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ON DIVISOR FUNCTION OVER $\mathbb{Z}[i]$

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. We investigate the distribution of values of the function $\tau(\alpha)$ of Gaussian integers α whose norms belong to an arithmetic progression and construct an asymptotical formula for summatory function for $\tau(\alpha)$.

1. Introduction

As usual, $\mathbb{N}, \mathbb{Z}$ denote the set of natural and integer numbers, respectively. Denote by $\mathbb{Z}[i] := \{a+bi \mid a, b \in \mathbb{Z},\ i^2 = -1\}$ the ring of the Gaussian integers. We write $N(\alpha) = a^2 + b^2$, $\mathrm{Sp}(\alpha) = 2a$ for $\alpha = a + bi \in \mathbb{Z}[i]$; $\mathbb{Z}_q$ (respectively, $\mathbb{Z}_q[i]$) is $\mathbb{Z}/q\mathbb{Z}$ (respectively, $\mathbb{Z}[i]/q\mathbb{Z}[i]$), $\mathbb{Z}_q^* = \{a \in \mathbb{Z}_q \mid (a, q) = 1\}$, $\mathbb{Z}_q^*[i] = \{\alpha \in \mathbb{Z}_q[i] \mid (\alpha, q) = 1\}$; γ is the Euler constant; for $(a, q) = 1$ we designate through $\overline{a}$ a solution of the congruence $ax \equiv 1 \pmod{q}$; the Vinogradov symbol "$\ll$" means the same as the Landau symbol "O".

Let $\tau(n)$ denote the classical divisor function, which is defined by

$$\tau(n) = \sum_{d|n} 1 \qquad (n \in \mathbb{N}),$$

Mathematics Subject Classification: 11M06, 11M37 where the sum runs over all natural divisors d of n. The Weil bound for the Kloosterman sums implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $q \ll x^{\frac{2}{3}-\varepsilon}$

$$\sum_{n\equiv \ell \pmod q} \tau(n) = \frac{x}{q}\prod_{p|q}\left(1-\frac{1}{p}\right)\times \tag{1}$$

$$\times\left(\log x + 2\gamma - 1 + 2\prod_{p|q}\left(1-\frac{1}{p}\right)^{-1}\sum_{d|q}\frac{\mu(d)\log d}{d}\right) + O\left(\frac{x^{1-\delta}}{q}\right)$$

(see C. Hooley [5], D. Heath-Brown [4]).

O. Gunjavy [2], M. Jutila [6,7], W. Banks, R. Heath-Brown, I. Shparlinski [1] considered the sums of divisors with weighted additive character

$$\sum_{n\leq x}\tau(n)e^{2\pi i\frac{hn}{q}}.$$

Similar results have been achieved for $\rho(n)$ instead of $\tau(n)$, where $\rho(n)$ is n-th Fourier coefficient of Maass wave form, or the number of the representations of n in form $n = Q(u,v)$, where Q is a binary positive quadratic form.

P. Varbanets, U. Zhanbyrbaeva [10] studied the divisor function over the ring of the Gaussian integers $\mathbb{Z}[i]$ and the following asymptotic formula was obtained

$$\sum_{\substack{\alpha\equiv\alpha_0 \pmod \beta\\ N(\alpha)\leq x}} \tau(\alpha) = C_0(\beta)\frac{x}{N(\beta)} + C_1(\beta)\frac{x}{N(\beta)} + O\left(\frac{x^{\frac{3}{5}+\varepsilon}}{N(\beta)^{\frac{2}{5}}}\right), \tag{2}$$

where

$$C_0(\beta) = \pi^2 N((\alpha_0,\beta))\overline{\varphi}\left(\frac{\beta}{(\alpha_0,\beta)}\right)N^{-1}(\beta)\tau((\alpha_0,\beta)),$$

$$C_1(\beta) = \pi^2\sum_{\delta\ |\ \beta}\left(2\gamma - 1 + 2\frac{L'(1,\chi_4)}{L(1,\chi_4)} + 2\sum_{\wp\ |\ \frac{\beta}{\delta}}\frac{\log N(\wp)}{N(\wp)-1}\right)\cdot\prod_{\wp\ |\ \frac{\beta}{\delta}}(1-N(\wp)^{-1}),$$

$\overline{\varphi}(\alpha)$ denotes the Euler function in $\mathbb{Z}[i]$.

The asymptotic formula (2) is analogous to the asymptotic formula of the Dirichet divisor problem in an arithmetic progression (1).

The geometry of the Gaussian integers is "more rich" than the geometry of the natural numbers and thus we can consider the distribution of values of the divisor function not only in an arithmetic progression, but in narrow sectorial region also. Moreover, we can study $\tau(\alpha)$ not only in an arithmetic progression $\alpha \equiv \alpha_0 \pmod{\gamma}$, but in an arithmetic progression of norms $N(\alpha) \equiv \ell \pmod{q}$. Such problem does not have an analogue in the rational case.

