_{d\in A(m)} f(d)f_A^{-1}(m/d) = e(m)$$

for all positive integers m.

Let $\emptyset \neq T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi : T \to \mathbb{Z}^+$ be a mapping satisfying the following conditions:

(1.10) For each $n \in \mathbb{Z}^+$, $\psi(x,y) = n$ has a finite number of solutions.

(1.11) If $(x,y) \in T$, then $(y,x) \in T$ and $\psi(x,y) = \psi(y,x)$.

(1.12) The statements "$(x,y) \in T, (\psi(x,y),z) \in T$" and "$(y,z) \in T, (x,\psi(y,z)) \in T$" are equivalent; if one of these conditions holds, we have $\psi(\psi(x,y),z) = \psi(x,\psi(y,z))$.

If $f, g \in F$, then the ψ-product of f and g denoted by $f\psi g \in F$ is defined by

$$(f\psi g)(n) = \sum_{\psi(x,y)=n} f(x)g(y) \tag{1.13}$$

for all $n \in \mathbb{Z}^+$. The binary operation ψ in (1.13) is due to D,H. Lehmer [6].

It is easily seen that $(F, +, \psi)$ is a commutative ring (cf. [6]). Clearly, the Dirichlet and unitary convolutions arise special cases of the ψ-convolutions. Let $\psi(x, y) = xy$ for all $(x, y) \in T$. If $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ then ψ in (1.13) reduces to the Dirichlet convolution. If $T = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (x, y) = 1\}$, then ψ reduces to the unitary convolution [2]. More generally, if $T = \bigcup_{n=1}^{\infty} \{(d, n/d) : d \in A(n)\}$, where A is Narkiewicz convolution [10], then ψ reduces to the A-convolution. Thus the binary operation in (1.13) is more general than that of Narkiewicz's A-convolution.

If ψ satisfies (1.10)-(1.12) then the binary operation ψ in (1.13) is said to be multiplicativity preserving, if $f\psi g$ is multiplicative whenever f and g are (cf. [17]).

If ψ is multiplicativity preserving, then the following ψ-analogue of the identical equation in (1.5) has been established by V. Sitaramaiah and M.V. Subbarao [19] placing some mild conditions on ψ (see [19], Theorem 3): if f is multiplicative, then for any $(m, n) \in T$ we have

$$f(\psi(m, n)) = \sum_{\substack{a,x,b,y \\ \psi(a,x)=m \\ \psi(b,y)=n}} f(x)f(y)f^{-1}(\psi(a, b))G(a, b), \tag{1.14}$$

f^{-1} being the inverse of f with respect to ψ and $G(a, b)$ is as given in (1.7).

The conditions placed on ψ (for precise conditions see Theorem 2.7 in Section 2 of the present paper) are general enough to contain (1.5) as a special case and in general the A-analogue of the identical equation given in (1.9). However, in all these cases $\psi(x, y) = xy \ \forall x, y \in T$. Only one example of a function ψ was constructed by V. Sitaramaiah and M.V. Subbarao (cf. [19], Remark 3.2) for which the ψ-analogue of the identical equation holds and also $\psi(x, y) \neq xy$.

The A-analogue of the identical equation given in (1.9) might have suggested the authors [20] to study regular ψ-convolutions; the binary operation in (1.13) is called a regular ψ-convolution (cf. [20]) if ψ satisfies the conditions (1.1)-(1.3).

An attempt was made in [20] to characterize regular ψ-convolutions; a less effective characterization was obtained in that paper (see Theorem 2.11 in Section 2 of the present paper). The ψ-convolutions that admit the ψ-analogue of the identical equation (1.14) (under some mild conditions on ψ, see Theorem 2.7) have been subsequently characterized by V. Sitaramaiah and M.V. Subbarao and have been named as *Lehmer-Narkiewicz convolutions* for some reasons (cf. [20], Remark 4.3). So the ψ-analogue in (1.140 is valid when ψ is a Lehmer-Narkiewicz convolution.

It is interesting to note that it has been proved by J.L. Nicolas and V. Sitaramaiah (cf. [11], Theorem 5.1) that the only multiplicativity preserving ψ-convolutions that admit the ψ-analogue of the identical equation given in (1.14) satisfying $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ and with respect to which every multiplicative function is invertible are Lehmer-Narkiewicz convolutions.

Since a regular ψ-convolution is a generalization of a regular Narkiewicz A-convolution and in view of the A-analogue of the identical equation in (1.9), one may expect that the ψ-analogue of the identical equation could be possible when ψ is a *regular convolution.* One of the main objectives of the present paper is to prove that this guess is a valid one (see Section 3, Theorem 3.1). We prove this by showing that if $\psi(x,y) \geq \max\{x,y\}$ $\forall x,y \in T$ then the regular ψ-convolutions and the Lehmer-Narkiewicz convolutions (L-N convolutions) are one and the same. Since a useful characterization of L-N convolutions has been established (cf. [20], Corollary 4.1), it follows immediately that regular ψ-convolutions satisfying $\psi(x,y) \geq \max\{x,y\}$ $\forall x,y \in T$ have a useful characterization on the tree of its descendants, the regular Narkiewicz A-convolutions.

In Section 4 we study ψ-multiplicative functions, when ψ is a Lehmer-Narkiewicz convolution and establish several properties of these similar to those in the case of A-convution established by K.L. Yocom [27]. In addition to these, we discuss some results on ψ-multiplicative functions that are characteristic of Lehmer-Narkiewicz convolutions.

In Section 5 we discuss the ψ-analogue of Busche-Ramanujan identity (cf. [8]) when ψ is a Lehmer-Narkiewicz convolution. These results show that the Lehmer-Narkiewicz convolutions are favourites of classical results.

2. Preliminaries

The following results (Lemmas 2.1 and 2.2) describe necessary and sufficent conditions concerning the existence of unity and inverses in $(F,+,\psi)$.

Lemma 2.1. (cf. [16], Theorem 2.2) *Let $(F,+,\psi)$ be a commutative ring and $\psi(x,y) \geq \max\{x,y\}$ for all $x,y \in T$. Then $(F,+,\psi)$ possesses the unity if and only if for each $k \in \mathbb{Z}^+$ $\psi(x,k)=k$ has a solution. In such a case if g stands for the unity, then for each $k \in \mathbb{Z}^+$*

$$g(k) = \begin{cases} 1 - \sum\limits_{\substack{\psi(x,k)=k \\ x<k}} g(x) & \text{if } \psi(k,k)=k, \\ 0 & \text{if } \psi(k,k) \neq k. \end{cases} \tag{2.1}$$

Remark. It has been established by J.L. Nicolas and V. Sitaramaiah (cf. [12], Theorem 3.1) that if $(F,+,\psi)$ is a commutative ring, it possesses unity if and only if for each $k \in \mathbb{Z}^+$ $\psi(x,k)=k$ has a solution.

