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A JOINT LIMIT THEOREM FOR HURWITZ–FUNCTIONS WITH ALGEBRAIC IRRATIONAL PARAMETERS

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To the memory of Professor M.V. Subbarao

Abstract. We prove a limit theorem in the sense of weak convergence of probability measures on the r-dimensional complex plane for a collection of Hurwitz zeta-functions with algebraic irrational parameters.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and α be a fixed real number satisfying $0 < \alpha \leq 1$. The Hurwitz zeta-function $\zeta(s, \alpha)$ with a parameter α is for $\sigma > 1$ defined by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and by analytic continuation elsewhere, except for a simple pole at $s = 1$ with residue 1. If $\alpha = 1$, then $\zeta(s, \alpha)$ reduces to the Riemann zeta-function $\zeta(s)$.

The first limit theorems in the sense of weak convergence of probability measures for the Hurwitz zeta-function were obtained in [1], for the further development see [7]. However, these results considered $\zeta(s, \alpha)$ with transcendental or rational parameter α. For the case of an algebraic parameter α a new

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method was needed, and this case was treated in [6], [8] and [9]. We will recall the latter results.

Denote by $meas\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and define, for $T > 0$,

$$\nu_T^t(...) = \frac{1}{T}\text{meas}\{t \in [0, T] : ...\},$$

where in place of dots a condition satisfied by t is to be written, and the sign t in ν_T^t indicates that the measure is taken over $t \in [0; T]$. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. We also need the following topological structure.

Suppose that α is an algebraic irrational number. Define the system

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\},$$

where $\mathbb{N}_0$ is the set of non-negative integers. J.W.S. Cassels in [3] proved that at least 51 percent of the elements of $L(\alpha)$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Let $I(\alpha)$ be a maximal linearly independent subset of $L(\alpha)$. We assume that $I(\alpha) \neq L(\alpha)$, because otherwise we would have the same situation as in the case of transcendental α. Define $D(\alpha) = L(\alpha) \setminus I(\alpha)$. Then, for any element $d_m \in D(\alpha)$, the system $\{d_m\} \bigcup I(\alpha)$ is already linearly dependent over $\mathbb{Q}$, and therefore, there exists a finite set of elements $i_{m_1}, ..., i_{m_n} \in I(\alpha)$ such that

$$k_0(m)d_m + k_1(m)i_{m_1} + ... + k_n(m)i_{m_n} = 0$$

with some non-zero integers $k_j(m)$, $j = 0, 1, ..., n$. Hence, we find that

$$m + \alpha = (m_1 + \alpha)^{-\frac{k_1(m)}{k_0(m)}} \cdot ... \cdot (m_n + \alpha)^{-\frac{k_n(m)}{k_0(m)}}. \tag{1}$$

Now let $\mathcal{M}(\alpha) = \{m : \log(m + \alpha) \in I(\alpha)\}$ and $\mathcal{N}(\alpha) = \{m : \log(m + \alpha) \in D(\alpha)\}$. Define

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \gamma = \{s \in \mathbb{C} : |s| = 1\}$ is the unit circle for all $m \in \mathcal{M}(\alpha)$. Since γ is compact, it follows from the Tikhonov theorem that the torus Ω with product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to γ_m. If $m \in \mathcal{N}(\alpha)$ and relation (1) is true, then we define

$$\omega(m) = \omega(m_1)^{-\frac{k_1(m)}{k_0(m)}} ... \omega(m_n)^{-\frac{k_n(m)}{k_0(m)}}, \tag{2}$$

where the principal values of the roots are taken. For $\sigma > \frac{1}{2}$ let

$$\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}.$$

Then $\zeta(\sigma, \alpha, \omega)$, for $\sigma > \frac{1}{2}$, is a complex-valued random variable on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. For a region G in the complex plane, denote by $H(G)$ the space of analytic functions on G equipped with the topology of uniform convergence on compacta. Let $D_1 = \{s \in \mathbb{C} : \sigma > 1\}$ and $D_2 = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Then $\zeta(s, \alpha, \omega)$ is an $H(D_1)$-valued random element as well as an $H(D_2)$-valued random element defined on $(\Omega, \mathcal{B}(\Omega), m_H)$. Now we are able to state limit theorems for $\zeta(s, \alpha)$.

Theorem 1. [8] *Suppose that α is algebraic irrational, and that $\sigma > \frac{1}{2}$. Then the probability measure*

$$\nu_T^t\big(\zeta(\sigma + it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random variable $\zeta(\sigma, \alpha, \omega)$ as $T \to \to \infty$.