From (2), it follows that

$$\sum_{\substack{N(\alpha)\equiv\ell(\text{mod } q)\\ N(\alpha)\le x}} \tau(\alpha) = \sum_{\substack{\alpha_0\in\mathbb{Z}_q^*[i]\\ N(\alpha_0)\equiv\ell(\text{mod } q)}} \sum_{\substack{\alpha\equiv\alpha_0(\text{mod } q)\\ N(\alpha)\le x}} \tau(\alpha) =$$

$$= \Im(\ell,q)\left(C_0(q)\frac{x}{q^2}\log\frac{x}{q^2} + C_1(q)\frac{x}{q^2}\right) + O\left(x^{\frac{3}{5}+\varepsilon}q^{\frac{1}{5}}\right), \tag{3}$$

where $\Im(\ell,q)$ is the number of solutions of the congruence

$$N(\alpha) \equiv \ell \pmod{q}, \quad \alpha \in \mathbb{Z}_q^*[i].$$

It is well known that

$$\Im(\ell,q) = a_q q \prod_{p|q}\left(1 - \frac{\chi_4(p)}{p}\right), \quad a_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{2}, \\ 1 & \text{if } q \equiv 2 \pmod{4}, \\ 2 & \text{if } q \equiv 0 \pmod{4}. \end{cases}$$

From here it follows that the asymptotic formula (3) is non-trivial for $q \ll x^{\frac{1}{3}-\varepsilon}$.

Our aim is to expand a region of non-triviality of the asymptotic formula (3). Moreover, we obtain an asymptotic formula for

$$\sum_{N(\alpha)\le x} \tau(\alpha) e^{2\pi i N\left(\frac{\alpha_0\alpha}{\beta}\right)}.$$

Our main results are as follows.

Theorem 1. *Let $\ell, q \in \mathbb{N}$, $(\ell,q) = 1$. For $x \to \infty$ and $q \ll x^{\frac{2}{5}-\varepsilon}$ the following asymptotic formula*

$$\sum_{\substack{d\in\mathbb{Z}[i]\\ N(\alpha)\equiv\ell(\mathrm{mod}\ q)\\ N(\alpha)\le x}} \tau(\alpha) = A(x,q) + O\left(x^{\frac{1}{2}+\varepsilon}q^{\frac{1}{4}}\right)$$

holds, where

$$A(x,q) = \frac{\pi\varepsilon_q x}{\varphi(q)} \prod_{\substack{p|q\\ p\equiv 1(4)}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p|q\\ p\equiv 3(4)}} \left(1-\frac{1}{p^2}\right) \times$$

$$\times \left\{\log x + 2\left(\gamma + \frac{L'(1,\chi_4)}{L(1,\chi_4)}\right) + 2\prod_{\substack{p|q\\ p\equiv 1(4)}} \frac{\log p}{p-1} + 2\prod_{\substack{p|q\\ p\equiv 3(4)}} \frac{\log p}{p^2-1} + \varepsilon'_q \log 2\right\},$$

$$\varepsilon_q = \begin{cases} 1 & \text{if } q\equiv 1\ (2),\\ \frac{1}{2} & \text{if } q\equiv 0\ (2), \end{cases} \qquad \varepsilon'_q = \begin{cases} 0 & \text{if } q\equiv 1\ (2),\\ 0 & \text{if } q\equiv 0\ (2). \end{cases}$$

Theorem 2. *Let S_φ denote a sector $\varphi_1 \le \arg\alpha \le \varphi_2$. Then for $\ell, q \in \in \mathbb{N}$, $(\ell,q)=1$ if $\varphi \gg x^{-\frac{1}{2}+\varepsilon}\cdot q^{\frac{5}{4}}$, we have*

$$\sum_{\substack{\alpha\in S_\varphi\\ N(\alpha)\equiv\ell(\mathrm{mod}\ q)\\ N(\alpha)\le x}} \tau(\alpha) = \frac{2\varphi}{\pi}A(x,q) + O\left(x^{\frac{1}{2}+\varepsilon}q^{\frac{1}{4}}\right) + O\left(\varphi\frac{x^{1-\varepsilon}}{q}\right).$$

Theorem 3. *Let α_0, β be Gaussian integers, $(\alpha_0,\beta)=1$. Then for $x\to\infty$ and $N(\beta) \ll x^{\frac{1}{4}-\varepsilon}$*

$$\sum_{N(\alpha)\le x} \tau(\alpha) e^{2\pi i N\left(\frac{\alpha_0\alpha}{\beta}\right)} = C_1(\beta)\frac{x\log x}{N(\beta)} + C_2(\beta)x + O\left(x^{\frac{3}{4}+\varepsilon}\right) + O\left(x^{\frac{1}{2}+\varepsilon}N(\beta)\right),$$

where $C_i(\beta)$ are computable constants, $N(\beta)^{-\varepsilon} \ll C_i(\beta) \ll N(\beta)^{\varepsilon}$, $i=1,2$.

