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FUNCTIONAL EQUATIONS INVOLVING QUASI–ARITHMETIC MEANS AND THEIR GAUSS COMPOSITION

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This article is dedicated
to the memory of Professor M.V. Subbarao

Abstract. Let $I \subset \mathbb{R}$ be a nonvoid open interval. Let $M_i : I^2 \longrightarrow I$ ($i = 1, 2, 3$) be weighted quasi-arithmetic means with the property

$$M_3 = M_1 \otimes M_2\,,$$

where $\otimes$ denote the Gauss composition of M_1 and M_2. We consider the following two functional equations for the unknown $f : I \longrightarrow \mathbb{R}$:

$$(1) \qquad f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \quad (x, y \in I),$$

$$(2) \qquad 2f(M_3(x,y)) = f(x) + f(y) \quad (x, y \in I).$$

It is known, that all solutions of (2) are solutions of (1), too. We give a complete characterization for the means M_i ($i = 1, 2, 3$) so that arbitrary solutions of (1) also satisfy (2).

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1. Introduction

Let $I \subset \mathbb{R}$ be a non-void open interval. A function $M : I^2 \to I$ is called a *strict mean* on I if it is continuous on I^2 and, for all $x, y \in I$ with $x \neq y$,

$$\min\{x, y\} < M(x, y) < \max\{x, y\}.$$

It is obvious that $M(x, x) = x$ for all $x \in I$. Let $M_i : I^2 \to I$ $(i = 1, 2)$ be strict means on I and $x, y \in I$. Consider the sequences (x_n) and (y_n) defined by the Gauss iteration in the following way:

$$\begin{aligned} x_1 &:= x, & y_1 &:= y, \\ x_{n+1} &:= M_1(x_n, y_n), & y_{n+1} &:= M_2(x_n, y_n) \quad (n \in \mathbb{N}). \end{aligned}$$

It is known (see [3], [8]) that the sequence of the intervals

$$I_n(x, y) := [\min\{x_n, y_n\}, \max\{x_n, y_n\}] \quad (n \in \mathbb{N})$$

is decreasing, and the intersection of these intervals is a singleton whose unique element is denoted by $M_1 \otimes M_2(x, y)$. Then the function $M_1 \otimes M_2 : I^2 \to I$ so defined is a strict mean on I, and the *invariance equation*

$$(IE) \qquad M_1 \otimes M_2(M_1(x, y), M_2(x, y)) = M_1 \otimes M_2(x, y)$$

holds for all $x, y \in I$ ([3], [8], [14], [16]). The strict mean $M_1 \otimes M_2$ constructed in this way is called the *Gauss composition* of the strict means M_1 and M_2.

In this paper we discuss the following problem of [7]. Let M_1 and M_2 be two strict means on I and let $f : I \to \mathbb{R}$ be a function such that functional equation

$$(1) \qquad f(M_1(x, y)) + f(M_2(x, y)) = f(x) + f(y)$$

holds for all $x, y \in I$. Our main purpose is to find further conditions on the means M_1 and M_2 under which the equation

$$(2) \qquad 2f(M_1 \otimes M_2(x, y)) = f(x) + f(y)$$

is equivalent to (1). The invariance equation (IE) immediately implies that if $f : I \to \mathbb{R}$ satisfies (2) for all $x, y \in I$, then f is also a solution of (1). Indeed, due to (IE), the repeated application of (2) yields

$$\begin{aligned} f(x) + f(y) &= 2f(M_1 \otimes M_2(x, y)) = \\ &= 2f(M_1 \otimes M_2(M_1(x, y), M_2(x, y))) = f(M_1(x, y)) + f(M_2(x, y)). \end{aligned}$$

Therefore the basic problem is to find conditions on the means involved so that arbitrary solutions of (1) also satisfy (2).

In Section 2 we consider the case when the two means are weighted arithmetic means. The case when M_1, M_2 and $M_1 \otimes M_2$ are weighted quasi-arithmetic means is considered in Section 3.