Theorem 2. [6] *Suppose that α is algebraic irrational. Then the probability measure*

$$\nu_T^\tau\big(\zeta(s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to the distribution of the $H(D_1)$-valued random element $\zeta(s, \alpha, \omega)$ as $T \to \infty$.

Theorem 3. [9] *Suppose that α is algebraic irrational. Then the probability measure*

$$\nu_T^\tau\big(\zeta(s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(D_2)),$$

converges weakly to the distribution of the $H(D_2)$-valued random element $\zeta(s, \alpha, \omega)$ as $T \to \infty$.

The aim of this paper is a generalization of Theorem 1 for a collection of Hurwitz zeta-functions. Let $\alpha_1, ..., \alpha_r$ be distinct algebraic irrational numbers, $0 < \alpha_j < 1$, and, for $\sigma > \frac{1}{2}$,

$$\zeta(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha_j)^s}, \quad j = 1, ..., r.$$

Let $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$ and $\underline{\sigma} = (\sigma_1, ..., \sigma_r)$. Define on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ the probability measure $P_{T,\underline{\sigma},\underline{\alpha}}$ by

$$P_{T,\underline{\sigma},\underline{\alpha}}(A) = \nu_T^t\big(\big(\zeta(\sigma_1 + it, \alpha_1), ..., \zeta(\sigma_r + it, \alpha_r)\big) \in A\big).$$

Moreover, let

$$\Omega^{(r)} = \prod_{j=1}^{r} \Omega_j,$$

where

$$\Omega_j = \prod_{m \in \mathcal{M}(\alpha_j)} \gamma_m,$$

with $\gamma_m = \gamma$ for $m \in \mathcal{M}(\alpha_j)$, $j = 1, ..., r$. Then $\Omega^{(r)}$ is also a compact topological Abelian group, and we obtain a probability space $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}), m_H^{(r)})$, where $m_H^{(r)}$ is the probability Haar measure on $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}))$. Note that $m_H^{(r)}$ is the product of the Haar measures on Ω_j, $j = 1, ..., r$. Denote by $\underline{\omega} = (\omega_1, ..., \omega_r)$ the element of Ω^r, where $\omega_j \in \Omega_j$, $j = 1, ..., r$, and define on the probability space $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}), m_H^{(r)})$ a $\mathbb{C}^r$-valued random element $\zeta(\underline{\sigma}, \underline{\alpha}, \underline{\omega})$, for $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$, by

$$\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}) = (\zeta(\sigma_1, \alpha_1, \omega_1), ..., \zeta(\sigma_r, \alpha_r, \omega_r)),$$

where

$$\zeta(\sigma_j, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)}{(m + \alpha_j)^{\sigma_j}}, \quad j = 1, ..., r,$$

and

$$\omega_j(m) = \begin{cases} \text{projection of } \omega_j \in \Omega \text{ to } \gamma_m, & \text{if } m \in \mathcal{M}(\alpha_j), \\ \text{relation of type (2)}, & \text{otherwise}, \end{cases}$$

$j = 1, ..., r$. Denote by $P_{\underline{\sigma},\underline{\alpha}}$ the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega})$, that is

$$P_{\underline{\sigma},\underline{\alpha}}(A) = m_H^{(r)}\left(\underline{\omega} \in \Omega^{(r)} : \zeta(\underline{\sigma}, \underline{\alpha}, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Theorem 4. *Suppose that $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers such that the set*

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$*, and that* $\min\limits_{1\le j\le r}\sigma_j > \frac{1}{2}$*. Then the probability measure* $P_{T,\underline{\sigma},\underline{\alpha}}$ *converges weakly to the measure* $P_{\underline{\sigma},\underline{\alpha}}$ *as* $T\to\infty$*.*

Example. Let $\alpha_j = p_j^{-1/2}$, where p_j denotes the j-th prime number (in ascending order). Then the hypotheses of Theorem 4 are satisfied.