2. Some lemmas

Let $q \geq 1$ be a natural number. For every Dirichlet character $\chi_q \pmod q$ we denote the function $\chi(q;*)$ on $\mathbb{Z}[i]$:

$$\chi(q,\alpha) = \chi_q(N(\alpha)), \quad \alpha \in \mathbb{Z}[i].$$

Obviously, $\chi(q,\alpha)$ is a character of the residue group modulo q in $\mathbb{Z}[i]$. The following lemma holds.

Lemma 1. *Only the real characters χ_q can generate principal character $\chi(q,*)$. If q is odd or $q = 2q_1$, q_1 is odd, then a nonprincipal character χ_q generates a nonprincipal character $\chi(q,*)$. If $q = 2^k q_1$, $k \geq 2$, $(q_1, 2) = 1$, then a real character χ_q generates a principal character $\chi(q,*)$ only in two cases*

$$\chi_q = \chi_4 \chi_{0q_1} \quad \textit{or} \quad \chi_q = \chi'_8 \chi_{0q_1},$$

where χ_4 is the nonprincipal character modulo 4, χ'_8 is the nonprincipal character modulo 8, defined by equalities

$$\chi'_8(1) = \chi'_8(5) = 1, \quad \chi'_8(-1) = \chi'_8(-5) = -1,$$

and χ_{0q_1} is a principal character modulo q_1.

Let Ξ be the Hecke character of second kind

$$\Xi(\alpha) = e^{4mi\arg\alpha} \chi(q,\alpha) \quad (\alpha \in \mathbb{Z}[i],\ m \in \mathbb{Z}).$$

If $m = 0$, $\chi(q,*)$ is a principal character mod q, we denote $\Xi = \Xi_0$ and call it the principal Hecke character.

For the Hecke zeta-function

$$Z(s,\Xi) = \sum_{\alpha \in \mathbb{Z}[i]} \frac{\Xi(\alpha)}{N(\alpha)^s}, \quad (\Re\, s > 1)$$

the assertion holds.

Lemma 2. *If $\Xi \neq \Xi_0$, then $Z(s,\Xi)$ is an entire function. If $\Xi = \Xi_0$, then $Z(s,\Xi_0)$ is analytic in the complex plane except at $s = 1$, where it has a simple pole with residue*

$$\pi \cdot \varepsilon_q \prod_{\substack{p|q \\ p\equiv 1(4)}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p|q \\ p\equiv 3(4)}} \left(1-\frac{1}{p^2}\right), \quad \varepsilon_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod 2, \\ \frac{1}{2} & \text{if } q \equiv 0 \pmod 2. \end{cases}$$

Moreover, for a primitive character $\chi(q,*)$ *the functional equation*

$$Z(s,\Xi) = \kappa(\chi_q)\left(\frac{q}{\pi}\right)^{1-2s} \frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)} Z(1-s,\overline{\Xi})$$

holds, where $\kappa(\chi_q) = q^{-1} \sum_{\alpha\in\mathbb{Z}_q^*[i]} \chi_q(N(\alpha))e^{\pi i \mathrm{Sp}\left(\frac{\alpha}{q}\right)}$

(here $\overline{\Xi}(\alpha) = \chi_q\overline{(N(\alpha))}e^{-4mi\arg\,\alpha}$*).*

The assertion of these lemmas is well-known.

Lemma 3 (see [11], Theorem 3). *Let* $\alpha,\beta \in \mathbb{Z}[i]$, $\ell, q \in \mathbb{N}$, $(\ell,q)=1$, $(\alpha,\beta,q)=1$. *Then*

$$K(\alpha,\beta;\ell,q) = \sum_{u,v\in\mathbb{Z}_q[i]} e^{\pi i \mathrm{Sp}\left(\frac{\alpha u+\beta v}{q}\right)} = O\left(q^{\frac{3}{2}+\varepsilon}\right).$$

The constant on the right hand side implied by O term does not depend on α,β,ℓ,q.

Lemma 4 (see [3], formula (12)). *Let* $D \subset \mathbb{Z}[i]$ *and let*

$$S_\varphi := \left\{\alpha \in \mathbb{Z}[i] \mid 0 \le \varphi_1 \le \arg\,\alpha \le \varphi_2 \le \frac{\pi}{2}, \quad \varphi_2-\varphi_1=\varphi\right\}.$$

Then for any function $a(\alpha)$, $\alpha \in D$ *and integer* $M_0 > 1$ *we have*

$$\sum_{\alpha\in D\cap S_\varphi} a(\alpha) =$$

$$= \frac{2\varphi}{\pi}\sum_{\alpha\in D} a(\alpha) + O\left(\frac{1}{M_0}\sum_{\alpha\in D}|a(\alpha)|\right) + O\left(\varphi\sum_{m=1}^{M_0}\left|\sum_{\alpha\in D} a(\alpha)e^{4mi\arg\alpha}\right|\right).$$