2. Weighted arithmetic means

In the following we consider the original problem in the case $M_1(u,v) := := pu + (1-p)v$ and $M_2(u,v) := qu + (1-q)v$ $(u, v \in J; 0 < p < 1, 0 < q < 1$, where $J \subset \mathbb{R}$ is a non-void open interval). Then by the invariance equation (IE) we have

$$(3) \qquad M_1 \otimes M_2(u,v) = \frac{q}{1-p+q}u + \frac{1-p}{1-p+q}v \quad (u, v \in J),$$

i.e. M_1, M_2 and $M_1 \otimes M_2$ are weighted arithmetic means on J.

Theorem 1 (Maksa [13]). *Let $0 < p < 1$, $0 < q < 1$ with $p + q \neq 1$. If $g : J \to \mathbb{R}$ is a solution of the functional equation*

$$(4) \qquad g(pu + (1-p)v) + g(qu + (1-q)v) = g(u) + g(v) \quad (u, v \in J),$$

then g is a constant function on J.

Proof. In [13] this theorem was presented by Maksa without proof. Now, we give here the proof of Maksa.

By a theorem of Páles [15] (see Daróczy-Maksa [6]) there exists a unique $\bar{g} : \mathbb{R} \to \mathbb{R}$, such that

$$\bar{g}(pu + (1-p)v) + \bar{g}(qu + (1-q)v) = \bar{g}(u) + \bar{g}(v)$$

for all $u, v \in \mathbb{R}$ and

$$\bar{g}(u) = g(u) \quad \text{if} \quad u \in J.$$

By a theorem of Székelyhidi [17] there exist $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 0, 1, 2)$ k-additive and symmetric functions such that

$$\bar{g}(u) = A_2(u,u) + A_1(u) + A_0 \quad (u \in \mathbb{R}).$$

If we insert this previous representation of $\bar{g}$ into the extension of the original equation (4), replacing u and v by ru and rv respectively, where r is an arbitrary rational number and $u, v \in \mathbb{R}$, then we obtain

$$\begin{aligned} rA_2\,(pu + (1-p)\,v, pu + (1-p)\,v) + A_1\,(pu + (1-p)\,v) + \\ + rA_2\,(qu + (1-q)\,v, qu + (1-q)\,v) + A_1\,(qu + (1-q)\,v) = \\ = rA_2\,(u,u) + A_1\,(u) + rA_2\,(v,v) + A_1\,(v)\,. \end{aligned}$$

Whence taking the limit $r \to 0$ we get the relation

$$A_1\,(pu + (1-p)\,v) + A_1\,(qu + (1-q)\,v) = A_1\,(u) + A_1\,(v)$$

valid for all $u, v \in \mathbb{R}$, i.e.

$$A_1\,((p+q-1)\,(u-v)) = 0.$$

In this case $p + q \neq 1$, therefore we have $A_1\,(u) = 0$ for all $u \in \mathbb{R}$. Now we prove, that A_2 is zero for all $u, v \in \mathbb{R}$, i.e. $\bar{g}$ and g are constant functions. For A_2 we have

$$\begin{aligned} A_2\,(pu + (1-p)\,v, pu + (1-p)\,v) + \\ + A_2\,(qu + (1-q)\,v, qu + (1-q)\,v) = A_2\,(u,u) + A_2\,(v,v) \end{aligned} \tag{5}$$

for all $u, v \in \mathbb{R}$. From (5) with $u = 0$ (or $v = 0$) we obtain

$$A_2\,((1-p)\,v, (1-p)\,v) + A_2\,((1-q)\,v, (1-q)\,v) = A_2\,(v,v) \quad (v \in \mathbb{R}) \tag{6}$$

and

$$A_2\,(pu, pu) + A_2\,(qu, qu) = A_2\,(u,u) \quad (u \in \mathbb{R})\,. \tag{7}$$