Lemma 2.2. (cf. [15], also see [17], Remark 1.1) *Let ψ satisfy (1.10)-(1.12) and $\psi(x,y) \geq \max\{x,y\}$ for all $x,y \in T$. For each $k \in \mathbb{Z}^+$ let the equation $\psi(x,k)=k$ have a solution so that the unity exists in $(F,+,\psi)$. Let g denote the unity. Then $f \in F$ is invertible with respect to ψ if and only if*

$$S_f(k) \overset{def}{=} \sum_{\psi(x,k)=k} f(x) \neq 0$$

for all $k \in \mathbb{Z}^+$. In such a case, this inverse denoted by $f^{-1}(k)$ can be computed by

$$f^{-1}(1) = \frac{1}{f(1)},$$

and for $k>1$

$$f^{-1}(k) = (S_f(k))^{-1}\left[g(k) - \sum_{\substack{\psi(x,y)=k \\ y<k}} f(x)f^{-1}(y)\right].$$

The following results (Lemmas 2.3 and 2.4) give a characterization of multiplicativity preserving ψ-functions:

Lemma 2.3. (cf. [18], Theorem 3.1) *Let ψ satisfy (1.10)-(1.12) and $\psi(x,y) \geq \max\{x,y\}$ for all $x,y \in T$. Suppose that the binary operation ψ in (1.13) is multiplicativity preserving. If $x = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $y = \prod_{i=1}^{r} p_i^{\beta_i}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, α_i and β_i are non-negative integers, we have*

(a) $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i=1,2,\ldots,r$.

(b) For each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in$ $\in T$ there is a unique non-negative integer $\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$ such that $\psi(p^\alpha, p^\beta) = p^{\theta_p(\alpha,\beta)}$.

(c) If $(x,y) \in T$, then

$$\psi(x,y) = \prod_{i=1}^{r} p_i^{\theta_{p_i}(\alpha_i,\beta_i)}. \tag{2.2}$$

Further, $\theta_p(\alpha,\beta)$ satisfies

(d) For each integer $\gamma \geq 0$, $\theta_p(\alpha,\beta)=\gamma$ has a finite number of solutions.

(e) $\theta_p(\alpha,\beta)=0$ if and only if $\alpha=\beta=0$.

(f) $\theta_p(\alpha,\beta)=\theta_p(\beta,\alpha)$.

(g) For non-negative integers α,β,γ and for any prime p the statements "$(p^\beta,p^\gamma)\in T$, $(p^\alpha,p^{\theta_p(\beta,\gamma)})\in T$" and "$(p^\alpha,p^\beta)\in T$ and $(p^{\theta_p(\alpha,\beta)},p^\gamma)\in T$" are equivalent; when one of these conditions holds we have

$$\theta_p(\alpha,\theta_p(\beta,\gamma))=\theta_p(\theta_p(\alpha,\beta),\gamma).$$

Lemma 2.4. (cf. [18], Theorem 3.2) *Let $T\subseteq\mathbb{Z}^+\times\mathbb{Z}^+$ be such that*

(a) $(x,y)\in T$ if and only if $(y,x)\in T$.

(b) If x and y are as given in Lemma 2.3, then $(x,y)\in T$ if and only if $(p_i^{\alpha_i},p_i^{\beta_i})\in T$ for $i=1,2,\dots,r$.

If for $(x,y)\in T$, $\psi(x,y)$ is defined by (2.2) and $\theta_p(\alpha,\beta)$ satisfies (e), (f) and (g) of Lemma 2.3, then ψ is multiplicativity preserving and for each $k\in\mathbb{Z}^+$, $\psi(x,k)=k$ has a solution.

Lemma 2.5. (cf. [18], Theorem 3.3) *Let ψ be given as in Lemma 2.4 and $\psi(x,y)\geq\max\{x,y\}$ for all $(x,y)\in T$. If M denotes the set of all multiplicative functions which are invertible with respect to ψ, then (M,ψ) is a commutative ring in which the function g defined in (2.1) is the identity.*

Remark 2.6. If ψ and θ_p are as given in Lemma 2.4, clearly $\psi(x,y)\geq$ $\geq\max\{x,y\}$ for all $(x,y)\in T$ is equivalent to saying that $\theta_p(\alpha,\beta\}\geq\max\{\alpha,\beta\}$ for all non-negative integers α and β such that $(p^\alpha,p^\beta)\in T$. In such a case, it is clear that $\psi(x,y)=n$ implies that $x|n$ and $y|n$; it may also be noted that if $\psi(1,n)=n$ for all $n\in\mathbb{Z}^+$, $\psi(x,y)=xy$ whenever $(x,y)=1$. (See also [17], Lemmas 2.1 and 2.2.)

Theorem 2.7. (cf. [19], Theorem) *Let T,ψ and θ_p be as in Lemma 2.4 and $\psi(x,y)\max\{x,y\}$ for all $(x,y)\in T$. Further we assume that for each prime p we have*

$$(\theta_p(\alpha,\beta)=\theta_p(\alpha,\gamma)) \text{ implies that } (\beta=\gamma), \tag{2.3}$$

(2.4)

$$(\theta_p(\alpha,\beta)=\theta_p(\gamma,\delta)) \text{ implies that } \begin{cases} \alpha=\theta_p(\gamma,c) \text{ for some } c\geq 0 \\ \text{or } \beta=\theta_p(\delta,d) \text{ for some } d\geq 0. \end{cases}$$

If f is multiplicative, then the identity in (1.14) holds.

Definition 2.8. Let ψ be multiplicativity preserving with $\psi(x,y) \geq \geq \max\{x,y\}$ for all $(x,y) \in T$ and $\psi(1,k) = k$ for all $k \in \mathbb{Z}^+$. Let T and θ_p be as in Lemma 2.4. Then ψ is called a Lehmer-Narkiewicz convolution or simply $L-N$ convolution if θ_p satisfies (2.3) and (2.4) for all primes p.

Lemma 2.9. (cf. [20], Lemma 3.1) *Let ψ be a regular convolution and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. Then for each $k \in \mathbb{Z}^+$, $(1,k) \in T$ and $\psi(1,k) = k$. Hence the function e defined in (1.6) is the unity in $(F,+,\psi)$ (also $\theta_p(0,\alpha) = \alpha$ for all non-negative integers α).*

Lemma 2.10. (cf. [20], Lemma 3.2) *Let ψ be a regular convolution and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. We fix a prime p and write θ for θ_p, where ψ, T and θ_p are as given in Lemma 2.4. Let*

$$S_{p\alpha} = S_\alpha = \{a \geq 0 : \theta(a,b) = \alpha \text{ for some } b \geq 0\}, \tag{2.5}$$

and μ_ψ denote the inverse of the constant function 1 with respect to ψ. Then we have the following:

(a) If $S_\alpha = \{0,\alpha\}$, then $\{(x,y) : \theta(x,y) = \alpha\} = \{(0,\alpha),(\alpha,0)\}$ and $\mu_\psi(p^\alpha) = -1$.

(b) If $S_\alpha = \{0, a_1, a_2, \ldots, a_r, a_{r+1}$ with $0 < a_1 < a_2 < \ldots < a_{r+1} = \alpha\}$ then $\mu_\psi(p^{a_1}) = -1$ and $\mu_\psi(p^{a_i}) = 0$ for $i = 2,3,\ldots,r+1$. Also $\{(x,y) : \theta(x,y) = a_1\} = \{(0,a_1),(a_1,0)\}$, $a_1 \in S_{a_i}$ and $\theta(a_1,y) = a_i$ has a unique solution in S_{a_i} for $i = 2,3,\ldots,r+1$.