3. Proof of Theorems 1 and 2

For every Dirichlet character χ_q and $m = 0$ we have

$$Z^2(s,\Xi) = \sum_{\alpha} \frac{\Xi(\alpha)\tau(\alpha)}{N(\alpha)^s}, \qquad (\Re\ s > 1).$$

Hence,

$$\sum_{\substack{\alpha \\ N(\alpha)\equiv \ell (\mathrm{mod}\ \ q)}} \frac{\tau(\alpha)}{N(\alpha)^s} = \frac{1}{\varphi(q)} \sum_{\chi_q} \overline{\chi}_q(\ell) Z^2(s,\Xi). \tag{4}$$

Now, taking into account that in (4) $N(\alpha)$ runs on an arithmetic progression, we obtain for $T > 1$, $c = 1 + \frac{1}{\log x}$:

$$\sum_{\substack{N(\alpha)\equiv \ell (q) \\ N(\alpha)\le x}} \tau(\alpha) =$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{1}{\varphi(q)} \sum_{\chi_q} \overline{\chi}_q(\ell) Z^2(s,\Xi) - \frac{c(\ell)}{l^s} \right) \frac{x^s}{s} ds + O\left(\frac{x \log^2 x}{Tq} \right) + O(x^{\varepsilon}),$$

where $c(\ell) = \sum\limits_{N(\alpha)=\ell} \tau(\alpha) = \sum\limits_{mn=\ell} r(m) r(n) = O(\ell^{\varepsilon})$, $r(n)$ is the number of solutions of the equation $n = u^2 + v^2$, $u, v \in \mathbb{Z}$, the constants implied by O terms depend only on ε, $\varepsilon > 0$ is arbitrarily small.

Shifting the line of integration to $\Re\ s = -\delta$ $(0 < \delta < \frac{1}{4})$ we have

$$\sum_{\substack{N(\alpha)\equiv \ell (q) \\ N(\alpha)\le x}} \tau(\alpha) = \sum_{s=0,1} \mathrm{res} \left(\left[\frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(\bar{\ell}) Z^2(s,\Xi) - \frac{c(\ell)}{\ell^s} \right] \frac{x^s}{s} \right) +$$

$$+ \frac{1}{2\pi i} \int_{-\delta-iT}^{-\delta+iT} \left(\frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(\bar{\ell}) Z^2(s,\Xi) - \frac{c(\ell)}{\ell^s} \right) \frac{x^s}{s} ds + O\left(\frac{x \log^2 x}{Tq} \right) + O(x^{\varepsilon}) =$$

$$= \frac{1}{\varphi(q)} \operatorname*{res}_{s=1} \left(Z^2(s,\Xi) \frac{x^s}{s} \right) + \frac{1}{\varphi(q)} \sum_{\chi_q} \chi(\bar{\ell}) Z^2(0,\Xi) + \tag{5}$$

$$+\frac{1}{\varphi(q)}\sum_{\alpha}\sum_{q_1|q}\sum_{\chi_{q_1}}{}^{*}\chi_{q_1}(\bar{\ell})\kappa^2(\chi_{q_1})\times$$

$$\times\int\limits_{-\delta-iT}^{-\delta+iT}\left(\frac{q_1}{\pi}\right)^{2-4s}\frac{\Gamma^2(1-s)}{\Gamma^2(s)}\cdot\frac{\tau(\alpha)}{N(\alpha)^{1-s}}\chi_{q_1}(\overline{N(\alpha)})\frac{x^s}{s}ds+$$

$$+O\left(\max_{-\delta\le\sigma\le 1+c}\left(\left|\frac{1}{\varphi(q)}\sum_{\chi_q}\chi_q(\bar{\ell})Z^2(s,\Xi)-\frac{c(\ell)}{\ell^s}\right|\cdot\frac{x^\sigma}{T}\right)\right)+$$

$$+O\left(\frac{x\log^2 x}{Tq}\right)+O(x^\varepsilon).$$

Since $Z(s,\Xi_0)=4\zeta(s)L(s,\chi_4)\prod\limits_{\wp|q}(1-N(\wp)^{-s})$, where $\wp$ denotes a general Gaussian prime number, we have

$$\text{(6)}\qquad \operatorname*{res}_{s=1}\left(\frac{1}{\varphi(q)}Z^2(s,\Xi_0)\frac{x^s}{s}\right)=\frac{\pi\varepsilon_q x}{\varphi(q)}\prod_{\substack{p|q\\ p\equiv 1(4)}}\left(1-\frac{1}{p}\right)^2\prod_{\substack{p|q\\ p\equiv 3(4)}}\left(1-\frac{1}{p^2}\right)\times$$

$$\times\left\{\log x+2\left(\gamma+\frac{L'(1,\chi_4)}{L(1,\chi_4)}\right)+2\sum_{\substack{p|q\\ p\equiv 1(4)}}\frac{\log p}{p-1}+2\sum_{\substack{p|q\\ p\equiv 3(4)}}\frac{\log p}{p^2-1}+\varepsilon_q'\log 2\right\}=$$

$$=A(x,q),$$

say, where

$$\varepsilon_q'=\begin{cases}1 & \text{if } q\equiv 0 \pmod 2,\\ 0 & \text{if } q\equiv 1 \pmod 2.\end{cases}$$