From (5), by the biadditivity and symmetry of A_2, we have

$$\begin{aligned} A_2\,(pu, pu) + A_2\,(qu, qu) + A_2\,((1-p)\,v, (1-p)\,v) + A_2\,((1-q)\,v, (1-q)\,v) + \\ + 2A_2\,(pu, (1-p)\,v) + 2A_2\,(qu, (1-q)\,v) = A_2\,(u,u) + A_2\,(v,v) \end{aligned}$$

for all $u, v \in \mathbb{R}$. From this, by (5) and (6), it follows

$$A_2\,(pu, (1-p)\,v) + A_2\,(qu, (1-q)\,v) = 0 \quad (u, v \in \mathbb{R})\,,$$

i.e.

$$A_2\,((p+q)\,u, v) = A_2\,(pu, pv) + A_2\,(qu, qv) \quad (u, v \in \mathbb{R})\,. \tag{8}$$

For all $u, v \in \mathbb{R}$ we have

$$A_2(pu, pv) = \frac{1}{4}\left[A_2(p(u+v), p(u+v)) - A_2(p(u-v), p(u-v))\right]$$

and

$$A_2(qu, qv) = \frac{1}{4}\left[A_2(q(u+v), q(u+v)) - A_2(q(u-v), q(u-v))\right].$$

By these equations and by (7) we have

$$\begin{aligned} A_2(pu, pv) + A_2(qu, qv) &= \frac{1}{4}\left[A_2(p(u+v), p(u+v)) + \right. \\ &\quad + A_2(q(u+v), q(u+v)) - A_2(p(u-v), p(u-v)) - \\ &\left. -A_2(q(u-v), q(u-v))\right] = \frac{1}{4}\left[A_2(u+v, u+v) - A_2(u-v, u-v)\right] = \\ &= A_2(u, v). \end{aligned}$$

From the above equation by (8) we obtain

$$A_2((p+q)u, v) = A_2(u, v) \quad (u, v \in \mathbb{R}),$$

i.e. $A_2((p+q-1)u, v) = 0$. By $p+q \neq 1$ we have that A_2 is zero for all $u, v \in \mathbb{R}$.

Theorem 2 (Daróczy, Lajkó, Lovas, Maksa, Páles [5]). *Let $0 < p < 1$. We consider the functional equation*

$$g(pu + (1-p)v) + g((1-p)u + pv) = g(u) + g(v) \quad (u, v \in J). \tag{9}$$

Then we have two possibilities:

(a) If p is algebraic, and $\frac{1-p}{p}$ and $-\frac{1-p}{p}$ are not algebraic conjugates, then any solution of (9) is a solution of the Jensen functional equation

$$2g\left(\frac{u+v}{2}\right) = g(u) + g(v) \quad (u, v \in J). \tag{10}$$

(b) If either p is transcendental, or p is algebraic, such that $\frac{1-p}{p}$ and $-\frac{1-p}{p}$ are algebraic conjugates, then there exists a solution $g : J \to \mathbb{R}$ of (9) which is not a solution of the Jensen equation (10).

For the assertion (b) see Daróczy-Maksa-Páles [7].

3. Weighted quasi–arithmetic means

Let $I \subset \mathbb{R}$ be a non-void open interval. Denote the class of all *continuous and strictly monotonic* functions defined on I by $CM(I)$. A function $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I if there exist a number $p \in \,]0,1[$ and a function $\varphi \in CM(I)$ such that

$$M(x,y) = \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)) =: A_{\varphi,p}(x,y) \tag{11}$$

for all $x, y \in I$. The number p is called the *weight* and the function φ is called the *generating function* of the mean (11) in I. If $p = \frac{1}{2}$ then we say M is a quasi-arithmetic mean on I ([10], [1], [12]). It is obvious that weighted quasi-arithmetic means are strict means on I.

In the following we need the following very important result of J. Jarczyk [11].