Theorem 2.11. (cf. [20], Theorem 3.1) *We fix a prime p and write θ for θ_p. Let $S_0 = \{0\}$ and $\theta(\alpha,\beta) = 0$ if and only if $\alpha = \beta = 0$. Also, $\theta(\alpha,0) = \theta(0,\alpha) = \alpha$ for every $\alpha \in \mathbb{Z}^+$. For each $\alpha \in \mathbb{Z}^+$ let S_α denote a finite set of integers containing $\{0,\alpha\}$ and with the property that $S_x \subseteq S_\alpha$ for every $x \in S_\alpha$. If $S_\alpha = \{0,\alpha\}$, define $\theta(x,y) = \alpha$ if and only if $(x,y) \in \{(0,\alpha)\}$, $\{(\alpha,0)\}$. If $S_\alpha = \{0,a_1,\ldots,a_r,a_{r+1}\}$ with $0 < a_1 < \ldots < a_r < a_{r+1} = \alpha$, let $S_{a_1} = \{0,a_1\}$ and $\theta(x,y) = a_1$ if and only if $(x,y) \in \{(0,a_1),(a_1,0)\}$. Suppose that for $i = 2,3,\ldots,r+1$ (i) $a_1 \in S_{a_i}$; (ii) solutions of $\theta(x,y) = a_i$ are chosen from $S_{a_i} \times S_{a_i}$ in such a way that (a) $\theta(a_1,y) = a_i$ has a unique solution in S_{a_i}; (b) $S_{a_i} = \{x : \theta(x,y) = a_i \text{ for some } y\}$; (c) θ is associative; (d) $\theta(x,y) \geq \geq \max\{x,y\}$ and (e) $\theta(x,y) = \theta(y,x)$ whenever θ is defined. Let T and ψ be defined as in Lemma 2.4. Then ψ is a regular convolution, $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ and e is the unity of the commutative ring $(F,+,\psi)$. Also, if we define a prime power p^α for $\alpha \in \mathbb{Z}^+$ to be ψ-primitive if $S_\alpha = \{0,\alpha\}$, then*

$$\mu_\psi(p^\alpha) = \begin{cases} -1 & \text{if } p^\alpha \text{ is } \psi\text{-primitive,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Conversely, every regular ψ convolution satisfying $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ can be obtained in this way.

Theorem 2.12. (cf. [20], Corollary 4.1) *For each prime p let π_p denote a class of subsets of non-negative integers such that*

(i) *the union of all members of π_p is the set of non-negative integers;*

(ii) *each member of π_p contains zero;*

(iii) *no two members of π_p contain a positive integer in common.*

If $S \in \pi_p$ and $S = \{a_0, a_1, a_2, \ldots\}$ with $0 = a_0 < a_1 < a_2 < \ldots$, we define $\theta_p(a_i, a_j) = a_{i+j}$, if a_i, a_j and $a_{i+j} \in S$ (i and j need not to be distinct). If ψ and T are as given in Lemma 2.4, then ψ is an L-N convolution and is also a regular convolution. Also, every L-N convolution can be obtained in this way.

It is clear from the above result that there are infinitely many L-N convolutions. If $\theta_p(x,y) = x + y$ for all x, y such that $(p^x, p^y) \in T$, then π_p should consist of arithmetic progressions. Thus, Theorem 2.12 reduces to the characterization theorem of regular Narkiewicz convolutions (see Section 1, Theorem 1.2).

3. Regular ψ-convolutions and Lehmer-Narkiewicz convolutions

In the following Lemmas 3.1, 3.2 and 3.3 we assume that ψ is a regular convolution and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. We first prove

Lemma 3.1. *We fix a prime p. Let $\theta = \theta_p$ and S_α be as given in (2.5). If $S_\alpha = \{a_0 < a_1 < \ldots < a_r < a_{r+1} = \alpha\}$ with $a_0 = 0$, then for $0 \leq i \leq r$,*

$$(a) \qquad S_{a_i} = \{a_0, a_1, \ldots, a_i\}$$

and

$$(b) \qquad \theta(a_1, a_i) = a_{i+1}.$$

Proof. First we prove that (b) implies (a). We assume that (b) is true. By Lemma 2.10, $S_{a_1} = \{a_0, a_1\}$. Hence (a) is true when $i = 1$. We assume that (a) is true for some i, $1 \leq i \leq r - 1$. By (b), $a_i \in S_{a_i+1}$ and so $S_{a_i} \subseteq S_{a_{i+1}}$. Since $a_{i+1} \in S_{a_{i+1}}$ and $S_{a_{i+1}} \subseteq \{a_0, a_1, \ldots, a_{i+1}\}$, we obtain that $S_{a_{i+1}} = \{a_0, a_1, \ldots, a_{i+1}\}$. The induction is complete and (a) follows. We now prove (b). This is true for $i = 0$. We shall prove (b) for $i = 1$. Since $a_1 \in S_{a_2}$, we can find $x \in S_{a_2}$ such that $\theta(a_1, x) = a_2$. $x = a_0 = 0$ is not possible since by Lemma 2.9 $\theta(0, k) = k$ for all positive integers k. So $x = a_1$ or $x = a_2$. If

$x = a_1$ we are through. We prove that $x = a_2$ is not possible. On the contrary suppose that $x = a_2$ so that

$$a_2 = \theta(a_1, a_2) = \theta(a_1, \theta(a_1, a_2)) = \theta((\theta(a_1, a_1), a_2).$$

Hence $\theta(a_1, a_1) = a_1$ or a_2 since $\theta(a_1, a_1) \geq a_1 > 0$. $\theta(a_1, a_1) = a_1$ is not possible since $\theta(a_1, 0) = a_1$ and by Lemma 2.10 $\theta(a_1, y) = a_1$ has a unique solution in S_{a_1}. Hence $\theta(a_1, a_1) = a_2$. Again this is not possible since we assumed that $\theta(a_1, a_2) = a_2$ and by Lemma 2.10 $\theta(a_1, y) = a_2$ has a unique solution in S_{a_2} and $a_1 \in S_{a_2}$. Hence $\theta(a_1, a_1) = a_2$. Hence (b) is true for $i = 1$. We assume (b) for $1 \leq i \leq k$, where $k < r + 1$. We prove (b) for $i = k + 1$. Since $a_1 \in S_{a_{k+2}}$ we can find $a_j \in S_{a_{k+2}}$ such that $\theta(a_1, a_j) = a_{k+2}$. By our induction hypothesis $j \leq k$ is not possible. Hence $j = k+1$ or $k+2$. If $j = k+1$ we are through. Suppose $j = k + 2$. Hence

$$a_{k+2} = \theta(a_1, a_{k+2}) = \theta(a_1, \theta(a_1, a_{k+2})) = \theta(\theta(a_1, a_1), a_{k+2}) = \theta(a_2, a_{k+2}) =$$

$$= \theta(a_2, \theta(a_1, a_{k+2})) = \theta(\theta(a_1, a_2), a_{k+2}) = \theta(a_3, a_{k+2}).$$

Continuing in this way we obtain $a_{k+2} = \theta(a_{k+1}, a_{k+2})$. Therefore

$$a_{k+2} = \theta(a_{k+1}, \theta(a_1, a_{k+2})) = \theta(\theta(a_1, a_{k+1}), a_{k+2}).$$

Since $\theta(a_1, a_{k+1}) \geq a_{k+1}$ and $\theta(a_1, a_{k+1}) \in S_{a_{k+2}}$, $\theta(a_1, a_{k+1}) = a_{k+1}$ or a_{k+2}. Since $\theta(a_1, a_k) = a_{k+1}$ and $\theta(a_1, y) = a_{k+1}$ has a unique solution in $S_{a_{k+1}}$, it follows that $\theta(a_1, a_{k+1}) = a_{k+1}$ is not possible. Hence $\theta(a_1, a_{k+1}) = a_{k+2}$. This is a contradiction to our assumption that $\theta(a_1, a_{k+2})$ (since $\theta(a_1, y) = a_{k+2}$ has a unique solution in $S_{a_{k+2}}$. This completes the proof of Lemma 3.1.

