A JOINT LIMIT THEOREM 
FOR HURWITZ–FUNCTIONS 
WITH ALGEBRAIC IRRATIONAL PARAMETERS

A. Laurinčikas (Vilnius, Lithuania) 
J. Steuding (Würzburg, Germany)

To the memory of Professor M.V. Subbarao

Abstract. We prove a limit theorem in the sense of weak convergence of probability measures on the \( r \)-dimensional complex plane for a collection of Hurwitz zeta-functions with algebraic irrational parameters.

1. Introduction

Let \( s = \sigma + it \) be a complex variable, and \( \alpha \) be a fixed real number satisfying \( 0 < \alpha \leq 1 \). The Hurwitz zeta-function \( \zeta(s, \alpha) \) with a parameter \( \alpha \) is for \( \sigma > 1 \) defined by

\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
\]

and by analytic continuation elsewhere, except for a simple pole at \( s = 1 \) with residue 1. If \( \alpha = 1 \), then \( \zeta(s, \alpha) \) reduces to the Riemann zeta-function \( \zeta(s) \).

The first limit theorems in the sense of weak convergence of probability measures for the Hurwitz zeta-function were obtained in [1], for the further development see [7]. However, these results considered \( \zeta(s, \alpha) \) with transcendental or rational parameter \( \alpha \). For the case of an algebraic parameter \( \alpha \) a new

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method was needed, and this case was treated in [6], [8] and [9]. We will recall the latter results.

Denote by \( \text{meas}\{A\} \) the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \), and define, for \( T > 0 \),

\[
\nu_T(...)=\frac{1}{T}\text{meas}\{t\in[0,T]:\ldots\},
\]

where in place of dots a condition satisfied by \( t \) is to be written, and the sign \( t \) in \( \nu_T \) indicates that the measure is taken over \( t \in [0; T] \). Let \( \mathcal{B}(S) \) stand for the class of Borel sets of the space \( S \). We also need the following topological structure.

Suppose that \( \alpha \) is an algebraic irrational number. Define the system

\[ L(\alpha) = \{ \log(m+\alpha) : m \in \mathbb{N}_0 \} \]

where \( \mathbb{N}_0 \) is the set of non-negative integers. J.W.S. Cassels in [3] proved that at least 51 percent of the elements of \( L(\alpha) \) are linearly independent over the field of rational numbers \( \mathbb{Q} \). Let \( I(\alpha) \) be a maximal linearly independent subset of \( L(\alpha) \). We assume that \( I(\alpha) \neq L(\alpha) \), because otherwise we would have the same situation as in the case of transcendental \( \alpha \). Define \( D(\alpha) = L(\alpha) \setminus I(\alpha) \). Then, for any element \( d_m \in D(\alpha) \), the system \( \{d_m\} \cup I(\alpha) \) is already linearly dependent over \( \mathbb{Q} \), and therefore, there exists a finite set of elements \( i_{m_1}, \ldots, i_{m_n} \in I(\alpha) \) such that

\[
k_0(m)d_m+k_1(m)i_{m_1}+\ldots+k_n(m)i_{m_n}=0
\]

with some non-zero integers \( k_j(m), \quad j = 0, 1, \ldots, n \). Hence, we find that

\[
m+\alpha=(m_1+\alpha)^{-\frac{k_1(m)}{k_1(m)}} \cdot \ldots \cdot (m_\nu+\alpha)^{-\frac{k_\nu(m)}{k_\nu(m)}},
\]

Now let \( \mathcal{M}(\alpha) = \{ m : \log(m+\alpha) \in I(\alpha) \} \) and \( \mathcal{N}(\alpha) = \{ m : \log(m+\alpha) \in D(\alpha) \} \). Define

\[
\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,
\]

where \( \gamma_m = \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \) is the unit circle for all \( m \in \mathcal{M}(\alpha) \). Since \( \gamma \) is compact, it follows from the Tikhonov theorem that the torus \( \Omega \) with product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on \( (\Omega, \mathcal{B}(\Omega)) \) the probability Haar measure \( m_H \) exists, and we obtain
a probability space \((\Omega, \mathcal{B}(\Omega), m_H)\). Denote by \(\omega(m)\) the projection of \(\omega \in \Omega\) to \(\gamma_m\). If \(m \in \mathbb{N}(\alpha)\) and relation (1) is true, then we define

\[
\omega(m) = \omega(m_1)^{\frac{k_1(m)}{k_0(m)}} \cdots \omega(m_n)^{\frac{k_n(m)}{k_0(m)}},
\]

where the principal values of the roots are taken. For \(\sigma > \frac{1}{2}\) let

\[
\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m + \alpha)^s}.
\]