Theorem 3. *Let $\varphi, \psi, \gamma \in CM(I)$ and let $p, q, r \in \,]0,1[$. The weighted quasi-arithmetic means $M_1 := A_{\varphi,p}$, $M_2 := A_{\psi,q}$ and $M_3 := A_{\gamma,r}$ on I satisfy*

$$M_3 = M_1 \otimes M_2$$

on I^2 if and only if the following two conditions are fulfilled:
(i)

$$r = \frac{q}{1-p+q},$$

(ii) either there exist $a, c \in \mathbb{R} \setminus \{0\}$ and $b, d \in \mathbb{R}$ such that

$$\varphi(x) = a\gamma(x) + b, \quad \psi(x) = c\gamma(x) + d \quad (x \in I),$$

or $r = \frac{1}{2}$, $p + q = 1$ and

$$\varphi(x) = a\exp(t\gamma(x)) + b, \quad \psi(x) = c\exp(-t\gamma(x)) + d \quad (x \in I),$$

with some $t \in \mathbb{R} \setminus \{0\}$.

For quasi-arithmetic means see the paper of Daróczy-Páles [8]. The second result is due to P. Burai [2] which is a generalization of a theorem of Ebanks [9].

Theorem 4. *Let $0 < p < 1$ and let $J \subset \mathbb{R}_+$ be a non-void open interval. Then the functional equations*

$$g(pu + (1-p)v) + g\left(\frac{uv}{pu + (1-p)v}\right) = g(u) + g(v) \quad (u, v \in J)$$

and

$$2g\left(\sqrt{uv}\right) = g(u) + g(v) \quad (u, v \in J)$$

are equivalent.

For $p = \frac{1}{2}$ see Ebanks [9] and Daróczy-Ebanks-Páles [4].

Definition 1. Let M_1, M_2, M_3 be weighted quasi-arithmetic means on I such that

$$M_3 = M_1 \otimes M_2. \tag{12}$$

We say that the triplet $(M_1,\ M_2,\ M_3)$ is an *exceptional case* (EC), if M_1, M_2, M_3 have a common generating function and M_1 is weighted by $p \in \,]0,1[$, M_2 is weighted by $(1-p) \in \,]0,1[$ and M_3 is weighted by $\frac{1}{2}$ (from this it follows (12)), where either p is transcendental, or p is algebraic, such that $\frac{1-p}{p}$ and $-\frac{1-p}{p}$ are algebraic conjugates.

The main result is the following

Theorem 5. *Let M_1, M_2, M_3 be weighted quasi-arithmetic means on I with (12).*

(1) *If the triplet (M_1, M_2, M_3) is not an exceptional case (NEC), then the functional equations (1) and (2) are equivalent.*

(2) *If the triplet (M_1, M_2, M_3) is an exceptional case (EC), then there exists a solution of the functional equation (1) which is not a solution of the functional equation (2).*

Proof. The weighted quasi-arithmetic means M_1, M_2, M_3 on I satisfying identity $M_3 = M_1 \otimes M_2$ (i.e. (12)) are derived from Theorem 3 of J. [11] (for quasi-arithmetic means see Daróczy-Páles [8]). We have two cases:

(a) There exists $\gamma \in CM(I)$ and $p, q, r \in \,]0,1[$ with $r = \frac{q}{1-p+q}$ such that

$$\begin{gathered} M_1(x,y) = A_{\gamma,p}(x,y), \quad M_2(x,y) = A_{\gamma,q}(x,y), \\ M_3(x,y) = A_{\gamma,r}(x,y) \end{gathered} \tag{13}$$

hold for all $x, y \in I$.

(b) There exist $\gamma \in CM(I)$, $p \in \,]0,1[$ and a constant $t \neq 0$ $(t \in \mathbb{R})$ such that

$$\begin{gathered} M_1(x,y) = A_{\exp(t\gamma),p}(x,y), \quad M_2(x,y) = A_{\exp(-t\gamma),1-p}(x,y), \\ M_3(x,y) = A_{\gamma,\frac{1}{2}}(x,y) \end{gathered} \tag{14}$$

hold for all $x, y \in I$.