Moreover,

$$\text{(7)}\qquad \operatorname*{res}_{s=0}\left[\left(\frac{1}{\varphi(q)}\sum_{\chi_q}Z^2(s,\Xi)-\frac{c(\ell)}{\ell^s}\right)\frac{x^s}{s}\right]=O(x^\varepsilon).$$

Now, by definition of $\kappa(\chi_q)$, we infer

$$\chi_{q_1}(\bar{\ell})\sum_{q_1|q}\sum_{\chi_{q_1}}{}^{*}\chi_{q_1}(\overline{N(\alpha)})\kappa^2(q_1)q_1^{2-4s}=$$

$$= \sum_{q_1|q} q_1^{-4s} {\sum_{\chi_{q_1}}}^* \chi_{q_1}(\overline{\ell N(\alpha)}) \left(\sum_{\beta\in\mathbb{Z}_q[i]} e^{\pi i \mathrm{Sp}\left(\frac{\beta}{q_1}\right)} \right)^2 =$$

$$= \sum_{q_1|q} q_1^{-4s} \sum_{\beta_1,\beta_2\in\mathbb{Z}_{q_1}[i]} e^{\pi i \mathrm{Sp}\left(\frac{\beta_1+\beta_2}{q}\right)} {\sum_{\chi_{q_1}}}^* \chi_{q_1}(N(\beta_1\beta_2)\overline{N(\alpha)}\cdot\overline{\ell}) =$$

$$(8) \qquad = \sum_{q_1|q} q_1^{-4s} \cdot \frac{1}{q_1^2} \sum_{\beta_1,\beta_2\in\mathbb{Z}_q[i]} e^{\pi i \mathrm{Sp}\left(\frac{\beta_1 q_1+\beta_2 q_2}{q}\right)} \sum_{\chi_q} \chi_q(N(\beta_1\beta_2\overline{\alpha}\overline{\ell})) =$$

$$= \varphi(q) \sum_{q_1|q} q_1^{-4s} \sum_{\substack{\beta_1,\beta_2\in\mathbb{Z}_{q_1}[i] \\ N(\beta_1\beta_2)\equiv \ell N(\alpha)\ (q_1)}} e^{\pi i \mathrm{Sp}\left(\frac{\beta_1+\beta_2}{q_1}\right)} = \varphi(q)\sum_{q_1|q} q_1^{-4s} E_{q_1}(\ell N(\alpha)),$$

where

$$q_2 = \frac{q}{q_1}, \qquad E_{q_1}(h) = \sum_{\substack{\beta_1,\beta_2\in\mathbb{Z}_{q_1}[i] \\ N(\beta_1\beta_2)\equiv h \pmod{q_1}}} e^{\pi i \mathrm{Sp}\left(\frac{\beta_1+\beta_2}{q}\right)}.$$

Hence, from (5)-(8) we get

$$(9) \qquad \sum_{\substack{N(\alpha)\equiv \ell \pmod{q} \\ N(\alpha)\le x}} \tau(\alpha) =$$

$$= A(x,q) + \frac{1}{\pi^2} \sum_{\alpha} \sum_{q_1|q} E_{q_1}(\ell N(\alpha)) \frac{\tau(\alpha)}{N(\alpha)} \cdot \frac{1}{2\pi i} \int_{-\delta-iT}^{-\delta+iT} \frac{\Gamma^2(1-s)}{\Gamma(s)} \frac{y^s}{s} ds +$$

$$+ O\left(\max_{-\delta\le\sigma\le c} \left(\left| \frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(\overline{\ell}) Z^2(s,\Xi) - \frac{c(\ell)}{\ell^s} \right| \cdot \frac{x^\sigma}{T} \right) \right) +$$

$$+ O\left(\frac{x\log^2 x}{Tq} \right) + O\left(x^\varepsilon\right),$$

where $y = \frac{\pi^4 x N(\alpha)}{q_1^4}$.

Let

$$I(\alpha) = \frac{1}{2\pi i} \int_{-\delta-iT}^{-\delta+iT} \frac{\Gamma^2(1-s)}{\Gamma^2(s)} \cdot \frac{y^s}{s} ds.$$

The integral $I(\alpha)$ can be computed by standard way using the estimates of exponential integrals by the first and second derivative tests. We have

$$I(\alpha) = \tag{10}$$

$$= \begin{cases} \frac{y^{\frac{3}{8}}}{4\sqrt{2\pi}} e^{-\frac{\pi i}{4}} e^{-iy^{\frac{1}{4}}} + O\left(y^{\frac{1}{4}} \min\left(\left(\log \frac{x}{N(\alpha)}\right)^{-1}, T^{\frac{1}{2}}\right)\right) + \\ \quad +O\left(T^{1+4\delta}(q^4x^{-1})^{\delta}N^{-\delta}(\alpha)\right) & \text{if } N(\alpha) \le X, \\ y^{-\delta}\log T \min\left(T^{\frac{3}{2}+4\delta}, T^{1+4\delta}\left(\log\frac{N(\alpha)}{x}\right)^{-1}\right) & \text{if } N(\alpha) > X, \end{cases}$$

where $X = \left(\frac{4}{\pi}\right)^4 \frac{T^4q^4}{x}$.