Then \(\zeta(\sigma, \alpha, \omega)\), for \(\sigma > \frac{1}{2}\), is a complex-valued random variable on the probability space \((\Omega, \mathcal{B}(\Omega), m_H)\). For a region \(G\) in the complex plane, denote by \(H(G)\) the space of analytic functions on \(G\) equipped with the topology of uniform convergence on compacta. Let \(D_1 = \{s \in \mathbb{C}: \sigma > 1\}\) and \(D_2 = \{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1\}\). Then \(\zeta(s, \alpha, \omega)\) is an \(H(D_1)\)-valued random element as well as an \(H(D_2)\)-valued random element defined on \((\Omega, \mathcal{B}(\Omega), m_H)\).

Now we are able to state limit theorems for \(\zeta(s, \alpha)\).

**Theorem 1.** [8] Suppose that \(\alpha\) is algebraic irrational, and that \(\sigma > \frac{1}{2}\). Then the probability measure

\[
\nu_T^1(\zeta(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),
\]

converges weakly to the distribution of the random variable \(\zeta(\sigma, \alpha, \omega)\) as \(T \to \infty\).

**Theorem 2.** [6] Suppose that \(\alpha\) is algebraic irrational. Then the probability measure

\[
\nu_T^2(\zeta(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_1)),
\]

converges weakly to the distribution of the \(H(D_1)\)-valued random element \(\zeta(s, \alpha, \omega)\) as \(T \to \infty\).

**Theorem 3.** [9] Suppose that \(\alpha\) is algebraic irrational. Then the probability measure

\[
\nu_T^3(\zeta(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_2)),
\]

converges weakly to the distribution of the \(H(D_2)\)-valued random element \(\zeta(s, \alpha, \omega)\) as \(T \to \infty\).

The aim of this paper is a generalization of Theorem 1 for a collection of Hurwitz zeta-functions. Let \(\alpha_1, \ldots, \alpha_r\) be distinct algebraic irrational numbers, \(0 < \alpha_j < 1\), and, for \(\sigma > \frac{1}{2}\),

\[
\zeta(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.
\]
Let $\alpha = (\alpha_1, ..., \alpha_r)$ and $\sigma = (\sigma_1, ..., \sigma_r)$. Define on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ the probability measure $P_{T, \sigma, \alpha}$ by

$$P_{T, \sigma, \alpha}(A) = \nu_T^t\left(\left(\zeta(\sigma_1 + it, \alpha_1), ..., \zeta(\sigma_r + it, \alpha_r)\right) \in A\right).$$

Moreover, let

$$\Omega^{(r)} = \prod_{j=1}^r \Omega_j,$$

where

$$\Omega_j = \prod_{m \in M(\alpha_j)} \gamma_m,$$

with $\gamma_m = \gamma$ for $m \in M(\alpha_j)$, $j = 1, ..., r$. Then $\Omega^{(r)}$ is also a compact topological Abelian group, and we obtain a probability space $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}), m_H^{(r)})$, where $m_H^{(r)}$ is the probability Haar measure on $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}))$. Note that $m_H^{(r)}$ is the product of the Haar measures on $\Omega_j$, $j = 1, ..., r$. Denote by $\omega = (\omega_1, ..., \omega_r)$ the element of $\Omega^r$, where $\omega_j \in \Omega_j$, $j = 1, ..., r$, and define on the probability space $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}), m_H^{(r)})$ a $\mathbb{C}^r$-valued random element $\zeta(\sigma, \alpha, \omega)$, for $\min{\min{\sigma_j}} > \frac{1}{2}$, by

$$\zeta(\sigma, \alpha, \omega) = (\zeta(\sigma_1, \alpha_1, \omega_1), ..., \zeta(\sigma_r, \alpha_r, \omega_r)),$$

where

$$\zeta(\sigma_j, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)}{(m + \alpha_j)^{\sigma_j}}, \quad j = 1, ..., r,$$

and

$$\omega_j(m) = \begin{cases} \text{projection of } \omega_j \in \Omega \text{ to } \gamma_m, & \text{if } m \in M(\alpha_j), \\ \text{relation of type (2)}, & \text{otherwise}, \end{cases} \quad j = 1, ..., r.$$ 

