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A SIMPLY–OBTAINED UPPER BOUND FOR $q(n)$

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. Using simple analytic methods, we obtain an upper bound for $q(n)$, the number of partitions of the natural number n into distinct parts (or into odd parts).

1. Introduction

If n is a natural number, let $p(n), q(n)$ denote respectively the number of unrestricted partitions of n, the number of partitions of n into distinct parts (or into odd parts). Using the circle method, G.H. Hardy and S. Ramanujan [3] obtained the asymptotic formula

$$p(n) \sim \frac{1}{n\sqrt{48}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1}$$

By similar methods, P. Hagis [2] obtained the following asymptotic formula for $q(n)$:

$$q(n) \sim 18^{\frac{1}{4}}(24n+1)^{-\frac{3}{4}} \exp\left(\frac{\pi}{12}\sqrt{48n+2}\right). \tag{2}$$

As is mentioned in [1] (Chapter 14), using more elementary methods, van Lint [4] obtained the following upper bound for $p(n)$:

$$p(n) < \frac{\pi}{\sqrt{6(n-1)}} \exp\left(\pi\frac{\sqrt{2n}}{3}\right). \tag{3}$$

The purpose of this note is to obtain, in similar fashion, an upper bound for $q(n)$, namely

$$q(n) < \frac{\pi}{\sqrt{12(n-1)}} \exp\left(\frac{\pi^2}{12} + \pi\frac{\sqrt{n-1}}{3}\right). \tag{4}$$

2. The main result

Theorem.

$$q(n) < \frac{\pi}{\sqrt{12(n-1)}} \exp\left(\frac{\pi^2}{12} + \pi\frac{\sqrt{n-1}}{3}\right) \quad .$$

Proof. We begin with the generating function

$$\sum_{n\geq 0} q(n)x^n = F(x) = \prod_{n\geq 1}(1-x^{2n-1})^{-1},$$

where x is a real variable and $0 < x < 1$. Taking logarithms, we have

$$\log F(x) = -\sum_{n\geq 1}\log(1-x^{2n-1}) = \sum_{n\geq 1}\sum_{m\geq 1}\frac{(x^{2n-1})^m}{m} =$$

$$\sum_{m\geq 1}\frac{1}{m}\sum_{n\geq 1}(x^{2n-1})^m = \sum_{m\geq 1}\frac{1}{m}\left(\frac{x^m}{1-x^{2m}}\right) = \sum_{m\geq 1}\frac{1}{m}\left(\frac{x^m}{1-x^m}\right)\left(\frac{1}{1+x^m}\right).$$

Recall from the proof of (3) that for $0 < x < 1$, we have

$$\frac{1}{m}\left(\frac{x^m}{1-x^m}\right) < \frac{1}{m^2}\left(\frac{x}{1-x}\right).$$

Therefore we have

$$\log F(x) < \sum_{m\geq 1}\frac{1}{m^2}\left(\frac{x}{1-x}\right)\left(\frac{1}{1+x^m}\right) < \sum_{m\geq 1}\frac{1}{m^2}\left(\frac{x}{1-x}\right)\left(\frac{1}{1+x}\right),$$

that is

$$\log F(x) < \sum_{m\geq 1} \frac{1}{m^2}\left(\frac{x}{1-x^2}\right) = \frac{x}{1-x^2}\sum_{m\geq 1}\frac{1}{m^2} = \frac{\pi^2}{6}\left(\frac{x}{1-x^2}\right).$$

Also, we have

$$q(n)\sum_{k=n}^{\infty} x^k \leq \sum_{m=n}^{\infty} q(m)x^m < F(x),$$

so that

$$q(n)\left(\frac{x^n}{1-x}\right) < F(x).$$

This implies

$$\log q(n) < \log(1-x) + n\log\frac{1}{x} + \frac{\pi^2}{6}\left(\frac{x}{1-x^2}\right).$$

Let $x = (1+t)^{-1}$, so that $1-x = xt$ and $\frac{1}{x} = 1+t$, where $0 \leq t < \infty$. Then we have

$$\log q(n) < \log t + \log x + n\log(1+t) + \frac{\pi^2}{6}\left(\frac{t+1}{(t+1)^2-1}\right),$$

that is

$$\log q(n) < \log t + (n-1)\log(1+t) + \frac{\pi^2}{6}\left(\frac{t+1}{(t+1)^2-1}\right),$$

which implies

$$\log q(n) < \log t + (n-1)t + \frac{\pi^2}{6}\left(\frac{t+1}{(t+1)^2-1}\right).$$

Let $v = (t+1)^2 - 1$, so that $t+1 = \sqrt{1+v}$, where $0 \leq v < \infty$. Now

$$(1+v)^{\frac{1}{2}} < 1 + \frac{v}{2},$$

so we have

$$\log q(n) < \log\frac{v}{2} + (n-1)\frac{v}{2} + \frac{\pi^2}{6}\left(\frac{1+\frac{v}{2}}{v}\right).$$

Let

$$g(v) = \log \frac{v}{2} + (n-1)\frac{v}{2} + \frac{\pi^2}{6}\left(\frac{1+\frac{v}{2}}{v}\right).$$

Then

$$g'(v) = \frac{1}{v} + \frac{n-1}{2} - \frac{\pi^2}{6v^2} = \frac{3(n-1)v^2 + 6v - \pi^2}{6v^2}.$$

Note that $g'(v) = 0$ when

$$v = v_0 = \frac{-6 + \sqrt{36 + 12(n-1)\pi^2}}{6(n-1)}.$$

Furthermore, $g'(v) < 0$ for $v < v_0$ and $g'(v) > 0$ for $v > v_0$. Therefore $g(v)$ must have a minimum value when at $v = v_0$. Let

$$v_1 = \frac{\pi}{3\sqrt{n-1}}.$$

Then

$$\log q(n) < g(v_1) = \log \frac{\pi}{\sqrt{12(n-1)}} + \frac{n-1}{2}\left(\frac{\pi}{3\sqrt{n-1}} + \frac{\pi^2}{6}\right).$$

The conclusion now follows.

References

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