Further, by Lemma 3, we have

$$\frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(\bar{\ell}) Z^2(s, \Xi) - \frac{c(\ell)}{\ell^s} \ll \tag{11}$$

$$\ll \begin{cases} \sum\limits_{\substack{N(\alpha)\equiv \ell(q) \\ N(\alpha)>q}} \frac{\tau(\alpha)}{N(\alpha)^{1+\varepsilon}} \ll q^{-1+\varepsilon} & \text{if } s = 1+\varepsilon, \\ \sum\limits_{\alpha} \frac{|E_q(\ell N(\alpha))|}{N(\alpha)^{1+\varepsilon}} \left|\frac{\Gamma^2(1-s)}{\Gamma^2(s)}\right| \ll q^{\frac{3}{2}+\varepsilon}(|t|+1)^{2(1+2\delta)} & \text{if } s = -\delta + iT. \end{cases}$$

Hence, by the Phragmen-Lindelöf theorem we obtain

$$\max_{\substack{-\delta\le\sigma\le 1+\varepsilon \\ |\Im s|=T}} \left| \frac{1}{\varphi(q)} \sum_{\chi_q} \chi_q(\bar{\ell}) Z^2(s,\Xi) - \frac{c(\ell)}{\ell^3} \right| \cdot \frac{x^\sigma}{T} \ll q^{\frac{3}{2}+5\delta} x^{-\sigma} T^{1+4\delta} + \frac{x^{1+\varepsilon}}{Tq}. \tag{12}$$

Thus, by combining (5), (6), (9)-(12) and taking $\delta = \frac{\varepsilon}{5}$, $T = \frac{x^{\frac{1}{2}}}{q^{\frac{5}{4}}}$, by simple computations we obtain the assertion of Theorem 1

$$\sum_{\substack{N(\alpha)\equiv\ell(\mathrm{mod}\ q) \\ N(\alpha)\le x}} \tau(\alpha) = A(x,q) + x^{\frac{1}{2}+\varepsilon} q^{\frac{1}{4}},$$

where $A(x,q)$ is defined in (6).

Let $m \neq 0$. Denote

$$\tau^{(m)}(\alpha) = \tau(\alpha) e^{4mi\arg\alpha}$$

and consider the Hecke zeta-function with character

$$\Xi(\alpha) = \chi_q(N(\alpha))e^{4mi\arg\alpha}.$$

We have

$$\sum_{\substack{\alpha \\ N(\alpha)\equiv\ell(\mathrm{mod}\ q)}} \frac{\tau^{(m)}(\alpha)}{N(\alpha)^s} = \frac{1}{\varphi(q)}\sum_{\chi_q}\chi_q(\bar{\ell})Z^2(s,\Xi).$$

Thus, repeating the argument of the proof of Theorem 1, we obtain

$$\sum_{\substack{N(\alpha)\equiv\ell(\mathrm{mod}\ q) \\ N(\alpha)\le x}} \tau^{(m)}(\alpha) = \tag{13}$$

$$= \frac{1}{\pi^2}\sum_{\alpha}\sum_{q_1|q}E_{q_1}(\ell N(\alpha))\frac{\tau^{(m)}(\alpha)}{N(\alpha)}\cdot\frac{1}{2\pi i}\int\limits_{-\delta-iT}^{-\delta+iT}\frac{\Gamma^2(2|m|+1-s)}{\Gamma^2(2|m|+s)}\cdot\frac{y^s}{y}ds+$$

$$+O\left(\max_{-\delta\le\sigma\le c}\left(\left|\frac{1}{\varphi(q)}\sum_{\chi_q}\chi_q(\bar{\ell})Z^2(s,\Xi)-\ell^{-s}\sum_{n(\alpha)=\ell}\tau^{(m)}(\alpha)\right|\frac{x^\sigma}{T}\right)\right)+$$

$$+O\left(\frac{x^{1+\varepsilon}}{Tq}\right)+O\left(x^\varepsilon\right).$$

The integrals

$$I_m(\alpha) = \frac{1}{2\pi i}\int\limits_{-\delta-iT}^{-\delta+iT}\frac{\Gamma^2(2|m|+1-s)}{\Gamma^2(2|m|+s)}\cdot\frac{x^s}{s}ds$$

can be computed similarly as the integrals $I(\alpha)$ (in the estimates we write $(T^2+m^2)^{\frac{1}{2}}$ instead of T).