Lemma 3.2. *If $S_\alpha = \{a_0 < a_1 < \ldots < a_r = \alpha\}$ with $a_0 = 0$, then $\theta(a_i, a_j) = a_{i+j}$ if $0 \leq i, j \leq r$ and $i + j \leq r$.*

Proof. Since $\theta(a_i, a_j) = a_{i+j}$ when $i = 0$ or $j = 0$, we can assume that $i > 0$ and $j > 0$. By Lemma 3.1 the conclusion is true when $i + j = 2$. We assume that the conclusion holds for all pairs i and j with $2 \leq i + j < r$. Let now $i + j = r$. Since $a_i \in S_\alpha$, we can find $a_k \in S_\alpha$ such that $\theta(a_i, a_k) = a_r = \alpha$. Clearly, $1 \leq k \leq r - 1$. For $k = r$ we have

$$a_r = \theta(a_i, a_k) = \theta(a_i, a_r) = \theta(a_i, \theta(a_1, a_{r-1})) = \theta(a_1, \theta(a_i, a_{r-1})).$$

Since $a_r = \theta(a_1, a_{r-1})$, we obtain $a_{r-1} = \theta(a_i, a_{r-1})$. Continuing in this way, if $i + 1 \leq t \leq r$, we obtain $\theta(a_i, a_{r-t}) = a_{r-t}$. By induction hypothesis $\theta(a_i, a_{r-t}) = a_{i+r-t}$. Hence $i = 0$. But $i > 0$. Hence $k = r$ is not possible so that $1 \leq k \leq r - 1$. We now prove that $i + k \leq r$. If $i + k > r$, we have

$$a_r = \theta(a_i, a_k) = \theta(\theta(a_1, a_{i-1}), a_k) = \theta(a_1, \theta(a_{i-1}, a_k)),$$

so that $\theta(a_{i-1}, a_k) = a_{r-1}$. Continuing as above, if $i+k-r < j \le i$, we obtain $\theta(a_{i-j}, a_k) = a_{r-j}$. Since $i-j+k < r$, by our induction hypothesis,

$$a_{r-j} = \theta(a_{i-j}, a_k) = a_{i-j+k}.$$

Hence $r-j = i-j+k$ or $i+k = r$. This is a contradiction to our assumption that $i+k > r$. Hence $i+k \le r$. If $i+k \le r-1$,

$$a_r = \theta(a_i, a_k) = a_{i+k} \le a_{r-1} < a_r,$$

a contradiction. Hence $i+k = r$ so that $k = j$ and $\theta(a_i, a_j) = a_{i+j}$.

Lemma 3.3. *Let $S_\alpha = \{a_0 < a_1 < \ldots < a_r = \alpha\}$ with $a_0 = 0$. If $\theta(a_i, a_j) = a_k$, where $0 \le i, j \le k \le r$, then $i + j \le k$.*

Proof. We can assume that i, j and k are positive. Suppose $i+j > k$. We have

$$a_k = \theta(a_i, a_j) = \theta(\theta(a_1, a_{i-1}), a_j) = \theta(a_1, \theta(a_{i-1}, a_j)).$$

Hence $\theta(a_{i-1}, a_j) = a_{k-1}$. Continuing as above, if $i+j-k \le t \le i$ (so that $i-t+j < k$) we obtain $\theta(a_{i-t}, a_j) = a_{k-t}$. Since $i-t+j < k \le r$, by Lemma 3.2, $\theta(a_{i-t}, a_j) = a_{i-t+j}$. Hence $k-t = i-t+j$ or $i+j = k$. This contradicts $i+j > k$. Hence $i+j \le k$.

Lemma 3.4. *If $S_\alpha = \{0 = a_0 < a_1 < \ldots < a_k = \alpha\}$ and a_i, a_j and $\theta(a_i, a_j) \in S_\alpha$, then $\theta(a_i, a_j) = a_{i+j}$.*

Proof. It follows from Lemmas 3.2 and 3.3.

Theorem 3.1. *Let ψ satisfy (1.10)-(1.12) and $\psi(x, y) \ge \max\{x, y\}$ for all $(x, y) \in T$. Then ψ is a regular convolution if and only if it is a Lehmer-Narkiewicz convolution (L-N convolution).*

Proof. If ψ is a Lehmer-Narkiewicz convolution then it is a regular convolution. This follows from Theorem 2.12.

Now we assume that ψ is a regular convolution. In view of Lemma 2.9, (2.6), Lemmas 3.2 and 3.3, to prove that ψ is an $L-N$ convolution it is enough to verify (2.3) and (2.4). We fix a prime p and write $\theta = \theta_p$. Let

$$\theta(\alpha, \beta) = \theta(\alpha, \gamma) = t, \tag{3.1}$$

where α, β and γ are non-negative integers. If $t = 0$ by (e) of Lemma 2.3, $\alpha = \beta = \gamma = 0$. So $\beta = \gamma$. Let $t > 0$. Let

$$S_t = \{a_0, a_1, \ldots, a_r = t\} \tag{3.2}$$

with $0 = a_0 < a_1 < \ldots < a_r$. Since α, β and $\gamma \in S_t$, we can assume that $\alpha = a_i, \beta = a_j$ and $\gamma = a_k$. By Lemmas 3.2 and 3.3, $\theta(\alpha, \beta) = a_{i+j}$ and $\theta(\alpha, \gamma) = a_{i+k}$. From (3.1) we must have $i + j = i + k$, so that $j = k$. Hence $\beta = \gamma$. This proves (2.3). We now prove (2.4). Suppose

$$\theta(\alpha, \beta) = \theta(\gamma, \delta) = t, \tag{3.3}$$

where α, β, γ and δ are non-negative integers. If $t = 0$, (2.4) follows trivially since in that case $\alpha = \beta = \gamma = \delta = 0$. Let $t > 0$. If S_t is given by (3.2) we can assume that $\alpha = a_i$, $\beta = a_j$, $\gamma = a_k$ and $\delta = a_\ell$. By Lemma 3.4 and (3.3) we have $i + j = k + \ell$. Both $i > k$ and $j > \ell$ cannot hold simultaneously. If $i \leq k$, we express $j = (k - i) + \ell$ so that $\beta = \theta(a_\ell, a_{k-i}) = \theta(\delta, c)$ with $c = a_{k-i} \geq 0$. Similarly if $j \leq \ell, \alpha = \theta(\gamma, c)$ with $c = a_{\ell-j} \geq 0$. Hence (2.4) follows. Thus ψ is an $L - N$ convolution.

Remark 3.5. If $\alpha \in \mathbb{Z}^+$ and for each prime p, $S_{p,\alpha} = \{0 = a_0 < a_1 < < \ldots < a_k = \alpha\}$, it follows from Lemmas 3.1-3.3 and Theorem 3.1 that if ψ is a regular convolution (satisfying $\psi(x, y) \geq \max\{x, y\}$ for all $(x, y) \in T$) with $\theta_p = \theta$, then for $0 \leq \ell \leq k$ the solutions of $\theta(x, y) = a_\ell$ are precisely $\{(a_i, a_j) : i + j = \ell,\ i, j \geq 0\}$.

Remark 3.6. In view of Lemma 3.1 the function θ, given in Example 3.1 of [20], p.139, is inadequate to define a regular ψ-convolution.

4. ψ-multiplicative functions

We start with

Definition 4.1. Let ψ satisfy (1.10)-(1.13). An arithmetic function f is called ψ-multiplicative if

$$f(\psi(m, n)) = f(m)f(n) \tag{4.1}$$

for all $(m, n) \in T$.

The concept of ψ-multiplicative functions is due to D.H. Lehmer [6]. The zero function and the constant function 1 are two trivial examples of ψ-multiplicative functions.