2. A limit theorem on $\Omega^{(r)}$

We start the proof of Theorem 4 with a limit theorem on $\Omega^{(r)}$. For $A \in \mathcal{B}(\Omega^{(r)})$, define

$$Q_T(A) =$$
$$= \nu_T^t\left(\left(\{(m+\alpha_1)^{it} : m \in \mathcal{M}(\alpha_1)\}, ..., \{(m+\alpha_r)^{it} : m \in \mathcal{M}(\alpha_r)\}\right) \in A\right).$$

Lemma 5. *Suppose that the numbers* $\alpha_1, ..., \alpha_r$ *satisfy the hypotheses of Theorem 4. Then the probability measure* Q_T *converges weakly to the Haar measure* $m_H^{(r)}$ *as* $T\to\infty$*.*

Proof. The dual group of $\Omega^{(r)}$ is isomorphic to

$$\bigoplus_{j=1}^{r}\bigoplus_{m\in\mathcal{M}(\alpha_j)}\mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha_j)$ and $j = 1, ..., r$. The element

$$\underline{k} = \left((k_{m1})_{m\in\mathcal{M}(\alpha_r)}, ..., (k_{mr})_{m\in\mathcal{M}(\alpha_r)}\right) \in \bigoplus_{j=1}^{r}\bigoplus_{m\in\mathcal{M}(\alpha_j)}\mathbb{Z}_{mj}$$

acts on $\Omega^{(r)}$ by

$$\underline{\omega} \mapsto \underline{\omega}^{\underline{k}} = \prod_{j=1}^{r}\prod_{m\in\mathcal{M}(\alpha_j)}\omega_j^{k_{mj}}(m),$$

where only a finite number of integers k_{mj} are distinct from zero. Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is

$$g_T(\underline{k}) = \int\limits_{\Omega^{(r)}}\prod_{j=1}^{r}\prod_{m\in\mathcal{M}(\alpha_j)}\omega_j^{k_{mj}}(m)dQ_T,$$

where, as above, only a finite number of integers k_{mj} are distinct from zero. Thus, by the definition of Q_T we have

$$g_T(\underline{k}) = \frac{1}{T}\int\limits_0^T \prod_{j=1}^r \prod_{m\in\mathcal{M}(\alpha_j)} (m+\alpha_j)^{ik_{mj}t}dt =$$

$$= \frac{1}{T}\int\limits_0^T \exp\left\{ it\sum_{j=1}^r \sum_{m\in\mathcal{M}(\alpha_j)} k_{mj}\log(m+\alpha_j)\right\} dt.$$

Since the set $\bigcup\limits_{j=1}^r I(\alpha_j)$ is linearly independent over $\mathbb{Q}$, hence we easily find that

$$g_T(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ \dfrac{\exp\left\{ iT\sum\limits_{j=1}^r \sum\limits_{m\in\mathcal{M}(\alpha_j)} k_{mj}\log(m+\alpha_j)\right\} - 1}{iT\sum\limits_{j=1}^r \sum\limits_{m\in\mathcal{M}(\alpha_j)} k_{mj}\log(m+\alpha_j)}, & \text{otherwise.} \end{cases}$$

Therefore,

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Hence, by general continuous theorems on locally compact groups, see, for example [5], Theorem 1.4.2, we obtain that the measure Q_T converges weakly to $m_H^{(r)}$ as $T \to \infty$.

3. Joint limit theorems for absolutely convergent series

For fixed $\sigma_{1j} > \frac{1}{2}$ and $m, n \in \mathbb{N}_0$, we put

$$v_j(m,n,\alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_{1j}}\right\},$$

and define, for $\sigma > \frac{1}{2}$,

$$\zeta_{n,j}(s,\alpha_j) = \sum_{m=0}^{\infty} \frac{v_j(m,n,\alpha_j)}{(m+\alpha_j)^s}, \quad j=1,...,r.$$

In [8] it was observed that the latter series converges absolutely for $\sigma > \frac{1}{2}$. Since $|\omega_j(m)| = 1$, the series

$$\zeta_{n,j}(s,\alpha_j,\omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_j(m,n,\alpha_j)}{(m+\alpha_j)^s}, \quad j=1,...,r$$

is absolutely convergent in the region $\sigma > \frac{1}{2}$.

This section is devoted to the weak convergence of the probability measures

$$P_{T,n,\underline{\sigma},\underline{\alpha}}(A) = \nu_T^t\big((\zeta_{n,1}(\sigma_1+it,\alpha_1),...,\zeta_{n,r}(\sigma_r+it,\alpha_r)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

and

$$\hat{P}_{T,n,\underline{\sigma},\underline{\alpha}}(A) = \nu_T^t\big((\zeta_{n,1}(\sigma_1+it,\alpha_1,\hat{\omega}_1),...,\zeta_{n,r}(\sigma_r+it,\alpha_r,\hat{\omega}_r)) \in A\big),$$
$$A \in \mathcal{B}(\mathbb{C}^r),$$

where $\underline{\hat{\omega}} = (\hat{\omega}_1,...,\hat{\omega}_r)$ is a fixed element of $\Omega^{(r)}$.