Denote by $P_{\sigma, \alpha}$ the distribution of the random element $\zeta(\sigma, \alpha, \omega)$, that is

$$P_{\sigma, \alpha}(A) = m_H^{(r)}\left(\omega \in \Omega^{(r)} : \zeta(\sigma, \alpha, \omega) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

**Theorem 4.** Suppose that $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers such that the set

$$\bigcup_{j=1}^r I(\alpha_j)$$
is linearly independent over \( \mathbb{Q} \), and that \( \min_{1 \leq j \leq r} \sigma_j > \frac{1}{2} \). Then the probability measure \( P_{T,\mathbb{Q},\alpha} \) converges weakly to the measure \( P_{\mathbb{Q},\alpha} \) as \( T \to \infty \).

**Example.** Let \( \alpha_j = p_j^{-1/2} \), where \( p_j \) denotes the \( j \)-th prime number (in ascending order). Then the hypotheses of Theorem 4 are satisfied.

2. **A limit theorem on \( \Omega^{(r)} \)**

We start the proof of Theorem 4 with a limit theorem on \( \Omega^{(r)} \). For \( A \in \mathcal{B}(\Omega^{(r)}) \), define

\[
Q_T(A) = \nu_T^r \left( \{(m + \alpha_1)\uparrow : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{(m + \alpha_r)\uparrow : m \in \mathcal{M}(\alpha_r)\} \right) \in A.
\]

**Lemma 5.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) satisfy the hypotheses of Theorem 4. Then the probability measure \( Q_T \) converges weakly to the Haar measure \( m_H^{(r)} \) as \( T \to \infty \).

**Proof.** The dual group of \( \Omega^{(r)} \) is isomorphic to

\[
\bigoplus_{j=1}^r \bigoplus_{m \in \mathcal{M}(\alpha_j)} \mathbb{Z}_{m_j},
\]

where \( \mathbb{Z}_{m_j} = \mathbb{Z} \) for all \( m \in \mathcal{M}(\alpha_j) \) and \( j = 1, \ldots, r \). The element

\[
k = ((k_{m1})_{m \in \mathcal{M}(\alpha_1)}, \ldots, (k_{mr})_{m \in \mathcal{M}(\alpha_r)}) \in \bigoplus_{j=1}^r \bigoplus_{m \in \mathcal{M}(\alpha_j)} \mathbb{Z}_{m_j}
\]

acts on \( \Omega^{(r)} \) by

\[
\omega \mapsto \omega^k = \prod_{j=1}^r \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m),
\]

where only a finite number of integers \( k_{mj} \) are distinct from zero. Therefore, the Fourier transform \( g_T(k) \) of the measure \( Q_T \) is

\[
g_T(k) = \int_{\Omega^{(r)}} \prod_{j=1}^r \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m) dQ_T,
\]
where, as above, only a finite number of integers $k_{mj}$ are distinct from zero. Thus, by the definition of $Q_T$ we have

$$g_T(k) = \frac{1}{T} \int_0^T \prod_{j=1}^r \prod_{m \in M(\alpha_j)} (m + \alpha_j)^{ik_{mj}} dt =$$

$$= \frac{1}{T} \int_0^T \exp \left\{ \sum_{j=1}^r \sum_{m \in M(\alpha_j)} k_{mj} \log(m + \alpha_j) \right\} dt.$$

Since the set $\bigcup_{j=1}^r I(\alpha_j)$ is linearly independent over $\mathbb{Q}$, hence we easily find that

$$g_T(k) = \begin{cases} 1, & \text{if } k = 0, \\ \exp \left\{ \frac{iT \sum_{j=1}^r \sum_{m \in M(\alpha_j)} k_{mj} \log(m + \alpha_j)}{iT \sum_{j=1}^r \sum_{m \in M(\alpha_j)} k_{mj} \log(m + \alpha_j)} \right\}^{-1}, & \text{otherwise.} \end{cases}$$

Therefore,

$$\lim_{T \to \infty} g_T(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Hence, by general continuous theorems on locally compact groups, see, for example [5], Theorem 1.4.2, we obtain that the measure $Q_T$ converges weakly to $m_H^{(r)}$ as $T \to \infty$.