In the case of Dirichlet convolution a ψ-multiplicative function is nothing but a completely multiplicative function ($f \in F$ is said to be completely multiplicative if $f \not\equiv 0$ and $f(mn) = f(m)f(n)$ for all positive integers m and n). If ψ is the unitary convolution, a ψ-multiplicative function is simply a multiplicative function. If ψ is a regular Narkiewicz A-convolution, the concept of ψ-multiplicative functions reduces to that of A-multiplicative functions introduced and studied by K.L. Yocom [27].

Remark 4.2. If ψ is Dirichlet, unitary or in general a regular Narkiewicz A-convolution, every ψ-multiplicative function is also multiplicative. However, this is not true in general. For example if $T = \{(n,n) : n \in \mathbb{Z}^+\}$ and $\psi : T \to \mathbb{Z}^+$ is defined by $\psi(n,n) = n$ for each $n \in \mathbb{Z}^+$, then any $f \in F$ such that $f(n) = 1$ or 0 for each $n \in \mathbb{Z}^+$ is a ψ-multiplicative function. Hence not all such $f \in F$ need be multiplicative. As another example let $T = \{(x,y) \in \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \gamma(x) = \gamma(y)\}$. Let $\psi_E : T \to \mathbb{Z}^+$ be defined by $\psi_E(1,1) = 1$ and $\psi_E(x,y) = \prod_{i=1}^{r} p_i^{\alpha_i\beta_i}$, if $x = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $y = \prod_{i=1}^{r} p_i^{\beta_i}$ are the canonical representations of x and y. If $f \in F$ is defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 1, \\ |\mu(n)| & \text{if } n > 1, \end{cases}$$

where μ is the Möbius function, then f is ψ_E-multiplicative, but not multiplicative. ψ_E is the *exponential convolution* introduced by M.V. Subbarao [22].

We can prove the following

Lemma 4.3. *Let ψ be multiplicativity preserving, $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ and $\psi(1,n) = n$ for all $n \in \mathbb{Z}^+$. Then (i) every ψ-multiplicative function is also multiplicative; (ii) if every multiplicative function is ψ-multiplicative, then ψ must be the unitary convolution.*

Proof. (i) From Remark 2.6 whenever m and n are relatively prime, $\psi(m,n) = mn$. (ii) is not difficult to prove. Hence Lemma 4.3 follows.

In what follows we show that the results concerning the distributivity of A-multiplicative functions over A-products of arithmetic functions established by K.L. Yocom [27] can be extended to Lehmer-Narkiewicz convolutions.

Throughout the following we assume that ψ is a Lehmer-Narkiewicz convolution (that is a regular ψ-convolution satisfying $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$). We reserve the letter p (with or without suffixes) to denote prime numbers. We freely make use of Lemmas 3.1-3.3, Theorem 3.1 and the remark following it.

Let T, θ_p and ψ be given as in Lemma 2.4. We write $\theta = \theta_p$. Let $S_{p,\alpha} = S_\alpha$ be as given in (2.5). The least positive integer in S_α will be denoted by $\tau_p(\alpha) = \tau(\alpha)$ and the number of positive integers in S_α will be denoted by $r_p(\alpha) = r(\alpha)$. We also define $r(0) = 0$ and $\tau(0) = 0$. From Lemma 3.1 it follows that if x and y are positive integers in S_α, then $\tau(x) = \tau(y)$; also, from the same lemma, it follows that if $S_\alpha = \{0 = a_0 < a_1 < \ldots < a_k = \alpha\}$ then $r(a_\ell) = \ell$ for $1 \le \ell \le k$. We now prove

Theorem 4.1. *$f \in F$ is ψ-multiplicative if and only if f is multiplicative and for all $\alpha \in \mathbb{Z}^+$ and each p*

$$f(p^\alpha) = \left(f(p^{a_1})\right)^k, \tag{4.2}$$

where $a_1 = \tau(\alpha)$ and $k = r(\alpha)$.

Proof. Suppose that f is ψ-multiplicative. By Lemma 4.3 f is multiplicative. We prove (4.2) by induction on $r(\alpha)$. If $r(\alpha) = 1$, (4.2) follows immediately since $k = 1$ and $a_1 = \alpha$. We assume the truth of (4.2) whenever $r(\alpha) < t$, where t is a positive integer. Let $r(\alpha) = t$ and $S_\alpha = \{0 = a_0 < a_1 < a_2 < \ldots < a_{t-1} < a_t = \alpha\}$. Since f is ψ-multiplicative and $\theta(a_1, a_{t-1}) = a_t$, we have

$$f\left(p^\alpha\right) = f\left(p^{a_t}\right) = f\left(p^{\theta(a_1,a_{t-1})}\right) = f\left(p^{a_1}\right) f\left(p^{a_{t-1}}\right). \tag{4.3}$$

Since $r(a_{t-1}) = t - 1$ and $\tau(a_{t-1}) = a_1$, it follows from the induction hypothesis that $f\left(p^{a_{t-1}}\right) = f\left(p^{a_1}\right)^{t-1}$. Substituting this into (4.3) we obtain (4.2) with $k = t$. The induction is complete. Conversely, we assume that f is a multiplicative function and (4.2) holds. We need to prove (4.1) for all $(m, n) \in T$. By (2.2) and since f is multiplicative, it suffices to verify (4.1) when $m = p^\alpha$ and $n = p^\beta$, where α and β are positive integers as the cases $\alpha = 0$ or $\beta = 0$ can be disposed easily since $f(1) = 1$ and $\psi(1, x) = x$ for all $x \in \mathbb{Z}^+$. Since $\psi(p^\alpha, p^\beta) = p^{\theta(\alpha,\beta)}$, we need to verify that

$$f\left(p^{\theta(\alpha,\beta)}\right) = f(p^\alpha) f(p^\beta), \tag{4.4}$$

whenever $(p^\alpha, p^\beta) \in T$. Let $\gamma = \theta(\alpha, \beta)$. Then $\alpha, \beta \in S_\gamma = \{0 = a_0 < a_1 < \ldots < a_t = \gamma\}$ so that $\alpha = a_i$ and $\beta = a_j$ for some i and j with $0 < i, j \le t$. Since $\gamma = a_t = \theta(\alpha, \beta) = \theta(a_i, a_j)$, by Lemma 3.4 $\theta(a_i, a_j) = a_{i+j}$. Hence $t = i + j$. Hence

$$f\left(p^{\theta(\alpha,\beta)}\right) = \left(f(p^{a_1})\right)^{i+j} = f(p^\alpha) f(p^\beta).$$

Hence f is ψ-multiplicative. The proof of Theorem 4.1 is complete.

In the following for $f, g \in F$, fg stands for the pointwise product of f and g, that is $(fg)(x) = f(x)g(x)$ for all $x \in \mathbb{Z}^+$.

Theorem 4.2. *Let $f \in F$. The following statements are equivalent:*

(a) f is ψ-multiplicative.

(b) $f(g\psi h) = fg\psi fh$ for all $g, h \in F$.

(c) $f(g\psi g) = fg\psi fg$ for some ψ-multiplicative function g which is never zero.

(d) $f(1\psi 1) = f\psi f$.

(e) $f(1\psi g) = f\psi fg$ for some $g \in F$ which is positive.

Proof. The proofs of $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ are quite easy. If we prove $(e) \Rightarrow (a)$, then by taking $g(x) = 1$ for all $x \in \mathbb{Z}^+$ in this proof, the proof of $(d) \Rightarrow (a)$ follows. We shall now prove that $(e) \Rightarrow (a)$.