Theorem 6. *Let* $\min\limits_{1\le j\le r} \sigma_j > \frac{1}{2}$. *Suppose that the numbers* $\alpha_1,...,\alpha_r$ *satisfy the hypotheses of Theorem 4. Then there exists a probability measure* $P_{n,\underline{\sigma},\underline{\alpha}}$ *on* $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ *such that both the measures* $P_{T,n,\underline{\sigma},\underline{\alpha}}$ *and* $\hat{P}_{T,n,\underline{\sigma},\underline{\alpha}}$ *converge weakly to* $P_{n,\underline{\sigma},\underline{\alpha}}$ *as* $T \to \infty$.

Proof. Let the function $h_{n,\underline{\sigma},\underline{\alpha}} : \Omega^{(r)} \to \mathbb{C}^r$ be given by the formula

$$h_{n,\underline{\sigma},\underline{\alpha}}\left((\{\omega_1(m) : m \in \mathcal{M}(\alpha_1)\},...,\{\omega_r(m) : m \in \mathcal{M}(\alpha_r)\})\right) =$$
$$= \left(\sum_{m=0}^{\infty} \frac{v_1(m,n,\alpha_1)}{(m+\alpha_1)^{\sigma_1}\omega_1(m)},...,\sum_{m=0}^{\infty} \frac{v_r(m,n,\alpha_r)}{(m+\alpha_r)^{\sigma_r}\omega_r(m)}\right).$$

Since the series

$$\sum_{m=0}^{\infty} \frac{v_j(m,n,\alpha_j)}{(m+\alpha_j)^{\sigma_j}\omega_j(m)}, \quad j=1,...,r,$$

converges uniformly with respect to ω_j, the function $h_{n,\underline{\sigma},\underline{\alpha}}$ is continuous. Moreover, we see that

$$h_{n,\underline{\sigma},\underline{\alpha}}\left((\{(m+\alpha_1)^{it} : m \in \mathcal{M}(\alpha_1)\},...,\{(m+\alpha_r)^{it} : m \in \mathcal{M}(\alpha_r)\})\right) =$$

$$= \big(\zeta_{n,1}(\sigma_1+it,\alpha_1),...,\zeta_{n,r}(\sigma_r+it,\alpha_r)\big).$$

Therefore, Lemma 5 together with Theorem 5.1 of [2] shows that the probability measure $P_{T,n,\underline{\sigma},\underline{\alpha}}$ converges weakly to the measure $m_H^{(r)} h_{n,\underline{\sigma},\underline{\alpha}}^{-1}$ as $T \to \infty$.

Now let $\hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}} : \Omega^{(r)} \to \Omega^{(r)}$ be defined by

$$\hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}}\,((\{\omega_1(m): m \in \mathcal{M}(\alpha_1)\},...,\{\omega_r(m): m \in \mathcal{M}(\alpha_r)\})) =$$

$$= \left(\{\omega_1(m)\hat{\omega}_1^{-1}(m): m \in \mathcal{M}(\alpha_1)\},...,\{\omega_r(m)\hat{\omega}_r^{-1}(m): m \in \mathcal{M}(\alpha_r)\}\right).$$

Then we have that

$$\left(\sum_{m=0}^{\infty}\frac{\hat{\omega}_1(m)v_1(m,n,\alpha_1)}{(m+\alpha_1)^{\sigma_1}},...,\sum_{m=0}^{\infty}\frac{\hat{\omega}_r(m)v_r(m,n,\alpha_r)}{(m+\alpha_r)^{\sigma_r}}\right) =$$

$$= h_{n,\underline{\sigma},\underline{\alpha}}\left(\hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}}((\{(m+\alpha_1)^{it}: m \in \mathcal{M}(\alpha_1)\},...,\{(m+\alpha_r)^{it}: m \in \mathcal{M}(\alpha_r)\}))\right).$$

Thus, similarly to the case of the measure $P_{T,n,\underline{\sigma},\underline{\alpha}}$, we obtain that the measure $\hat{P}_{T,n,\underline{\sigma},\underline{\alpha}}$ converges weakly to $m_H^{(r)}(h_{n,\underline{\sigma},\underline{\alpha}}\hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}})^{-1}$ as $T \to \infty$. Since the Haar measure $m_H^{(r)}$ is invariant, we have

$$m_H^{(r)}(h_{n,\underline{\sigma},\underline{\alpha}}\ \hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}})^{-1} = \left(m_H^{(r)}\hat{h}_{\underline{\alpha},\ \underline{\hat{\omega}}}^{-1})\right) h_{n,\underline{\sigma},\underline{\alpha}}^{-1} = m_H^{(r)} h_{n,\underline{\sigma},\underline{\alpha}}^{-1},$$

and the theorem is proved.