3. Joint limit theorems for absolutely convergent series

For fixed $\sigma_{1j} > \frac{1}{2}$ and $m,n \in \mathbb{N}_0$, we put

$$v_j(m,n,\alpha_j) = \exp \left\{ - \left( \frac{m + \alpha_j}{n + \alpha_j} \right)^{\sigma_{1j}} \right\},$$
and define, for \( \sigma > \frac{1}{2} \),

\[
\zeta_{n,j}(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_j(m, n, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.
\]

In [8] it was observed that the latter series converges absolutely for \( \sigma > \frac{1}{2} \).

Since \( |\omega_j(m)| = 1 \), the series

\[
\zeta_{n,j}(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)v_j(m, n, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r
\]

is absolutely convergent in the region \( \sigma > \frac{1}{2} \).

This section is devoted to the weak convergence of the probability measures

\[
P_{T,n,\sigma,\alpha}(A) = \nu_{T}^{\sigma}((\zeta_{n,1}(\sigma_1 + it, \alpha_1), \ldots, \zeta_{n,r}(\sigma_r + it, \alpha_r)) \in A), \quad A \in \mathcal{B}(\mathbb{C}^r),
\]

and

\[
\hat{P}_{T,n,\sigma,\alpha}(A) = \nu_{T}^{\sigma}((\zeta_{n,1}(\sigma_1 + it, \alpha_1, \hat{\omega}_1), \ldots, \zeta_{n,r}(\sigma_r + it, \alpha_r, \hat{\omega}_r)) \in A), \quad A \in \mathcal{B}(\mathbb{C}^r),
\]

where \( \hat{\omega} = (\hat{\omega}_1, \ldots, \hat{\omega}_r) \) is a fixed element of \( \Omega^r \).

**Theorem 6.** Let \( \min_{1 \leq j \leq r} \sigma_j > \frac{1}{2} \). Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) satisfy the hypotheses of Theorem 4. Then there exists a probability measure \( P_{n,\sigma,\alpha} \) on \((\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))\) such that both the measures \( P_{T,n,\sigma,\alpha} \) and \( \hat{P}_{T,n,\sigma,\alpha} \) converge weakly to \( P_{n,\sigma,\alpha} \) as \( T \to \infty \).

**Proof.** Let the function \( h_{n,\sigma,\alpha} : \Omega^r \to \mathbb{C}^r \) be given by the formula

\[
h_{n,\sigma,\alpha}(\{(\omega_1(m) : m \in \mathcal{M}(\alpha_1)), \ldots, (\omega_r(m) : m \in \mathcal{M}(\alpha_r))\}) = \left( \sum_{m=0}^{\infty} \frac{v_1(m, n, \alpha_1)}{(m + \alpha_1)^s \omega_1(m)}, \ldots, \sum_{m=0}^{\infty} \frac{v_r(m, n, \alpha_r)}{(m + \alpha_r)^s \omega_r(m)} \right).
\]

Since the series

\[
\sum_{m=0}^{\infty} \frac{v_j(m, n, \alpha_j)}{(m + \alpha_j)^s \omega_j(m)}, \quad j = 1, \ldots, r
\]

converges uniformly with respect to \( \omega_j \), the function \( h_{n,\sigma,\alpha} \) is continuous. Moreover, we see that

\[
h_{n,\sigma,\alpha}(\{(m + \alpha_1)^u : m \in \mathcal{M}(\alpha_1)), \ldots, (m + \alpha_r)^u : m \in \mathcal{M}(\alpha_r))\}) = \]
Therefore, Lemma 5 together with Theorem 5.1 of [2] shows that the probability measure \( P_{T,n,\sigma,\alpha} \) converges weakly to the measure \( m_{h,n,\sigma,\alpha}^{-1} \) as \( T \to \infty \).