It gives for $m\neq 0$

$$\sum_{\substack{N(\alpha)\equiv\ell(\mathrm{mod}\ q) \\ N(\alpha)\le x}} \tau^{(m)}(\alpha) = q^{\frac{3}{2}+\varepsilon}|m|^{2+\varepsilon}+x^{\frac{1}{2}+\varepsilon}q^{\frac{1}{4}}. \tag{14}$$

Now we apply the formula of Lemma 4. We take

$$D := \{\alpha\in\mathbb{Z}[i]:\quad N(\alpha)\equiv\ell\ (\mathrm{mod}\ q),\quad N(\alpha)\le x\},$$

$a(\alpha)=\tau(\alpha)$, $M_0 = x^{\frac{1}{2}}-2\varepsilon q^{-\frac{5}{4}}$.

Hence, Theorem 1 together with (14) proves the assertion of Theorem 2.

4. Proof of Theorem 3

At first we formulate the theorem which is an analogue of theorem from [2] (for the divisor function $\tau(n)$).

Theorem A. *Let $\Delta = 1+\delta$, $0<\delta\ll 1$ and let the following conditions hold:*

"$g(x)$, $f(x)$ are defined on $[N,\Delta N]$ and $g'(x)$, $f''(x)$ are monotone functions and moreover $g(x)\ll g(N)$, $g'(N)\ll \frac{1}{N}g(N)$, $\frac{1}{N\delta}\ll |f'(N)|\ll$ $\ll |f'(x)|\ll |f'(N)|$, $|f'(N)|\ll |f'(x)+2xf''(x)|\ll |f'(N)|$, $|f'''(x)|\ll$ $\ll \frac{1}{N^2}|f'(N)|$ for $x\in[N,\Delta N]$".

Furthermore, let $\varphi(x)$ be the inverse function of $x\cdot(f'(x))^2$. Then, for an arbitrary small positive ε we have

$$\sum_{n\sim N} r(n)g(n)e^{2\pi i f(n)} =$$

$$=\sum_{m\sim N(f'(N))^2} r(m)g(\varphi(m))\sqrt{|\varphi'(m)|}e^{2\pi i(f(\varphi(m))-2\sqrt{m\varphi(m)})}+$$

$$+O\left(N^{\varepsilon}\left|\frac{g(N)}{f'(N)}\right|\right)+O(N^{\varepsilon}|g(N)|)+$$

$$+O\left(|g(N)|N^{\frac{1}{2}+\varepsilon}\min\left(\sqrt{|f'(N)|},\frac{1}{\sqrt{|f'(N)|}}\right)\right),$$

where

$$r(n)=\sum_{\substack{u,v\in\mathbb{Z}\\ u^2+v^2=n}}1,\qquad \omega_f=\begin{cases}1 & \text{if}\quad f'(N)(f'(n)+2Nf''(N))>0,\\ -i & \text{if}\quad f'(N)>0,\quad f'(N)+2Nf''(N)<0,\\ i & \text{if}\quad f'(N)<0,\quad f'(N)+2Nf''(N)>0,\end{cases}$$

$k\sim K$ denotes that k runs over the interval $[K,\Delta K]$.

This theorem can be proved by the application of van der Corput transformation of the exponential sum (see [9]) and the truncated formula for the error term $P(x)$ in the circle problem

$$P(x) = \sum_{n\le x} r(n) - \pi x =$$

$$= -\frac{x^{\frac{1}{4}}}{\pi} \sum_{n\le M} r(n) n^{-\frac{3}{4}} \cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right) + O(x^{\varepsilon}) + O\left(x^{\frac{1}{2}+\varepsilon} M^{-\frac{1}{2}}\right)$$

(here $1 \ll M \ll x^A$, $A > 0$ is any fixed constant).

Corollary. *Let $a, q \in N$, $(a,q) = 1$. Then the asymptotic formula*

$$\sum_{n\le x} r(n) e^{2\pi i \frac{an}{q}} = \frac{\pi x}{q} + O\left(x^{\frac{1}{2}+\varepsilon}\right) + O(qx^{\varepsilon}) \tag{15}$$

holds for $q \ll x^{\frac{1}{3}-\varepsilon}$ and an arbitrary small $\varepsilon > 0$.

Proof. The function $f(x) = \frac{ax}{q}$ satisfies all conditions of the Theorem A and, hence, for any $N \le x$

$$\sum_{n\sim N} r(n) e^{2\pi i \frac{an}{q}} = \sum_{m\sim N\cdot\frac{a^2}{q^2}} r(m) \cdot \frac{q}{a} e^{-2\pi i \frac{q}{a} m} + O\left(x^{\varepsilon} \cdot \frac{q}{a}\right) + O\left(x^{\frac{1}{2}+\varepsilon} \left(\frac{a}{q}\right)^{\frac{1}{2}}\right) =$$

$$= \frac{q}{a} \sum_{n_1\sim N_1} r(n_1) e^{2\pi i \frac{bn_1}{a}} + O\left(x^{\varepsilon}\frac{q}{a}\right) + \left(x^{\frac{1}{2}+\varepsilon}\left(\frac{a}{q}\right)^{\frac{1}{2}}\right),$$

where $N_1 = \dfrac{a^2}{q^2} N$, $1 \le b < a$, $b \equiv -q \pmod{a}$. We apply Theorem A again and then at most through $\log q$ iterations obtain

$$\sum_{n\sim N} r(n) e^{2\pi i \frac{an}{q}} = q \sum_{n\sim \frac{N}{q^2}} r(n) + O\left(x^{\varepsilon}\left(\frac{q}{a} + a\right)\right). \tag{16}$$