4. Approximation in the mean

For $\underline{z}^{(1)} = (z_1^{(1)},...,z_r^{(1)})$, $\underline{z}^{(2)} = (z_1^{(2)},...,z_r^{(2)}) \in \mathbb{C}^r$, let

$$\rho(\underline{z}^{(1)},\underline{z}^{(2)}) = \left(\sum_{j=1}^{r}\left|z_j^{(1)} - z_j^{(2)}\right|^2\right)^{\frac{1}{2}}.$$

Then ρ is a metric in the space $\mathbb{C}^r$ which induces its topology. In this section, we will prove some preliminary results in order to pass from Theorem 6 to Theorem 4.

For brevity, define

$$\underline{\zeta}(\underline{\sigma},\underline{\alpha},t)=(\zeta(\sigma_1+it,\alpha_1),...,\zeta(\sigma_r+it,\alpha_r)),$$

$$\underline{\zeta}_n(\underline{\sigma},\underline{\alpha},t)=(\zeta_{n,1}(\sigma_1+it,\alpha_1),...,\zeta_{n,r}(\sigma_r+it,\alpha_r)),$$

and

$$\underline{\zeta}(\underline{\sigma},\underline{\alpha},\underline{\omega},t)=(\zeta(\sigma_1+it,\alpha_1,\omega_1),...,\zeta(\sigma_r+it,\alpha_r,\omega_r)),$$

$$\underline{\zeta}_n(\underline{\sigma},\underline{\alpha},\underline{\omega},t)=(\zeta_{n,1}(\sigma_1+it,\alpha_1,\omega_1),...,\zeta_{n,r}(\sigma_r+it,\alpha_r,\omega_r)).$$

Theorem 7. *Suppose that* $\min\limits_{1\le j\le r}\sigma_j>\frac{1}{2}$ *and* $\alpha_1,...,\alpha_r$ *satisfy the hypotheses of Theorem 4. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T\rho\left(\underline{\zeta}(\underline{\sigma},\underline{\alpha},t),\underline{\zeta}_n(\underline{\sigma},\underline{\alpha},t)\right)dt=0$$

and, for almost all $\underline{\omega}\in\Omega^{(r)}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T\rho\left(\underline{\zeta}(\underline{\sigma},\underline{\alpha},\underline{\omega},t),\underline{\zeta}_n(\underline{\sigma},\underline{\alpha},\underline{\omega},t)\right)dt=0.$$

Proof. By Lemmas 6 and 9 of [8] we have, for each $j=1,...,r$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T|\zeta(\sigma_j+it,\alpha_j)-\zeta_n(\sigma_j+it,\alpha_j)|\,dt=0,$$

and, for almost all $\omega_j\in\Omega_j$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T|\zeta(\sigma_j+it,\alpha_j,\omega_j)-\zeta_n(\sigma_j+it,\alpha_j,\omega_j)|\,dt=0.$$

This and the definition of ρ prove the theorem.

5. Ergodicity

For $t \in \mathbb{R}$, define

$$a_{t,\underline{\alpha}} = \left\{ (\{(m+\alpha_1)^{-it} : m \in \mathcal{M}(\alpha_1)\}, ..., \{(m+\alpha_r)^{-it} : m \in \mathcal{M}(\alpha_r)\}) \right\}.$$

Now let $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ be the one-parameter family of transformations on $\Omega^{(r)}$ for $\underline{\omega} \in \Omega^{(r)}$ defined by

$$\varphi_{t,\underline{\alpha}}(\underline{\omega}) = a_{t,\underline{\alpha}}\underline{\omega}.$$

Then $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\Omega^{(r)}$. We recall that a set $A \in \mathcal{B}(\Omega^{(r)})$ is invariant with respect to the group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ if, for each t, the sets A and $A_t = \varphi_{t,\underline{\alpha}}(A)$ differ one from another by a set of zero $m_H^{(r)}$-measure. All invariant sets form a σ-subfield of the field $\mathcal{B}(\Omega^{(r)})$. The one-parameter group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only of sets having $m_H^{(r)}$-measure equal to 0 or 1.