Now let \( \hat{h}_{\alpha,\omega} : \Omega^\omega \to \Omega^\omega \) be defined by
\[
\hat{h}_{\alpha,\omega}((\{\omega_1(m) : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{\omega_r(m) : m \in \mathcal{M}(\alpha_r)\})) = \left(\{\omega_1(m)\hat{\omega}_1^{-1}(m) : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{\omega_r(m)\hat{\omega}_r^{-1}(m) : m \in \mathcal{M}(\alpha_r)\}\right).
\]
Then we have that
\[
\left(\sum_{m=0}^{\infty} \hat{\omega}_1(m)v_1(m, n, \alpha_1)(m + \alpha_1)^{\sigma_1}, \ldots, \sum_{m=0}^{\infty} \hat{\omega}_r(m)v_r(m, n, \alpha_r)(m + \alpha_r)^{\sigma_r}\right) = \hat{h}_{\alpha,\omega}(\{|(m + \alpha_1)^{\sigma_1} : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{|(m + \alpha_r)^{\sigma_r} : m \in \mathcal{M}(\alpha_r)\}|).
\]
Thus, similarly to the case of the measure \( P_{T,n,\sigma,\alpha} \), we obtain that the measure \( \hat{P}_{T,n,\sigma,\alpha} \) converges weakly to \( m_{H}^{-1}(h_{n,\sigma,\alpha}\hat{h}_{\alpha,\omega})^{-1} \) as \( T \to \infty \). Since the Haar measure \( m_{H}^{-1} \) is invariant, we have
\[
m_{H}^{-1}(h_{n,\sigma,\alpha}\hat{h}_{\alpha,\omega})^{-1} = \left(m_{H}^{-1}(\hat{h}_{\alpha,\omega})\right)^{-1} = m_{H}^{-1}(h_{n,\sigma,\alpha}^{-1}),
\]
and the theorem is proved.

4. Approximation in the mean

For \( \bar{z}^{(1)} = (z_1^{(1)}, \ldots, z_r^{(1)}) \), \( \bar{z}^{(2)} = (z_1^{(2)}, \ldots, z_r^{(2)}) \in \mathbb{C}^r \), let
\[
\rho(\bar{z}^{(1)}, \bar{z}^{(2)}) = \left(\sum_{j=1}^{r} |z_j^{(1)} - z_j^{(2)}|^2\right)^{\frac{1}{2}}.
\]
Then \( \rho \) is a metric in the space \( \mathbb{C}^r \) which induces its topology. In this section, we will prove some preliminary results in order to pass from Theorem 6 to Theorem 4.
A joint limit theorem for Hurwitz zeta-functions

For brevity, define

\[ \zeta(\sigma, \alpha, t) = (\zeta(\sigma_1 + it, \alpha_1), ..., \zeta(\sigma_r + it, \alpha_r)), \]
\[ \zeta_n(\sigma, \alpha, t) = (\zeta_{n,1}(\sigma_1 + it, \alpha_1), ..., \zeta_{n,r}(\sigma_r + it, \alpha_r)), \]

and

\[ \zeta(\sigma, \alpha, \omega, t) = (\zeta(\sigma_1 + it, \alpha_1, \omega_1), ..., \zeta(\sigma_r + it, \alpha_r, \omega_r)), \]
\[ \zeta_n(\sigma, \alpha, \omega, t) = (\zeta_{n,1}(\sigma_1 + it, \alpha_1, \omega_1), ..., \zeta_{n,r}(\sigma_r + it, \alpha_r, \omega_r)). \]

**Theorem 7.** Suppose that \( \min_{1 \leq j \leq r} \sigma_j > \frac{1}{2} \) and \( \alpha_1, ..., \alpha_r \) satisfy the hypotheses of Theorem 4. Then

\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(\sigma, \alpha, t), \zeta_n(\sigma, \alpha, t)) \, dt = 0 \]

and, for almost all \( \omega \in \Omega(r) \),

\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho(\zeta(\sigma, \alpha, \omega, t), \zeta_n(\sigma, \alpha, \omega, t)) \, dt = 0. \]

**Proof.** By Lemmas 6 and 9 of [8] we have, for each \( j = 1, ..., r \),

\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T |\zeta(\sigma_j + it, \alpha_j) - \zeta_n(\sigma_j + it, \alpha_j)| \, dt = 0, \]

and, for almost all \( \omega_j \in \Omega_j \),

\[ \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T |\zeta(\sigma_j + it, \alpha_j, \omega_j) - \zeta_n(\sigma_j + it, \alpha_j, \omega_j)| \, dt = 0. \]

This and the definition of \( \rho \) prove the theorem.
5. Ergodicity

For $t \in \mathbb{R}$, define

$$a_{t, \alpha} = \{(m + \alpha_1)^{-it} : m \in \mathcal{M}(\alpha_1)\}, \ldots,\{(m + \alpha_r)^{-it} : m \in \mathcal{M}(\alpha_r)\}.$$ 