We assume (e). Hence

$$f(n) \sum_{\psi(x,y)=n} g(y) = \sum_{\psi(x,y)=n} f(x)f(y)g(y) \tag{4.5}$$

for all positive integers n.

By taking $n = 1$ in (4.5) we obtain $f(1)g(1) = f^2(1)g(1)$ so that $f(1) = = f^2(1)$, since $g(1) > 0$. Hence $f(1) = 0$ or 1. If $f(1) = 0$ we show that $f \equiv 0$. Let $f(1) = 0$. Assume that $f(m) = 0$ for $1 \le m < n$. Consider the sum on the right hand side of (4.5). We have

$$\sum_{\psi(x,y)=n} f(x)f(y)g(y) = \sum_{\substack{\psi(x,y)=n \\ x=n, y=n}} f(x)f(y)g(y) = \text{Empty sum} = 0,$$

since $\psi(n, n) > n$ for $n \ge 2$. Equating this value zero to the left hand side of (4.5) we obtain that $f(n) = 0$ since $\sum\limits_{\psi(x,y)=n} g(y) > 0$. Hence $f \equiv 0$.

Let $f(1) = 1$. If $n = \prod\limits_{i=1}^{\ell} p_i^{\alpha_i}$, where $p_1, p_2, \ldots, p_\ell$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are non-negative integers with $\alpha_1 + \ldots + \alpha_\ell = t$, $t \ge 0$, then we prove that

$$f(n) = \prod_{i=1}^{\ell} f\left(p_i^{a_i}\right)^{k_i}, \tag{4.6}$$

where $a_i = \tau_{p_i}(\alpha_i)$ and $k_i = r_{p_i}(\alpha_i)$ for $i = 1, 2, \ldots, \ell$. (4.6) follows trivially if $t = 0$. We assume (4.6) whenever $\alpha_1 + \ldots + \alpha_\ell < t$, $t > 0$. Suppose that

$\alpha_1+\ldots+\alpha_\ell=t$. If $S_{\alpha_i}=\{0,\alpha_i\}$ for each $\alpha_i>0$, where $1\le i\le\ell$, then for such i, $a_i=\tau_{p_i}(\alpha_i)=\alpha_i$ and $k_i=r_{p_i}(\alpha_i)=1$. Hence (4.6) follows trivially. We can assume that $S_{\alpha_i}\ne\{0,\alpha_i\}$ for some $\alpha_i>0$. Without loss of generality we may assume that $\alpha_1>0$ and $S_{\alpha_i}\ne\{0,\alpha_1\}$. Let $S_{\alpha_1}=\{0=a_0<a_1<$ $<\ldots<a_{m-1}<a_m=\alpha_1\}$. Then $\theta_{p_1}(a_1,a_{m-1})=a_m=\alpha_1$. Temporarily, let $\beta_1=a_1,\beta_2=\alpha_2,\ldots,\beta_\ell=\alpha_\ell$ and $\gamma_1=a_{m-1},\gamma_2=0,\ldots,\gamma_\ell=0$. Then

$$\beta_1+\beta_2+\ldots+\beta_\ell<a_1+\alpha_2+\ldots+\alpha_\ell<\alpha_1+\alpha_2+\ldots+\alpha_\ell=t$$

and similarly

$$\gamma_1+\gamma_2+\ldots+\gamma_\ell=a_{m-1}<a_m=\alpha_1\le t.$$

Also $\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i$ for $i=1,2,\ldots,\ell$. Hence the sum

$$\Sigma=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\le i\le\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell<t\\ \gamma_1+\gamma_2+\ldots+\gamma_\ell<t}} g\left(p_1^{\gamma_1}p_2^{\gamma_2}\ldots p_\ell^{\gamma_\ell}\right)>0, \tag{4.7}$$

since the sum is non-empty and $g>0$.

We consider the sum on the right hand side of (4.6). We obtain

$$\sum_{\psi(x,y)=n} f(x)f(y)g(y)= \tag{4.8}$$

$$=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\le i\le\ell}} f\left(\prod_{i=1}^{\ell}p_i^{\beta_i}\right)f\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)=\Sigma_1+\Sigma_2,$$

say, where

$$\Sigma_1=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\le i\le\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell<t}} f\left(\prod_{i=1}^{\ell}p_i^{\beta_i}\right)f\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)$$

and

$$\Sigma_2=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\le i\le\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell=t}} f\left(\prod_{i=1}^{\ell}p_i^{\beta_i}\right)f\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right).$$

Since $\alpha_1+\alpha_2+\ldots+\alpha_\ell=t$ and for $i=1,2,\ldots,\ell$, $\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i$ implies that $\beta_i\leq\alpha_i$, the conditions under Σ_2 imply that $\beta_i=\alpha_i$ and $\gamma_i=0$ for $i=1,2,\ldots,\ell$. Hence

$$\Sigma_2=f(n). \tag{4.9}$$

In a similar way we write

$$\Sigma_1=\Sigma_3+\Sigma_4,$$

where

$$\Sigma_3=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\leq i\leq\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell<t\\ \gamma_1+\gamma_2+\ldots+\gamma_\ell<t}} f\left(\prod_{i=1}^{\ell}p_i^{\beta_i}\right)f\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)$$

and

$$\Sigma_4=\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\leq i\leq\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell<t\\ \gamma_1+\gamma_2+\ldots+\gamma_\ell=t}} f\left(\prod_{i=1}^{\ell}p_i^{\beta_i}\right)f\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right).$$

Adopting similar arguments used in simplifying the sum Σ_2, it can be shown that

$$\Sigma_4=f(n)g(n). \tag{4.10}$$

We consider the sum Σ_3. By our induction hypothesis, we obtain

$$\Sigma_3=\prod_{i=1}^{\ell}f\left(p_i^{a_i}\right)^{k_i}\sum_{\substack{\theta_{p_i}(\beta_i,\gamma_i)=\alpha_i\\ 1\leq i\leq\ell\\ \beta_1+\beta_2+\ldots+\beta_\ell<t\\ \gamma_1+\gamma_2+\ldots+\gamma_\ell<t}} g\left(\prod_{i=1}^{\ell}p_i^{\gamma_i}\right)=\prod_{i=1}^{\ell}f\left(p_i^{a_i}\right)^{k_i}\cdot\Sigma, \tag{4.11}$$

by (4.7). Putting (4.9)-(4.11) into (4.8), we obtain

$$\sum_{\psi(x,y)=n}f(x)f(y)g(y)=\prod_{i=1}^{\ell}f(p_i^{a_i})^{k_i}\cdot\Sigma+f(n)+f(n)g(n). \tag{4.12}$$

In a similar way we obtain

$$\sum_{\psi(x,y)=n}g(y)=1+g(n)+\Sigma. \tag{4.13}$$

Substituting (4.13) into the left hand side of (4.8), equating with (4.12), cancelling the like terms and finally cancelling the factor $\Sigma > 0$ we obtain (4.6). This completes the proof of Theorem 4.2.

The following theorems can be proved on lines similar to that of K.L. Yocom [27].

Theorem 4.3. *(a) If $f \in F$ is ψ-multiplicative and $f(1) \neq 0$, then*

$$(fg)^{-1} = fg^{-1} \tag{4.14}$$

for all $g \in F$ with $g(1) \neq 0$.

(b) If f is multiplicative and (4.14) holds with $g = 1$ so that $g^{-1} = \mu_\psi$, then f is ψ-multiplicative.