Lemma 8. *The one-parameter group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is ergodic.*

Proof. Let $\chi : \Omega^{(r)} \to \gamma$ be a character. Then

$$\chi(\underline{\omega}) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m),$$

where only a finite number of integers k_{mj} are distinct from zero. If χ is a non-principal character, then

$$\chi(a_{t,\underline{\alpha}}) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} (m+\alpha_j)^{-itk_{mj}}.$$

Since the set

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$, hence we find that there exists a $t_0 \neq 0$ such that $\chi(a_{t,\underline{\alpha}}) \neq 1$. The further proof runs in the same way as that of Lemma 7 from [8].

6. Proof of Theorem 4

Define on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ another probability measure

$$\hat{P}_{T,\ \underline{\sigma},\ \underline{\alpha}}(A) = \nu_T^t\big((\zeta(\sigma_1+it,\alpha_1,\omega_1),...,\zeta(\sigma_r+it,\alpha_r,\omega_r)) \in A\big),$$

where $(\omega_1,...,\omega_r) \in \Omega^{(r)}$.

Theorem 9. *Suppose that* $\min\limits_{1\le j\le r} \sigma_j > \frac{1}{2}$, *and that the numbers* $\alpha_1,...,\alpha_r$ *satisfy the hypotheses of Theorem 4. Then on* $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ *there exists a probability measure* $P_{\underline{\sigma},\underline{\alpha}}$ *such that both the measures* $P_{T,\underline{\sigma},\underline{\alpha}}$ *and* $\hat{P}_{T,\underline{\sigma},\underline{\alpha}}$ *converge weakly to* $P_{\underline{\sigma},\underline{\alpha}}$ *as* $T \to \infty$.

Proof. By Theorem 6, both the measures $P_{T,n,\underline{\sigma},\underline{\alpha}}$ and $\hat{P}_{T,n,\underline{\sigma},\underline{\alpha}}$ converge weakly to the same measure $P_{n,\underline{\sigma},\underline{\alpha}} \overset{def}{=} m_H^{(r)} h^{-1}_{n,\underline{\sigma},\underline{\alpha}}$ as $T \to \infty$. We will prove that the family of probability measures $\{P_{n,\underline{\alpha}} : n \in \mathbb{N}_0\}$ is tight (for the definition, see [2]).

By the definition of $P_{T,n,\underline{\sigma},\underline{\alpha}}$ and Chebyshev's inequality, for $M > 0$, we have

$$P_{T,n,\underline{\sigma},\underline{\alpha}}(\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z},\underline{0}) > M\}) =$$
$$= \nu_T^t\left(\rho(\underline{\zeta_n}(\underline{\sigma},\underline{\alpha},t),\underline{0}) > M\right) \le$$
$$\le \frac{1}{MT}\int\limits_0^T \rho(\underline{\zeta_n}(\underline{\sigma},\underline{\alpha},t),\underline{0})dt \le$$
$$\le \frac{1}{M}\left(\frac{1}{T}\int\limits_0^T\left(\sum_{j=1}^r |\zeta_{n,\ j}(\sigma_j+it,\alpha_j)|^2\right)dt\right)^{\frac{1}{2}} =$$

$$= \frac{1}{M}\left(\sum_{j=1}^r \frac{1}{T}\int\limits_0^T |\zeta_{n,\ j}(\sigma_j+it,\alpha_j)|^2\,dt\right)^{\frac{1}{2}}. \tag{3}$$

Since the series for $\zeta_{n,j}(\sigma_j+it,\alpha_j)$ converges absolutely, we have that

$$\limsup_{T\to\infty}\frac{1}{T}\int\limits_0^T |\zeta_{n,j}(\sigma_j+it,\alpha_j)|^2\,dt =$$

$$= \sum_{m=0}^{\infty} \frac{v_j^2(m,n,\alpha_j)}{(m+\alpha_j)^{2\sigma_j}} \leq \sum_{m=0}^{\infty} \frac{1}{(m+\alpha_j)^{2\sigma_j}} \stackrel{def}{=} R_j < \infty$$

for $j = 1, ..., r$. Therefore, in view of (3),

$$\limsup_{T\to\infty} P_{T,n,\underline{\sigma},\underline{\alpha}}(\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) > M\}) \leq$$