Now let $\{\varphi_{t, \alpha} : t \in \mathbb{R}\}$ be the one-parameter family of transformations on $\Omega^{(r)}$ for $\omega \in \Omega^{(r)}$ defined by

$$\varphi_{t, \alpha}(\omega) = a_{t, \alpha}\omega.$$ 

Then $\{\varphi_{t, \alpha} : t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\Omega^{(r)}$. We recall that a set $A \in \mathcal{B}(\Omega^{(r)})$ is invariant with respect to the group $\{\varphi_{t, \alpha} : t \in \mathbb{R}\}$ if, for each $t$, the sets $A$ and $A_t = \varphi_{t, \alpha}(A)$ differ one from another by a set of zero $m_H^{(r)}$-measure. All invariant sets form a $\sigma$-subfield of the field $\mathcal{B}(\Omega^{(r)})$. The one-parameter group $\{\varphi_{t, \alpha} : t \in \mathbb{R}\}$ is called ergodic if its $\sigma$-field of invariant sets consists only of sets having $m_H^{(r)}$-measure equal to $0$ or $1$.

**Lemma 8.** The one-parameter group $\{\varphi_{t, \alpha} : t \in \mathbb{R}\}$ is ergodic.

**Proof.** Let $\chi : \Omega^{(r)} \to \gamma$ be a character. Then

$$\chi(\omega) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m),$$

where only a finite number of integers $k_{mj}$ are distinct from zero. If $\chi$ is a non-principal character, then

$$\chi(a_{t, \alpha}) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} (m + \alpha_j)^{-itk_{mj}}.$$ 

Since the set

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$, hence we find that there exists a $t_0 \neq 0$ such that $\chi(a_{t, \alpha}) \neq 1$. The further proof runs in the same way as that of Lemma 7 from [8].
6. Proof of Theorem 4

Define on \((\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))\) another probability measure

\[
\hat{P}_{T, \mathcal{A}}(A) = \nu_T (\{\zeta(\sigma_1 + it, \alpha_1, \omega_1), \ldots, \zeta(\sigma_r + it, \alpha_r, \omega_r) \in A\}),
\]

where \((\omega_1, \ldots, \omega_r) \in \Omega^r\).

Theorem 9. Suppose that \(\min_{1 \leq j \leq r} \sigma_j > \frac{1}{2}\), and that the numbers \(\alpha_1, \ldots, \alpha_r\) satisfy the hypotheses of Theorem 4. Then on \((\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))\) there exists a probability measure \(P_{\mathcal{A}}\) such that both the measures \(P_{T, \mathcal{A}}\) and \(\hat{P}_{T, \mathcal{A}}\) converge weakly to \(P_{\mathcal{A}}\) as \(T \to \infty\).

Proof. By Theorem 6, both the measures \(P_{T, n, \mathcal{A}}\) and \(\hat{P}_{T, n, \mathcal{A}}\) converge weakly to the same measure \(P_{\mathcal{A}} \overset{\text{def}}{=} m^{(r)}(\mathcal{H}^{r-1}_n)\) as \(T \to \infty\). We will prove that the family of probability measures \(\{P_{n, \mathcal{A}} : n \in \mathbb{N}_0\}\) is tight (for the definition, see [2]).

By the definition of \(P_{T, n, \mathcal{A}}\) and Chebyshev’s inequality, for \(M > 0\), we have

\[
P_{T, n, \mathcal{A}}(\{z \in \mathbb{C}^r : \rho(z, 0) > M\}) = \nu_T (\{\rho(\zeta_n(z, \alpha, t), 0) > M\}) \leq \frac{1}{MT} \int_0^T \rho(\zeta_n(z, \alpha, t), 0) dt \leq \frac{1}{M} \left( \frac{1}{T} \int_0^T \left( \sum_{j=1}^r |\zeta_{n,j}(\sigma_j + it, \alpha_j)|^2 \right) dt \right)^{\frac{1}{2}}
\]

\[
= \frac{1}{M} \left( \frac{1}{T} \int_0^T \left( \sum_{j=1}^r |\zeta_{n,j}(\sigma_j + it, \alpha_j)|^2 \right) dt \right)^{\frac{1}{2}}.
\]

Since the series for \(\zeta_{n,j}(\sigma_j + it, \alpha_j)\) converges absolutely, we have that