Theorem 4.4. *If f is multiplicative, then f is ψ-multiplicative if and only if $f^{-1} = f\mu_\psi$.*

Theorem 4.5. *If f is multiplicative and $f(1) \neq 0$, then f is ψ-multiplicative if and only if $f^{-1}(p^\alpha) = 0$ for all prime powers $p^\alpha > 1$ that are not ψ-primitive.*

Theorem 4.6. *Let $g, G \in F$ and $g = G\psi\mu_\psi$. We have*

(a) if f is ψ-multiplicative with $f(1) \neq 0$ then

$$(fG)\psi f^{-1} = fg; \tag{4.15}$$

(b) if f is multiplicative and if (4.15) holds for some G with $G(1) = 1$ and $G(p^\alpha) \neq 1$ for all prime powers $p^\alpha > 1$, then f is ψ-multiplicative.

A characterization of Lehmer-Narkiewicz convolutions. It is interesting to note that the property of ψ-multiplicative functions derived in Theorem 4.1 is characteristic of Lehmer-Narkiewicz convolutions. More precisely let (i) ψ be multiplicativity preserving and (ii) $\psi(1, k) = k \ \forall k \in \mathbb{Z}^+$. We write $\theta = \theta_p$. For each positive integer α and any prime p let $S_\alpha = S_{p,\alpha}$ be as given in (2.5); $\tau_p(\alpha) = \tau(\alpha)$ and $r_p(\alpha) = r(\alpha)$ are as defined in the beginning of this section. We have

Theorem 4.7. *In the commutative ring $(F, +, \psi)$ suppose that a multiplicative function f is ψ-multiplicative if and only if (4.2) holds, that is for any prime p and any $\alpha \in \mathbb{Z}^+$*

$$f(p^\alpha) = (f(p^{a_1}))^k,$$

where $a_1 = \tau(\alpha)$ and $k = r(\alpha)$. Then ψ is a Lehmer-Narkiewicz convolution.

Proof. (I) First we prove that if α_i for $i = 1, 2, 3$ and 4 are such that $\theta(\alpha_1, \alpha_2) = \theta(\alpha_3, \alpha_4)$, then $\tau(\alpha_1) = \tau(\alpha_i)$ for $i = 2, 3$ and 4. Let $a_i = \tau(\alpha_i)$ for $i = 1, 2, 3$ and 4. For $1 \leq i \leq 4$ let $b_i = r(\alpha_i)$.

Let f be a multiplicative function (with $f(1) = 1$) defined at prime powers $p^x > 1$ by $f(p^x) = (f(p^t))^\tau$, where $t = \tau(x)$ and $r = r(x)$. We fix a prime p and we define the arithmetic function g by $g(x) = f(p^x)$ for all positive integers x. By our assumption f is ψ-multiplicative and hence we have

$$f\left(p^{\theta(\alpha_1,\alpha_2)}\right) = f(p^{\alpha_1})f(p^{\alpha_2}) = f(p^{a_1})^{b_1} f(p^{a_2})^{b_2} = g(a_1)^{b_1} g(a_2)^{b_2}$$

and

$$f\left(p^{\theta(\alpha_3,\alpha_4)}\right) = f(p^{\alpha_3})f(p^{\alpha_4}) = f(p^{a_3})^{b_3} f(p^{a_4}) = g(a_3)^{b_3} g(a_4)^{b_4}.$$

Hence

$$g(a_1)^{b_1} g(a_2)^{b_2} = g(a_3)^{b_3} g(a_4)^{b_4}$$

for all arithmetic functions g. Hence $a_1 = a_i$ for $i = 2, 3$ and 4.

(II) If $\theta(\alpha, \beta) = \theta(\alpha, \gamma)$ we show that $\beta = \gamma$. If $\alpha = 0$, trivially $\beta = \gamma$. We may assume that α is a positive integer. We define the multiplicative function h at prime powers $p^x > 1$ by

$$h(p^x) = p^{r(x)}.$$

If $a_1 = \tau(x)$, since $r(a_1) = 1$, we have

$$h(p^x) = (h(p^{a_1}))^{r(x)}.$$

Hence by our assumption h is a ψ-multiplicative function. Therefore

$$h(p^\alpha)h(p^\beta) = h\left(p^{\theta(\alpha,\beta)}\right) = h\left(p^{\theta(\alpha,\gamma)}\right) = h(p^\alpha)h(p^\gamma)$$

since $h(p^x) > 1$ for $x > 0$, cancelling $h(p^\alpha)$ on both sides we obtain

$$h(p^\beta) = h(p^\gamma). \tag{4.16}$$

From (4.16) it readily follows that $\beta = 0$ if and only if $\gamma = 0$. We can assume that β and γ are positive integers. Again from (4.16) we obtain

$$p^{r(\beta)} = h(p^\beta) = h(p^\gamma) = p^{r(\gamma)},$$

so that

$$r(\beta) = r(\gamma). \tag{4.17}$$

Let $t = \theta(\alpha, \beta) = \theta(\alpha, \gamma)$, where α, β and γ are positive integers. Let

$$S_t = \{0 < a_1 < a_2 < \ldots < a_s = t\}.$$

We claim that

$$S_{a_i} = \{0, a_1, a_2, \ldots, a_i\} \tag{4.18}$$

and

$$\theta(a_1, a_{i-1}) = a_i \quad \text{for} \quad 1 \le i \le s \tag{4.19}$$

with $a_0 = 0$. Clearly, (4.18) and (4.19) are true for $i = 1$. We can assume that $|S_t| \ge 3$. We assume (4.18) and (4.19) for some $i \ge 1$. If $2 \le j \le s-1$, since $a_1, a_j \in S_t$, $\theta(a_1, a_k) = \theta(a_j, a_k') = t$ for some $a_k, a_k' \in S_t$. Also, a_k and a_k' are positive integers. Hence, by case (I), $a_1 = \tau(a_1) = \tau(a_j)$ for $2 \le j \le s-1$. Hence $a_1 \in S_{a_{i+1}} \subseteq \{0, a_1, a_2, \ldots, a_{i+1}\}$. Hence we can find an $x \in S_{a_{i+1}}$, such that $\theta(a_1, x) = a_{i+1}$. By induction hypothesis x cannot be in $\{0, a_1, \ldots, a_{i-1}\}$. Suppose $x = a_{i+1}$ so that $\theta(a_1, a_{i+1}) = a_{i+1}$. Since h is ψ-multiplicative, we have

$$h(p^{a_{i+1}}) = h\left(p^{\theta(a_1, a_{i+1})}\right) = h(p^{a_1})h(p^{a_{i+1}}),$$

so that $h(p^{a_1}) = 1$. This cannot happen. Hence $x = a_i$. This implies that $a_i \in S_{a_{i+1}}$ and $\theta(a_1, a_i) = a_{i+1}$. The induction is complete.

Now $\theta(\alpha, \beta) = \theta(\alpha, \gamma) = t$ implies that $\beta, \gamma \in S_t$. Hence $\beta = a_k$ and $\gamma = a_\ell$ for some positive integers k and ℓ, $1 \le k, \ell \le s$. Since $S_{a_k} = \{0, a_1, \ldots, a_\ell\}$, $r(\beta) = r(a_k) = k$ and $r(\gamma) = r(a_\ell) = \ell$. From (4.17) we obtain $k = \ell$ and so $\beta = \gamma$.

(III) If $S_\alpha = \{0 = a_0 < a_1 < \ldots < a_s = \alpha\}$, as in Lemma 3.2, we can show that $\theta(a_i, a_j) = a_{i+j}$ if $0 \le i, j, i+j \le s$.