First we consider the case (a). By substituting the means M_1 and M_2 of (13) into equation (1), we have

$$f\left(\gamma^{-1}\left(p\gamma(x)+(1-p)\gamma(y)\right)\right)+f\left(\gamma^{-1}\left(q\gamma(x)+(1-q)\gamma(y)\right)\right)=$$

$$=f(x)+f(y) \tag{15}$$

for all $x,y\in I$. Now we define

$$u:=\gamma(x),\quad v:=\gamma(y)$$

where $u,v\in\gamma(I)=:J$ and J is non-void open interval. Furthermore, define

$$g(s):=f\circ\gamma^{-1}(s)\quad (s\in J). \tag{16}$$

Then it follows from (15) that for every $u,v\in J$

$$g(pu+(1-p)v)+g(qu+(1-q)v)=g(u)+g(v). \tag{17}$$

We consider two cases of (a):

(a1) $p+q\neq 1$,

(a2) $p+q=1$.

In the case (a1) by Theorem 1 of Maksa we have that g is a constant function, from which by (16) f is also a constant function on I. It is trivial, that any constant function is a solution of the functional equation (2), therefore equations (1) and (2) are equivalent.

In the case (a2) we have from (17)

$$g(pu+(1-p)v)+g((1-p)u+pv)=g(u)+g(v)\quad (u,v\in J) \tag{18}$$

and by (13) and by $r=\frac{1}{2}$ (2) yields

$$2f\left(\gamma^{-1}\left(\frac{\gamma(x)+\gamma(y)}{2}\right)\right)=f(x)+f(y)\quad (x,y\in I), \tag{19}$$

i.e. with (16) we obtain from (19)

$$2g\left(\frac{u+v}{2}\right)=g(u)+g(v)\quad (u,v\in J). \tag{20}$$

The equation (20) is the well-known Jensen equation on J ([12], [1]). Now by Theorem 2 of Daróczy-Lajkó-Lovas-Maksa-Páles [5] we have the following assertions:

(i) If $\frac{1-p}{p}$ is algebraic (i.e. p is algebraic), and $-\frac{1-p}{p}$ is not an algebraic conjugate of it, then any solution of (18) is a solution of Jensen's equation (20). From this we have that in this case equations (1) and (2) are equivalent.

(ii) If either p is transcendental or p is algebraic such that $\frac{1-p}{p}$ and $-\frac{1-p}{p}$ are algebraic conjugates, then there exists a solution of (18) which is not a solution of the Jensen equation (20). This is the exceptional case, i.e. in this case equations (1) and (20) are not equivalent.

In the second case (b), by substituting the means M_1 and M_2 of (14) into equation (1), we have that

$$\begin{aligned} f\circ\gamma^{-1}\left(\frac{1}{t}\log\left(pe^{t\gamma(x)}+(1-p)\,e^{t\gamma(y)}\right)\right)+\\ +f\circ\gamma^{-1}\left(-\frac{1}{t}\log\left((1-p)\,e^{-t\gamma(x)}+pe^{-t\gamma(y)}\right)\right)=f(x)+f(y)\end{aligned}\tag{21}$$

for all $x,y\in I$. Now define

$$u:=e^{t\gamma(x)},\quad v:=e^{t\gamma(y)},$$

where $u,v\in e^{t\gamma(I)}=:J\subset\mathbb{R}_+$, and J is a non-void open interval. Furthermore, define

$$g(s):=f\circ\gamma^{-1}\left(\frac{1}{t}\log s\right)\quad (s\in J\subset\mathbb{R}_+).$$

Then it follows from (21) that, for every $u,v\in J$,

$$g(pu+(1-p)\,v)+g\left(\frac{uv}{pu+(1-p)\,v}\right)=g(u)+g(v).\tag{22}$$

Applying Theorem 4 of Burai [2], (22) implies that

$$2g\left(\sqrt{uv}\right)=g(u)+g(v)\quad (u,v\in J).$$

Therefore,

$$2f\circ\gamma^{-1}\left(\frac{1}{t}\log\sqrt{e^{t\gamma(x)}e^{t\gamma(y)}}\right)=f(x)+f(y)\quad (x,y\in I).$$

Thus, we obtain, for all $x, y \in I$, that

$$2f\left(A_{\varphi,\frac{1}{2}}(x,y)\right) = f(x) + f(y) \quad (x, y \in I).$$

By (14) we know $A_{\varphi,\frac{1}{2}} = M_3$, i.e. f is a solution of (2).

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