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T |\zeta_{n,j}(\sigma_j + it, \alpha_j)|^2 dt =
\]
\[
\sum_{m=0}^{\infty} \frac{v_j^2(m, n, \alpha_j)}{(m + \alpha_j)^{2\sigma_j}} \leq \sum_{m=0}^{\infty} \frac{1}{(m + \alpha_j)^{2\sigma_j}} \text{def} R_j < \infty
\]

for \( j = 1, \ldots, r \). Therefore, in view of (3),

\[
\limsup_{T \to \infty} P_{T, n, \sigma, \alpha}(\{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) > M \}) \leq
\]

\[
\leq \sup_{n \in \mathbb{N}_0} \limsup_{T \to \infty} \frac{1}{M} \left( \sum_{j=1}^{r} \frac{1}{T} \int_{0}^{T} |\zeta_{n,j}(\sigma_j + it, \alpha_j)|^2 dt \right)^{\frac{1}{2}} \leq \frac{R}{M}
\]

with

\[
R = \left( \sum_{j=1}^{r} R_j \right)^{\frac{1}{2}} < \infty.
\]

For arbitrary \( \varepsilon > 0 \), let \( M = R\varepsilon^{-1} \). Then (4) yields the inequality

\[
\limsup_{T \to \infty} P_{T, n, \sigma, \alpha}(\{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) > M \}) \leq \varepsilon.
\]

Obviously, the function \( h : \mathbb{C}^r \to \mathbb{R} \) given by \( h(\tilde{z}) = \rho(\tilde{z}, 0) \) is continuous. This, Theorem 6 and Theorem 5.1 of [2] show that the probability measure

\[
\nu_T^{h^{-1}}(\rho(\tilde{z}, 0), t, 0) \in A), \quad A \in \mathcal{B}(\mathbb{R}),
\]

converges weakly to \( P_{n, \sigma, \alpha} h^{-1} \) as \( T \to \infty \). Since the set \( \{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) > M \} \) is open, this, Theorem 2.1 of [2] and (5) imply

\[
P_{n, \sigma, \alpha}(\{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) > M \}) \leq \liminf_{T \to \infty} P_{T, n, \sigma, \alpha}(\{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) > M \}) \leq \varepsilon.
\]

The set \( K_\varepsilon = \{ \tilde{z} \in \mathbb{C}^r : \rho(\tilde{z}, 0) \leq M \} \) is compact in \( \mathbb{C}^r \), and by (6)

\[
P_{n, \sigma, \alpha}(K_\varepsilon) \geq 1 - \varepsilon
\]

for all \( n \in \mathbb{N}_0 \). So, the tightness of the family \( \{ P_{n, \sigma, \alpha} : n \in \mathbb{N}_0 \} \) is proved. Hence, by the Prokhorov theorem [2] this family is relatively compact. Therefore, there exists a subsequence \( \{ P_{n_1, \sigma, \alpha} \} \subset \{ P_{n, \sigma, \alpha} \} \) such that \( P_{n_1, \sigma, \alpha} \) converges weakly to some measure \( Q_{\sigma, \alpha} \) on \( (\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r)) \) as \( n_1 \to \infty \). Let \( X_n, \sigma(\alpha) \) be a \( \mathbb{C}^r \)-valued random element with the distribution \( P_{n, \sigma, \alpha} \) and
let $\overset{D}{\to}$ denote the convergence in distribution. Then the weak convergence of $P_{n, \sigma, \alpha}$ to $P_{\sigma, \alpha}$ as $n \to \infty$ is equivalent to the relation

$$X_{n, \sigma, \alpha}(\xi) \overset{D}{\to}_{n \to \infty} Q_{\sigma, \alpha}.$$  

Let $\theta$ denote a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{\mathbb{P}})$ and uniformly distributed on $[0, 1]$. We set

$$X_{T, n, \alpha}(\sigma) = \zeta_n(\sigma, \alpha, \theta T).$$

Then the statement of Theorem 6 can be written in the form

$$X_{T, n, \alpha}(\sigma) \overset{D}{\to}_{T \to \infty} X_{n, \alpha}(\sigma).$$

Moreover, by the first assertion of Theorem 7, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P} \left( \xi \left( X_{T, n, \alpha}(\sigma), X_{T, \alpha}(\sigma) > \varepsilon \right) = \right.$$ 

$$\lim_{n \to \infty} \limsup_{T \to \infty} \nu_T \left( \xi \left( \zeta(\sigma, \alpha, t) \right), \zeta_n(\sigma, \alpha, t) > \varepsilon \right) \leq$$ 

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \xi \left( \zeta(\sigma, \alpha, t) \right), \zeta_n(\sigma, \alpha, t) dt = 0,$$

where

$$X_{T, \alpha}(\sigma) = \zeta(\sigma, \alpha, \theta T).$$

Now this, (7), (8) and Theorem 4.2 of [2] show that

$$X_{T, \sigma, \alpha}(\sigma) \overset{D}{\to}_{T \to \infty} Q_{\sigma, \alpha},$$

which is equivalent to the weak convergence of the measure $P_{T, \sigma, \alpha}$ to $P_{\sigma, \alpha}$ as $T \to \infty$.