(4) $$\leq \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{M} \left(\sum_{j=1}^{r} \frac{1}{T} \int_0^T |\zeta_{n,j}(\sigma_j + it, \alpha_j)|^2 \, dt \right)^{\frac{1}{2}} \leq \frac{R}{M}$$

with

$$R = \left(\sum_{j=1}^{r} R_j \right)^{\frac{1}{2}} < \infty.$$

For arbitrary $\varepsilon > 0$, let $M = R\varepsilon^{-1}$. Then (4) yields the inequality

(5) $$\limsup_{T\to\infty} P_{T,n,\underline{\sigma},\underline{\alpha}}(\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) > M\}) \leq \varepsilon.$$

Obviously, the function $h : \mathbb{C}^r \to \mathbb{R}$ given by $h(\underline{z}) = \rho(\underline{z}, \underline{0})$ is continuous. This, Theorem 6 and Theorem 5.1 of [2] show that the probability measure

$$\nu_T^t \left(\rho(\underline{\zeta_n}(\underline{\sigma}, \underline{\alpha}, t), \underline{0}) \in A \right), \quad A \in \mathcal{B}(\mathbb{R}),$$

converges weakly to $P_{n,\underline{\sigma},\underline{\alpha}} h^{-1}$ as $T \to \infty$. Since the set $\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) > M\}$ is open, this, Theorem 2.1 of [2] and (5) imply

(6)
$$P_{n,\underline{\sigma},\underline{\alpha}}(\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) > M\}) \leq \liminf_{T\to\infty} P_{T,n,\underline{\sigma},\underline{\alpha}}(\{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) > M\}) \leq \varepsilon.$$

The set $K_\varepsilon = \{\underline{z} \in \mathbb{C}^r : \rho(\underline{z}, \underline{0}) \leq M\}$ is compact in $\mathbb{C}^r$, and by (6)

$$P_{n,\underline{\sigma},\underline{\alpha}}(K_\varepsilon) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$. So, the tightness of the family $\{P_{n,\underline{\sigma},\underline{\alpha}} : n \in \mathbb{N}_0\}$ is proved. Hence, by the Prokhorov theorem [2] this family is relatively compact. Therefore, there exists a subsequence $\{P_{n_1,\underline{\sigma},\underline{\alpha}}\} \subset \{P_{n,\underline{\sigma},\underline{\alpha}}\}$ such that $P_{n_1,\underline{\sigma},\underline{\alpha}}$ converges weakly to some measure $Q_{\underline{\sigma},\underline{\alpha}}$ on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $n_1 \to \infty$. Let $X_{n,\,\underline{\alpha}}(\underline{\sigma})$ be a $\mathbb{C}^r$-valued random element with the distribution $P_{n,\underline{\sigma},\underline{\alpha}}$, and let $\xrightarrow{\mathcal{D}}$ denote the convergence in distribution. Then the weak convergence of $P_{n_1,\underline{\sigma},\underline{\alpha}}$ to $P_{\underline{\sigma},\underline{\alpha}}$ as $n_1 \to \infty$ is equivalent to the relation

$$X_{n_1,\underline{\alpha}}(\underline{\sigma}) \xrightarrow[n_1\to\infty]{\mathcal{D}} Q_{\underline{\sigma},\underline{\alpha}}. \tag{7}$$

Let θ denote a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$ and uniformly distributed on $[0,1]$. We set

$$X_{T,n,\underline{\alpha}}(\underline{\sigma}) = \underline{\zeta_n}(\underline{\sigma}, \underline{\alpha}, \theta T).$$

Then the statement of Theorem 6 can be written in the form

$$X_{T,n,\underline{\alpha}}(\underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\underline{\alpha}}(\underline{\sigma}). \tag{8}$$

Moreover, by the first assertion of Theorem 7, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho\left(X_{T,n,\underline{\alpha}}(\underline{\sigma}), X_{T,\underline{\alpha}}(\underline{\sigma})\right) > \varepsilon\right) =$$

$$= \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T^t \left(\rho\left(\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, t), \underline{\zeta}_n(\underline{\sigma}, \underline{\alpha}, t)\right) \geq \varepsilon\right) \leq$$

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \rho\left(\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, t), \underline{\zeta}_n(\underline{\sigma}, \underline{\alpha}, t)\right) dt = 0,$$

where

$$X_{T,\underline{\alpha}}(\underline{\sigma}) = \zeta(\underline{\sigma}, \underline{\alpha}, \theta T).$$

Now this, (7), (8) and Theorem 4.2 of [2] show that

$$X_{T,\underline{\alpha}}(\underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} Q_{\underline{\sigma},\underline{\alpha}}, \tag{9}$$

which is equivalent to the weak convergence of the measure $P_{T,\underline{\sigma},\underline{\alpha}}$ to $P_{\underline{\sigma},\underline{\alpha}}$ as $T \to \infty$.