(IV) Let

$$\theta(\alpha, \beta) = \theta(\gamma, \delta) = t.$$

As in the proof of Theorem 3.1, we can show that either $\alpha = \theta(\gamma, c)$ for some $c \ge 0$ and $\beta = \theta(\delta, d)$ for some $d \ge 0$. This completes the proof.

It is also interesting to note that the result in Theorem 4.4 can hold good in ψ-convolutions other than Lehmer-Narkiewicz convolutions. We write $\theta_p = \theta$ for each prime p. Define $\theta(0, \alpha) = \theta(\alpha, 0) = \theta(\alpha, \alpha) = \alpha$ for all non-negative integers α. Let T and ψ be as given by Lemma 2.4. Then $\psi(1, k) = k$ for all $k \in \mathbb{Z}^+$ and ψ is multiplicativity preserving. Also, $\psi(x, y) \ge \max\{x, y\}$ for all $(x, y) \in T$. Let f be multiplicative. If f is ψ-multiplicative, it is easy to verify that $f^{-1} = \mu_\psi f$. Conversely, let $f^{-1} = \mu_\psi f$. We show that f is ψ-multiplicative. For any prime power $p^\alpha > 1$ we have

$$(4.20)\quad 0 = e(p^\alpha) = \sum_{\theta(\beta,\gamma)=\alpha} f(p^\beta)f^{-1}(p^\gamma) = f^{-1}(p^\alpha) + f(p^\alpha) + f(p^\alpha)f^{-1}(p^\alpha).$$

Similarly

$$0 = e(p^\alpha) = \sum_{\theta(\beta,\gamma)=\alpha} \mu_\psi(p^\beta) = 1 + 2\mu_\psi(p^\alpha),$$

so that $\mu_\psi(p^\alpha) = -1/2$. Hence $f^{-1}(p^\alpha) = \mu_\psi(p^\alpha)f(p^\alpha) = -f(p^\alpha)/2$. Substituting this into (4.20), we obtain after simplification $f^2(p^\alpha) = f(p^\alpha)$. Since f is multiplicative, the verification of (4.1) reduces to the verification that $f\left(p^{\theta(x,y)}\right) = f(p^x)f(p^y)$ whenever $(p^x, p^y) \in T$; this identity is satisfied by all the admissible pairs $(x, y) \in \{(0, \alpha), (\alpha, 0), (\alpha, \alpha)\}$. Hence f is ψ-multiplicative.

5. Busche-Ramanujan identities

For a non-negative integer k and a positive integer n let $\sigma_k(n)$ denote the sum of the k-th powers of the divisors of n so that $\sigma_k(n) = \sum_{d|n} d^k$. Clearly, $\sigma_0(n) = d(n)$, the number of divisors of n and $\sigma_1(n) = \sigma(n)$, the sum of the divisors of n.

In 1906 E. Busche [1] stated the identity

$$(5.1)\qquad \sigma)k(m)\sigma_k(n) = \sum_{d|(m,n)} d^k \sigma_k(mn/d^2).$$

The identity

$$(5.2)\qquad \sigma_k(mn) = \sum_{d|(m,n)} \sigma_k(m/d)\sigma_k(n/d)\mu(d)d^k$$

was stated by Srinivasa Ramanujan (cf. [14], formula (11)) when $k = 0$; this has subsequently been verified by S. Chowla (J. Ind. Math. Soc., Notes and questions, **18** (1929), 87-88) for all positive integers k. Either of (5.1) and (5.2) is called *Busche-Ramanujan identity.*

In his celebrated paper R. Vaidyanathaswamy [25], among other things, succeeded in understanding the identities (5.1) and (5.2); he proved that these identities are special cases of a class of identities which he called as *Busche-Ramanujan identities*.

A multiplicative function f is said to admit a Busche-Ramanujan identity (cf. [25], p.646) if there exists a multiplicative function H such that

(5.3) $$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)H(d)$$

for all $m, n \in \mathbb{Z}^+$.

R. Vaidyanathaswamy (cf. [25]) completely characterized the multiplicative functions which admit Busche-Ramanujan identity. He proved (cf. [25], Theorem XXXV) that such functions are nothing but quadratic functions, that is the Dirichlet product of two completely multiplicative functions. These quadratic functions are also called specially multiplicative functions. This nomenclature is due to D.H. Lehmer [7].

As pointed out by P. Haukkanen (cf. [5], p.41), using the concepts of Vaidyanathaswamy [25], the inverse form of (5.3) can be represented as

(5.4) $$f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)H^{-1}(d),$$

where H^{-1} is the inverse of H with respect to Dirichlet convolution (see also [21], eq. (1.2)). It is of interest to note that K.G. Ramanathan [13] showed that a quadratic function f admits a Busche-Ramanujan identity of the form (5.3) with $H(n) = \mu(n)B(n)$, where B is the completely multiplicative function defined by $B(p) = f^2(p) - f(p^2)$ for all primes p. It is now clear that by taking $f(n) = \sigma_k(n) = (i_k D 1)(n)$ in (5.3) and (5.4), where $i_k(n) = n^k$ for all $n \in \mathbb{Z}^+$, we obtain (5.1) and (5.2).

Let ψ be a Lehmer-Narkiewicz convolution. An arithmetic function f is said to be a ψ-quadratic function if $f = g\ \psi\ h$, where g and h are ψ-multiplicative functions. In case ψ is the Dirichlet convolution, a ψ-quadratic function is a quadratic function. If ψ is a regular Narkiewicz convolution A, the notion of ψ-quadratic function reduces to that of A-specially multiplicative function introduced by P.J. McCarthy (cf. [9], p.174).

Following Vaidyanathaswamy [25], we say that a multiplicative function admits ψ-analogue of Busche-Ramanujan identity if we can find a multiplicative function H such that for all $(m, n) \in T$

(5.5) $$f(\psi(m,n)) = \sum_{\substack{\psi(x,z)=m \\ \psi(y,z)=n}} f(x)f(y)H(z).$$

On lines similar to that of P.J. McCarthy (cf. [9], Theorem 1.12, p.19), we can prove

Theorem 5.1. *The following statements are equivalent:*

(a) f is a ψ-quadratic function;

(b) f admits ψ-analogue of the Busche-Ramanujan identity (see (5.5));

(c) there is a ψ-multiplicative function B such that

$$f(m)f(n) = \sum_{\substack{\psi(a,d)=m \\ \psi(b,d)=n}} f(\psi(a,b))B(d)$$

for all $(m,n) \in T$;

(d) for any positive integer α, with $S_p(\alpha) = \{0 = a_0 < a_1 < \ldots < a_s < < a_{s+1} = \alpha\}$ and $r_p(\alpha) \geq 2$,

$$f(p^{a_{s+1}}) = f(p^{a_1})f(p^{a_s}) + f(p^{a_{s-1}})B(p^{a_1}),$$

where $B(p^{a_1}) = f(p^{a_2}) - f^2(p^{a_1})$;

(e)

$$f(\psi(m,n)) = \sum_{\substack{\psi(x,z)=m \\ \psi(y,z)=n}} f(x)f(y)\mu_\psi(z)B(z)$$

for all $(m,n) \in T$.

When ψ is a regular Narkiewicz convolution, the results in Theorem 5.1 reduce to the results obtained by P.J. McCarthy (cf. [9], pp.174 and 175). As pointed out by P. Haukkanen (cf. [5], p.42), while stating the A-analogues of (b) and (c) of Theorem 5.1, McCarthy [9] requires that $m, n \in A(r)$ for some positive integer r. But this is not enough. One must assume that $m, n \in A(mn)$ or simply m or $n \in A(mn)$.

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