It remains to obtain the same result for the measure $\hat{P}_{T, \sigma, \alpha}$. First we observe that in view of (9), the measure $P_{\sigma, \alpha}$ is independent on the sequence $P_{n, \sigma, \alpha}$. Thus, from the relative compactness of $\{P_{n, \sigma, \alpha}\}$ we have that

$$X_{n, \sigma, \alpha}(\sigma) \overset{D}{\to}_{n \to \infty} Q_{\sigma, \alpha}.$$
Now putting
\[ \hat{X}_{T,n,\alpha}(\sigma) = \zeta_n(\sigma, \alpha, \omega, \theta T) \]
and
\[ \hat{X}_{T,\alpha}(\sigma) = \zeta(\sigma, \alpha, \omega, \theta T), \]
and using (10), Theorem 6 and the second statement of Theorem 7, we obtain in a similar way as above that the measure \( \hat{P}_{T,\sigma,\alpha} \) also converges weakly to \( Q_{\sigma,\alpha} \) as \( T \to \infty \).

**Proof of Theorem 4.** By Theorem 9 it remains to check that the limit measure \( Q_{\sigma,\alpha} \) is the distribution \( P_{\sigma,\alpha} \) of the random element \( \zeta(\sigma, \alpha, \omega) \). This can be done by a standard argument. Let \( A \in \mathcal{B}(\mathbb{C}^r) \) be a continuity set of the measure \( Q_{\sigma,\alpha} \). Then we have by Theorem 2.1 of [2] that

\[ \lim_{T \to \infty} \nu_T^\alpha(\zeta(\sigma, \alpha, \omega, t) \in A) = Q_{\sigma,\alpha}(A) \]
for almost all \( \omega \in \Omega^{(r)} \). Let us fix the set \( A \), and on \( (\Omega^{(r)}, \mathcal{B}(\Omega^{(r)})) \) define the random variable \( \eta \) by

\[ \eta = \eta(\omega) = \begin{cases} 1 & \text{if } \zeta(\sigma, \alpha, \omega) \in A, \\ 0 & \text{otherwise}. \end{cases} \]

It is easily seen that the expectation \( E(\eta) \) of \( \eta \) is

\[ E(\eta) = \int_{\Omega^{(r)}} \eta d m_H^{(r)} = m_H^{(r)}(\{\omega \in \Omega^{(r)} : \zeta(\sigma, \alpha, \omega) \in A\}) = P_{\sigma,\alpha}(A). \]

Taking into account Lemma 8, the process \( \eta(\varphi_{t,\alpha}(\omega)) \) is ergodic. Therefore, by the classical Birkhoff-Khintchine theorem, see for example [4], we obtain that

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(\varphi_{t,\alpha}(\omega)) dt = E(\eta) \]
for almost all \( \omega \in \Omega^{(r)} \). However, by the definitions of \( \eta \) and \( \varphi_{t,\alpha} \),

\[ \frac{1}{T} \int_0^T \eta(\varphi_{t,\alpha}(\omega)) dt = \nu_T^\alpha(\zeta(\sigma, \alpha, \omega, t) \in A). \]

Now this, (13) and (12) yield
\[
\lim_{T \to \infty} \nu^T_t(\zeta(\sigma, \alpha, \omega, t) \in A) = P^{\omega}_T(A)
\]
for almost all \(\omega \in \Omega(\tau)\). Thus, by (11), \(Q^{\omega}_\sigma(A) = P^{\omega}_T(A)\) for all continuity sets \(A\) of \(Q^{\omega}_\sigma\), which is sufficient to deduce that \(Q^{\omega}_\sigma(A) = P^{\omega}_T(A)\) holds for all \(A \in B(\mathbb{C}^r)\). The proof of the theorem is completed.

References


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A. Laurinčikas
Dept. of Probability Theory and Number Theory
Vilnius University
Naugarduko 24
LT-03225 Vilnius, Lithuania

J. Steuding
Department of Mathematics
Würzburg University
Am Hubland
D-97074 Würzburg, Germany