It remains to obtain the same result for the measure $\hat{P}_{T,\underline{\sigma},\underline{\alpha}}$. First we observe that in view of (9), the measure $P_{\underline{\sigma},\underline{\alpha}}$ is independent on the sequence $P_{n_1,\underline{\sigma},\underline{\alpha}}$. Thus, from the relative compactness of $\{P_{n,\underline{\sigma},\underline{\alpha}}\}$ we have that

$$X_{n,\underline{\alpha}}(\underline{\sigma}) \xrightarrow[n\to\infty]{\mathcal{D}} Q_{\underline{\sigma},\underline{\alpha}}. \tag{10}$$

Now putting

$$\hat{X}_{T,n,\underline{\alpha}}(\underline{\sigma}) = \underline{\zeta_n}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}, \theta T)$$

and

$$\hat{X}_{T,\underline{\alpha}}(\underline{\sigma}) = \underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \omega, \theta T),$$

and using (10), Theorem 6 and the second statement of Theorem 7, we obtain in a similar way as above that the measure $\hat{P}_{T,\underline{\sigma},\underline{\alpha}}$ also converges weakly to $Q_{\underline{\sigma},\underline{\alpha}}$ as $T \to \infty$.

Proof of Theorem 4. By Theorem 9 it remains to check that the limit measure $Q_{\underline{\sigma},\underline{\alpha}}$ is the distribution $P_{\underline{\sigma},\underline{\alpha}}$ of the random element $\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega})$. This can be done by a standard argument. Let $A \in \mathcal{B}(\mathbb{C}^r)$ be a continuity set of the measure $Q_{\underline{\sigma},\underline{\alpha}}$. Then we have by Theorem 2.1 of [2] that

$$\lim_{T\to\infty} \nu_T^t \left(\zeta(\underline{\sigma}, \underline{\alpha}, \underline{\omega}, t) \in A\right) = Q_{\underline{\sigma},\ \underline{\alpha}}(A) \tag{11}$$

for almost all $\underline{\omega} \in \Omega^{(r)}$. Let us fix the set A, and on $(\Omega^{(r)}, \mathcal{B}(\Omega)^{(r)})$ define the random variable η by

$$\eta = \eta(\underline{\omega}) = \begin{cases} 1 & \text{if} \quad \underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the expectation $\mathbb{E}(\eta)$ of η is

$$\mathbb{E}(\eta) = \int\limits_{\Omega^{(r)}} \eta dm_H^{(r)} = m_H^{(r)}(\{\underline{\omega} \in \Omega^{(r)} : \underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}) \in A\}) = P_{\underline{\sigma},\underline{\alpha}}(A). \tag{12}$$

Taking into account Lemma 8, the process $\eta(\varphi_{t,\underline{\sigma}}(\underline{\omega}))$ is ergodic. Therefore, by the classical Birkhoff-Khintchine theorem, see for example [4], we obtain that

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \eta(\varphi_{t,\underline{\alpha}}(\underline{\omega}) dt = \mathbb{E}(\eta) \tag{13}$$

for almost all $\underline{\omega} \in \Omega^{(r)}$. However, by the definitions of η and $\varphi_{t,\underline{\alpha}}$,

$$\frac{1}{T} \int\limits_0^T \eta(\varphi_{t,\underline{\alpha}}(\underline{\omega}) dt = \nu_T^t(\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}, t) \in A).$$

Now this, (13) and (12) yield

$$\lim_{T\to\infty} \nu_T^t(\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}, t) \in A) = P_{\underline{\sigma},\underline{\alpha}}(A)$$

for almost all $\underline{\omega} \in \Omega^{(r)}$. Thus, by (11), $Q_{\underline{\sigma},\underline{\alpha}}(A) = P_{\underline{\sigma},\underline{\alpha}}(A)$ for all continuity sets A of $Q_{\underline{\sigma},\underline{\alpha}}$ which is sufficient to deduce that $Q_{\underline{\sigma},\underline{\alpha}}(A) = P_{\underline{\sigma},\underline{\alpha}}(A)$ holds for all $A \in \mathcal{B}(\mathbb{C}^r)$. The proof of the theorem is completed.

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