Hence,

$$\sum_{n\le x} r(n) e^{2\pi i \frac{an}{q}} = \frac{\pi x}{q} + O\left(\left(x^{\frac{1}{2}} + q\right) x^{\varepsilon}\right).$$

We are now in a position to prove Theorem 3. We set $N(\alpha_0)=a$, $N(\beta)=$ $=q$. Then we have

$$\sum_{N(\alpha)\le x}\tau(\alpha)e^{2\pi i N\left(\frac{\alpha\alpha_0}{\beta}\right)}=\sum_{\substack{\alpha_1,\alpha_2\in\mathbb{Z}[i]\\ N(\alpha_1\alpha_2)\le x}}e^{2\pi i\frac{aN(\alpha_1)N(\alpha_2)}{q}}=\sum_{mn\le x}r(m)r(n)e^{2\pi i\frac{amn}{q}}=$$

$$=2\sum_{d|q}\sum_{\substack{(m,q)=d\\ m\le x^{\frac12}}}r(m)\sum_{n\le\frac{x}{m}}r(n)e^{2\pi i\frac{amn}{q}}-\sum_{d|q}\sum_{\substack{m\le x^{\frac12}\\ (m,q)=1}}\sum_{n\le x^{\frac12}}r(m)r(n)e^{2\pi i\frac{amn}{q}}=$$

$$=2\sum\nolimits_1-\sum\nolimits_2, \tag{17}$$

say.

For $(m,q)=d$ we set $m=m_1d$, $q=q_1d$, $(m_1,q_1)=1$. So, from the Corollary we deduce

$$\sum\nolimits_1=\sum_{d|q}\sum_{m_1\le\frac{x^{\frac12}}{d}}r(m_1d)\sum_{n\le\frac{x}{m_1d}}r(n)e^{2\pi i\frac{am_1n}{q_1}}=$$

$$=\sum_{d|q}\left[\frac{\pi x}{dq_1}\sum_{m_1\le\frac{x^{\frac12}}{d}}\frac{r(m_1d)}{m_1}+O\left(\left(\sum_{\substack{m_1\le\frac{x^{\frac12}}{d}\\ (m_1,q_1)=1}}\left(\frac{x}{m_1d}\right)^{\frac12+\varepsilon}+q_1\left(\frac{x}{m_1d}\right)^{\varepsilon}\right)\right)\right]=$$

$$=\sum_{d|q}\left\{\left[\frac{\pi x}{q}\operatorname*{res}_{s=1}(\zeta(s)L(s,\chi_4))\sum_{d_1|d}\chi_4(d_1)\prod_{p|\frac{d}{d_1}}\left(1-\frac{\chi_4(d_1)}{p^s}\right)\frac{x^{s-1}}{s-1}\right]+\right.$$

$$\left.+O\left(\left(\frac{x^{\frac12}}{d}\right)^{\frac13+\varepsilon}\right)+O\left(x^{\frac34+\varepsilon}d^{-1-\varepsilon}\right)+O\left(qx^{\frac12+\varepsilon}\right)\right\}= \tag{18}$$

$$=A_1(q)\frac{x\log x}{q}+A_2(q)x+O\left(x^{\frac12+\varepsilon}q\right)+O\left(x^{\frac34+\varepsilon}\right),$$

where $A_1(q)$, $A_2(q)$ are computable constants, $q^{-\varepsilon} \ll A_i(q) \ll q^{\varepsilon}$, $i = 1, 2$. As before, we have

$$\sum\nolimits_2 = \sum_{d|q} \sum_{\substack{m_1 \le \frac{x^{\frac{1}{2}}}{d} \\ (m_1, q_1)=1}} r(m) \left(\frac{\pi x^{\frac{1}{2}}}{q_1} + O\left(x^{\frac{1}{4}+\varepsilon}\right) + O\left(q_1 x^{\varepsilon}\right) \right) = \tag{19}$$
$$= B(q)\frac{x}{q} + O\left(x^{\frac{3}{4}+\varepsilon}\right) + O\left(q_1 x^{\varepsilon}\right),$$

where $q^{-\varepsilon} \ll B(q) \ll q^{\varepsilon}$. Collecting together our estimates (17)-(19) we derive the desired result of Theorem 3.

5. Concluding remark

The result of Theorems 1 and 2 can be extended to the case of an arbitrary imaginary quadratic field. It would be interesting to improve the bounds of a nontrivial estimate for $q > x^{\frac{1}{4}}$ in Theorem 3.

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